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A Goodness-of-Fit Test for Rayleigh Distribution Based on Hellinger Distance

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Abstract In this paper, we introduce a new goodness-of-fit test for Rayleigh distribution based on Hellinger distance. In addition, some properties about the proposed test is presented. Then, new proposed test is compared with other goodness-of-fit tests for Rayleigh distribution in the literature in terms of power. Finally, we conclude that the entropy based tests demonstrate a good performance in terms of power and we can choose the Hellinger test as more powerful than the other competitor tests.

Keywords Entropy · Goodness-of-fit · Hellinger distance · Power · Rayleigh distribution

1 Introduction

The goodness-of-fit (GOF) tests for exponentiality have a large literature, but GOF tests for Rayleigh distribution (testing Rayleighity) have recently been considered. The GOF tests for the Rayleigh distribution were proposed in [2,3,5,12–14,17].

The Rayleigh distribution has been used in many areas of research, such as reliability, life testing and survival analysis. Modeling the lifetime of random phenomena has been another area of study for which the Rayleigh distribution has been significantly used. Being first introduced by Rayleigh [16], this statistical model was originally derived in connection with a problem in acoustics. It has been also used as the distance distribution between individuals in a spatial Poisson process and is useful in life-testing experiments, for its failure rate is a linear function of time. Dyer and

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Whisenand [7] established its importance in communication engineering and Polovko [15] stated that it is important in electro-vacuum devices. More details on the Rayleigh distribution can be found in [10].

The Rayleigh distribution has the following probability density function (pdf) and cumulative distribution function (cdf) respectively

$$f(x) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right), \qquad x > 0,\ \theta > 0,$$
$$F(x) = 1 - \exp\left(-\frac{x^2}{2\theta^2}\right), \qquad x > 0,\ \theta > 0.$$

It is easy to show that the maximum likelihood (ML) estimator of θ is

$$\hat{\theta} = \sqrt{\sum_{i=1}^{n} x_i^2/2n}.$$

It is important that $\hat{\theta}^2 = \sum x_i^2/2n$ is an unbiased ML estimate of θ^2, but $\hat{\theta}$ is a biased estimator of θ.

This article is organized as follows. In Sect. 2, a new test based on the Hellinger distance is developed and some theorems about its properties is presented. In Sect. 3, we compare the powers of the proposed test against competitors tests for each category of alternative distributions. In Sect. 4, the performance of the considered tests for two real data are evaluated. In Sect. 5, we appraise the ability of considered tests and we choose a better one.

2 New Proposed Test

In mathematics, metric functions are very important. Some measures proposed for determining the GOF do not possess all properties of a metric function. Therefore, they are rather called as divergence measures. For example, KL divergence is nonnegative and asymmetric. Also, it is easy to see that it does not satisfy the triangular inequality. Therefore it is not a metric function. Hence it must be interpreted as a pseudo-metric measure only.

Since we want to propose a new GOF test for Rayleigh distribution, we suggest using Hellinger distance which possesses all conditions of a metric function [9]. We guess using this measure leads to better performance in terms of power against other proposed tests in the statistical literature.

2.1 Hellinger Test

In this subsection, for constructing a new test based on Hellinger distance, we use the Vasicek's (1976) method to estimate the Shannon entropy. In this regard, proposed test will be belong to entropy based tests.

Let $X_1, ..., X_n$ be nonnegative; independent and identically distributed (iid) random variables from a continuous distribution function F with order statistics, $X_{(1)} \leq ... \leq X_{(n)}$. Let $f_0(x, \theta)$ denote a Rayleigh distribution, where θ is the unknown parameter. The hypotheses are as follow

$$H_0 : f(x) = f_0(x, \theta), \qquad H_1 : f(x) \neq f_0(x, \theta).$$

Compared with the KL divergence, Hellinger distance avoids stability problems when the denominator probability density function is zero. The Hellinger distance for discriminating between two hypotheses H_0 and H_1 for two density functions $f(x)$ and $f_0(x)$ is introduced by [9] as

$$D_H(f, f_0) = \frac{1}{2}\int_0^{\infty}\left(\sqrt{f(x)} - \sqrt{f_0(x)}\right)^2 dx, \tag{1}$$

It's very important that the Hellinger distance is symmetric and has all properties of a metric function. Since $D_H(f, f_0) \geq 0$ and the equality holds if and only if $f(x) = f_0(x)$, it motivates us to use Hellinger distance as a test statistic for checking Rayleigh distribution.

The Hellinger statistic for testing Rayleighity can be defined as below

$$DH_{mn} = \frac{1}{2n}\sum_{i=1}^{n}\frac{\left\{\sqrt{\left(\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right)^{-1}} - \sqrt{X_{(i)}e^{-X_{(i)}^2/2\hat{\theta}^2}/\hat{\theta}^2}\right\}^2}{\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}^{-1}},$$

where $X_{(i)} = X_{(1)}$ for $i < 1$ and $X_{(i)} = X_{(n)}$ for $i > n$.

By replacing f_0 in (1) we have

$$D_H(f, f_0) = \frac{1}{2}\int_0^{\infty}\left(\sqrt{f(x)} - \sqrt{\left(x/\theta^2\right)e^{-x^2/2\theta^2}}\right)^2 dx.$$

Under the null hypothesis $D_H(f, f_0) = 0$ and we expect large values of $D_H(f, f_0)$ under H_1. Based on the Vasicek's method [18] and supposing $F(x) = p$ we have

$$D_H(f, f_0) = \frac{1}{2}\int_0^{1}\left(\sqrt{\left(\frac{d}{dp}F^{-1}(p)\right)^{-1}} - \sqrt{\frac{F^{-1}(p)}{\theta^2}e^{-\left(F^{-1}(p)\right)^2/2\theta^2}}\right)^2 \times \left(\frac{d}{dp}F^{-1}(p)\right)dp.$$

Now, the proposed statistic easily will be derived by using following relation

$$
\begin{aligned}
f(x) &= \left(\frac{d}{dp}F^{-1}(p)\right)^{-1}, \\
&\simeq \left(\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right)^{-1}.
\end{aligned}
$$

As θ is a scale parameter of Rayleigh distribution we consider a scale transformation group as $G = \{g_c : g_c(X) = cX,\ c > 0\}$. Since

$$
DH_{mn}(g(X)) = DH_{mn}(X) \quad \forall X,\ \forall g \in G,
$$

Therefore, DH_{mn} is a scale-free statistic or invariant under G.

In addition, since the test statistic DH_{mn} is invariant under the scale transformations and the parameter space is transitive, the distribution of the proposed test statistic DH_{mn} is free of θ.

The least favorable conditions for an estimator is consistency the following theorem proves the consistency property of DH_{mn}.

Theorem 1 *Let F be an unknown continuous distribution on* $[0, \infty)$ *and F_0 be the Rayleigh distribution with unspecified parameter θ. Then under H_1, the test based on DH_{mn} is consistent.*

Proof We use [1] technique to prove consistency of DH_{mn}. As $n, m \to \infty$ and $m/n \to 0$, we have

$$
\begin{aligned}
\frac{2m}{n} &= F_n\left(X_{(i+m)}\right) - F_n\left(X_{(i-m)}\right) \simeq F\left(X_{(i+m)}\right) - F\left(X_{(i-m)}\right) \\
&\simeq \frac{f(X_{(i+m)}) + f(X_{(i-m)})}{2}\left(X_{(i+m)} - X_{(i-m)}\right) \\
&\simeq f(X_{(i)})\left(X_{(i+m)} - X_{(i-m)}\right),
\end{aligned}
$$

where F_n is the empirical distribution function. Since $\hat{\theta}$ is a consistent estimator of θ as $n \to \infty$ and based upon strong law of large numbers we have

$$
\begin{aligned}
DH_{mn} &= \frac{1}{2n}\sum_{i=1}^{n} \frac{\left\{\sqrt{\left(\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right)^{-1}} - \sqrt{X_{(i)}e^{-X_{(i)}^2/2\hat{\theta}^2}/\hat{\theta}^2}\right\}^2}{\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}^{-1}} \\
&\simeq \frac{1}{2n}\sum_{i=1}^{n} \frac{\left\{\sqrt{f(X_{(i)})} - \sqrt{X_{(i)}e^{-X_{(i)}^2/2\theta^2}/\theta^2}\right\}^2}{f(X_{(i)})} \\
&= \frac{1}{2n}\sum_{i=1}^{n} \frac{\left\{\sqrt{f(X_{(i)})} - \sqrt{f_0(X_{(i)})}\right\}^2}{f(X_{(i)})} \xrightarrow{a.s.} \frac{1}{2}E\left\{\frac{\left\{\sqrt{f(X)} - \sqrt{f_0(X)}\right\}^2}{f(X)}\right\}
\end{aligned}
$$

Table 1 Optimal values of m for various values of n

n	5–12	13–27	28–50	51–57	58–71	72–100	101–120	121–150	>150
m	2	3	4	5	6	7	8	9	10

$$= \frac{1}{2}\int \left\{\sqrt{f(x)} - \sqrt{f_0(x)}\right\}^2 dx$$
$$= D_H(f, f_0).$$

Thus, DH_{mn} is a consistent test under H_1. □

According to proved Theorem 1 and mentioned properties, the DH_{mn} test is a reasonable test for the Rayleigh distribution which has some good properties such as to be scale-free, invariancy and consistency.

2.2 Determining Optimum Values of *m*

The GOF test based on entropy involves choosing the best integer parameter m. Unfortunately, there is no choice criterion of m and in general it depends on n. Ebrahimi et al. [8] tabulated the values of m, which maximize the power of the test.

In fact, the optimum value of m is a value of n which leads to the smallest value of bias and mean square error (MSE). Therefore, we determined the optimum values of m based on 10,000 iterations whose results are presented in Table 1.

To determine optimum value of m for any value of n, we computed bias and MSE of H_{mn} for different values of m, 1 to $n/2$, by

$$\text{Bias} = \frac{1}{k}\sum_{i=1}^{k} H_{mn}^{(i)} - H(X), \quad \text{MSE} = \frac{1}{k}\sum_{i=1}^{k}\left\{H_{mn}^{(i)} - H(X)\right\}^2,$$

where $H(X) = 1 + \ln(\theta/\sqrt{2}) + \gamma/2$ is the entropy of Rayleigh distribution, γ is the Euler-Mascheroni constant which is already 0.57721 and k is number of iterations.

3 Power Study

In this section, we are going to compare performance of the proposed test against several alternative tests in terms of power. Moreover, considered tests are divided in entropy and non-entropy groups.

The GOF tests of Rayleighity in the statistical literature are as follows.

- Alizadeh et al. [2] proposed the GOF test based on Kullback-Leibler (KL) divergence for checking Rayleighity. Their proposed statistic is

$$KL_{mn} = -H_{mn} + 2\log(\hat{\theta}) - \frac{1}{n}\sum_{i=1}^{n}\log(X_i) + 1,$$

where $\hat{\theta}$ is the ML estimate of θ and H_{mn} is the Vasicek's estimator of entropy which is given by $H_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}$ where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$ and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics based on a random sample of size n.

- Baratpour and Khodadadi [3] based on cumulative KL, defined another GOF test for Rayleighity which is

$$CK_n = \frac{\sum_{i=1}^{n}\left(1-\frac{i}{n}\right)\ln\left(1-\frac{i}{n}\right)\left(X_{(i+1)} - X_{(i)}\right) + \sqrt{\frac{\pi}{2}}\sqrt{\sum_{i=1}^{n} X_i^3 \Big/ 3\sum_{i=1}^{n} X_i}}{\bar{X}},$$

- Meintanis and Iliopoulos [12] proposed the GOF test of Rayleighity based on the empirical Laplace transform which was defined as

$$\begin{aligned} L = \frac{n}{a} + \frac{\sqrt{2}}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}&\left\{\frac{1}{\hat{Y}_j + \hat{Y}_k + a} + \frac{\hat{Y}_j + \hat{Y}_k}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^2} + \frac{2\left(\hat{Y}_j\hat{Y}_k + 2\right)}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^3}\right. \\ &\left.+\frac{6\left(\hat{Y}_j + \hat{Y}_k\right)}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^4} + \frac{24}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^5}\right\} \\ &-2\sqrt{2}\sum_{j=1}^{n}\left\{\frac{1}{\left(\hat{Y}_j + a\right)} + \frac{\hat{Y}_j}{\left(\hat{Y}_j + a\right)^2} + \frac{2}{\left(\hat{Y}_j + a\right)^3}\right\}, \end{aligned}$$

where $a = 2\sqrt{2}$, $\hat{Y}_j = X_j/\hat{\theta}$ and $\hat{\theta}$ denotes the consistent estimator of θ.

- Safavinejad et al. [17], based on the empirical likelihood ratio methodology, proposed the following statistic

$$R_n = \frac{\min_{1\leq m<n^\delta}\prod_{j=1}^{n}\left\{\frac{2m}{n}\left(X_{(j+m)} - X_{(j-m)}\right)\right\}}{\left(\prod_{i=1}^{n} X_i/\hat{\theta}^{2n}\right)\exp\left\{-\sum_{i=1}^{n} X_i^2/2\hat{\theta}^2\right\}},$$

where $\hat{\theta}$ is the ML estimator of θ and $0 < \delta < 1$.

- Empirical distribution is a very important estimator in statistics and many statistical procedures depend on its performance. Some well known GOF tests belong to this family are considered as

Table 2 Comparing powers of considered tests for $n = 10$ and $\alpha = 0.05$ based on non-entropy and entropy tests separately

Test type	Tests	Dec. hazard		Inc. hazard		Non-mon. hazard	
		Chisq(1)	Wei(0.5, 1)	Chisq(3)	Beta(3,1)	Beta(1, 0.5)	Exp(2)
Non-entropy	L	**0.998**	**0.996**	0.420	0.252	0.140	0.412
	D	0.890	0.971	0.330	0.281	0.224	0.221
	V	0.820	0.956	0.220	0.351	0.348	0.402
	W^2	0.903	0.973	0.345	0.362	0.290	0.231
	U^2	0.833	0.956	0.259	0.340	0.345	0.426
	A^2	**0.977**	**0.995**	**0.464**	**0.417**	0.354	0.381
	S^*	0.894	0.970	0.350	0.357	0.341	0.236
Entropy	KL_{mn}	0.915	0.979	0.176	0.451	0.566	0.205
	CK_n	0.918	0.979	0.405	0.389	0.441	0.449
	R_n	0.718	0.911	**0.468**	**0.477**	**0.557**	**0.540**
	DH_{mn}	**0.964**	**0.992**	**0.471**	**0.476**	**0.573**	**0.711**

The greatest powers are in bold

$$D = \max(D_n^+, D_n^-), \quad V = D_n^+ + D_n^-,$$
$$W^2 = \sum_{i=1}^{n} \left\{ Z_{(i)} - \frac{2i-1}{2n} \right\}^2 + \frac{1}{12n}, \quad U^2 = W^2 - n\left(\bar{Z} - 0.5\right)^2,$$
$$A^2 = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left\{ \log(Z_{(i)}) + \log(1 - Z_{(n-i+1)}) \right\},$$
$$S^* = \sum_{i=1}^{n} \max \left\{ \left| Z_{(i)} - \frac{i}{n} \right|, \left| Z_{(i)} - \frac{i-1}{n} \right| \right\}.$$

where $D_n^+ = \max_i \left\{ i/n - Z_{(i)} \right\}$, $D_n^- = \max_i \left\{ Z_{(i)} - (i-1)/n \right\}$ and $Z_{(i)} = F(x_{(i)}; \hat{\theta})$; where $F(\cdot)$ is the true distribution of X. Meanwhile $\hat{\theta}$ is ML estimator of θ and considered empirical distribution base tests are Kolmogorov-Smirnov (D), Kuiper (V), Cramer-von Mises (W^2), Watson (U^2), Anderson-Darling (A^2) and Finkelstein-Schafers (S^*).

To compute power for each alternative, 10,000 samples of size n were generated. Then, the power was calculated by proportion of rejecting null hypothesis for simulated data from alternative distribution in some iterations.

The continuous alternative distributions are, therefore, classified in the following three classes :

- Monotonic decreasing hazard (Dec. Hazard) rate: Chisquare(1), Weibull(0.5,1),
- Monotonic increasing hazard (Inc. Hazard) rate: Chisquare(3), Beta(3,1),
- Non-Monotonic hazard (Non-Mon. Hazard) rate: Beta(1,0.5), Exponential(2).

Table 3 Comparing powers of considered tests for $n = 20$ and $\alpha = 0.05$ based on non-entropy and entropy tests separately

Test type	Tests	Dec. hazard		Inc. hazard		Non-mon. hazard	
		Chisq(1)	Wei(0.5, 1)	Chisq(3)	Beta(3,1)	Beta(1, 0.5)	Exp(2)
Non-entropy	L	**0.999**	**0.999**	0.623	0.469	0.160	0.413
	D	0.998	0.999	0.587	0.514	0.417	0.237
	V	0.979	0.999	0.490	0.658	0.668	0.701
	W^2	0.997	0.999	0.646	0.668	0.563	0.240
	U^2	0.999	0.999	0.632	0.609	0.649	0.716
	A^2	**0.997**	**0.999**	**0.707**	**0.612**	0.615	0.631
	S^*	0.993	0.999	0.631	0.708	0.615	0.250
Entropy	KL_{mn}	0.999	0.999	0.450	0.850	0.902	0.877
	CK_n	0.999	0.999	0.653	0.341	0.867	0.873
	R_n	0.880	0.987	**0.709**	**0.831**	**0.902**	**0.877**
	DH_{mn}	**0.999**	**0.999**	**0.717**	**0.851**	**0.932**	**0.940**

The greatest powers are in bold

Table 4 Comparing type I error of considered tests for $\alpha = 0.05$

Tests	$n = 10$	$n = 20$	$n = 30$	$n = 50$
L	0.046	0.042	0.046	0.049
D	0.048	0.062	0.041	0.066
V	0.049	0.052	0.064	0.040
W^2	0.056	0.056	0.045	0.046
U^2	0.050	0.066	0.054	0.049
A^2	0.045	0.051	0.056	0.044
S^*	0.049	0.046	0.048	0.040
KL_{mn}	0.040	0.046	0.055	0.054
CK_n	0.046	0.044	0.049	0.045
R_n	0.047	0.048	0.046	0.052
DH_{mn}	0.045	0.058	0.050	0.052

The power of the considered tests are compared to that of some other tests for Rayleighity against the same alternative which are tabulated in Tables 2 and 3.

Another important property of a test is type I error rate which is calculated as proportion of rejected null hypothesis for Rayleigh distribution. In order to, examine the type I error of considered tests we have conducted a Monte Carlo simulation and compared the results. In the simulation for each sample size, 10,000 replications were done and performance of all considered tests were evaluated at 5 percent level. Indeed, the values in the Table 4 are type I error rates of considered tests in assessing Rayleighity and we conclude type I error rates of all considered tests are about supposed significance level. So, they have acceptable performance in this field.

In order to have a more detailed analysis, the results are presented in Tables 2 and 3 under two groups as entropy and Non-entropy. According to these Tables we concluded that in decreasing hazard rate distributions between entropy type tests the most powerful tests in priority of order are DH_{mn}, CK_n, KL_{mn} and in the middle of Non-entropy tests the most powerful tests in order of preference are L, A^2, W^2. In addition, it is evident that the three most powerful tests in overall are respectively L, A^2, DH_{mn}.

We also concluded among increasing hazard rate distributions choosing the best test is difficult. It should be noticed that among entropy type tests the most powerful tests in order of preference are DH_{mn}, R_n, CK_n. Between Non-entropy type tests the most powerful tests in order of preference are A^2, S^*, L. Meanwhile, we can choose three most powerful tests in overall as DH_{mn}, R_n, A^2.

In addition to that between non-monotone hazard rate distributions although choosing the best test is difficult but among entropy type tests the most powerful tests in priority of order are DH_{mn}, R_n, KL_{mn}. In the middle of Non-entropy type tests the most powerful tests in order of preference are A^2, U^2, V. Moreover, it is interesting that three most powerful tests in overall are respectively DH_{mn}, R_n, KL_{mn} too.

Finally, we observed that among all considered tests the proposed DH_{mn} test which is based on a metric measure performed better than other all tests in three classes overall.

4 Application to Real Data

In this section, we analyze two data set to assess ability of considered tests.

Example 1 In this example we analyze the ball bearing data, which was given by [6] and represents the failure times of 25 ball bearings in endurance test. The observed failure times are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 67.80, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

For this data set, Lee [11] and Baratpour and Khodadadi [3] indicated that the one-parameter Rayleigh distribution provides a satisfactory fit.

Example 2 In this example we consider one average wind speed data analysis reported in [5]. This data represent 30 average daily wind speeds (in km/h) for the month of November 2007 recorded at Elanora Heights, a northeastern suburb of Sydney, Australia. The calculated data are 2.7, 3.2, 2.1, 4.8, 7.6, 4.7, 4.2, 4.0, 2.9, 2.9, 4.6, 4.8, 4.3, 4.6, 3.7, 2.4, 4.9, 4.0, 7.7, 10.0, 5.2, 2.6, 4.2, 3.6, 2.5, 3.3, 3.1, 3.7, 2.8, 4.0.

For this data set, Best et al. [5] and Alizadeh et al. [2] fitted the Rayleigh distribution successfully.

The probability values for some of more powerful tests in this paper of testing Rayleighity for considered tests are presented in Table 5. Considering the fact that the probability values of all tests are greater than 0.05 they are not significant at 5 percent significance level. Thus, we cannot reject the null hypothesis of Rayleighity. It means the ball bearing data and the average wind speed data truly follow Rayleigh distribution.

Table 5 The p-value of the most powerful tests in evaluating Rayleighity

Tests	KL_{mn}	CK_n	DH_{mn}	R_n	A^2
Example 1	0.881	0.889	0.884	0.910	0.781
Example 2	0.276	0.298	0.342	0.312	0.304

5 Conclusion

The aim of this paper is to evaluate the performance of the proposed test. So, we considered and compared eleven different GOF tests for evaluating Rayleighity. In order to evaluate the ability of mentioned tests in identifying Rayleigh distribution by a simulation study, the powers and type I error of the proposed test were computed under several alternatives and different sample sizes in three different classes of hazard rates. The results were tabulated in Tables 2 and 3 which are the powers of considered tests at significance level 0.05.

Finally, we concluded that DH_{mn} test which is a metric measure performed better than the other tests. Therefore, the results obtained from this study encourage us to use DH_{mn} test in all three classes for real data that we do not know type of hazard rate distribution.

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