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2-Nilpotent multipliers of a direct product of Lie algebras

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Abstract In this paper, we present an explicit formula for the 2-nilpotent multiplier of a direct product of two Lie algebras.

Keywords Schur multiplier · 2-Nilpotent multiplier · Free product · Free Lie algebra

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1 Introduction

Throughout the paper [,] denotes the Lie bracket. Let L be a Lie algebra analogues to the theory of the group (see, for instance [5,8,10]) the c-nilpotent multiplier of L defined as a factor Lie algebra

$$\mathcal{M}^{(c)}(L) = R \cap F^{c+1}/(R, F)^{c+1},$$

where F^{c+1} is the $(c+1)$-th term of the derived series of F, $(R, F)^1 = R$, $(R, F)^{c+1} = [(R, F)^c, F]$ and $L \cong F/R$ for a free Lie algebra F.

Similar to the group theory case, one may check that $\mathcal{M}^{(c)}(L)$ is abelian and independent of the choice of the free Lie algebra F. In the case that $c = 1$, $\mathcal{M}^{(1)}(L)$ is denoted by $\mathcal{M}(L)$,

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which is called the Schur multiplier of L, is more well known and in recent two decades has been studied by several authors in [1–4,6,7,9,11,12,19]. The reason for studying the c-nilpotent multiplier of Lie algebra as in the case of groups comes back to isoclinism theory of P. Hall. Recall that the isoclinism in groups (resp. Lie algebra) is an equivalence relation weaker than isomorphism, and lets us to classify groups (resp. Lie algebra) into suitable equivalence classes. More details can be found in [13–15,18].

Recently, some authors have tried to develop the group theory result to Lie algebras by checking non-obvious algebraic identities for the Schur multiplier. But in this direction, the 2-nilpotent multiplier wasn't considered enough. Obviously working on the 2-nilpotent multiplier enables us to know more about the structure of Lie algebras, such as 2-capability. Looking [1–4,6,7,9,11,12,19] show that the structure of the Schur multiplier respect to direct sum of two Lie algebras, may leads us to more result on the Schur multiplier of a Lie algebra.

In this paper, we are interested to give the structure of $\mathcal{M}^{(2)}(L)$, when $L \cong L_1 \oplus L_2$. This lets us to do more investigation on the 2-nilpotent multiplier of various Lie algebras. The similar formula for the behavior of the nilpotent multipliers of groups with respect to the direct product of two groups causes considerable progress in both computation and structural results for the nilpotent multipliers of groups. Therefore a similar procedure may be expected in the case of Lie algebras.

Let F be a free Lie algebra on the set $X = \{x_1, x_2, \ldots\}$. Then we may define the basic commutator on X inductively.

(i) The generators $x_1, x_2, \ldots, x_n$ are basic commutators of length one and ordered by setting $x_i < x_j$ if $i < j$.
(ii) If all the basic commutators d_i of length less than t have been defined and ordered, then we may define the basic commutators of length t to be all commutators of the form $[d_i, d_j]$ such that the sum of lengths of d_i and d_j is t, $d_i > d_j$, and if $d_i = [d_s, d_t]$, then $d_j \geq d_t$. The basic commutators of length t follow those of lengths less than t. The basic commutators of the same length can be ordered in any way, but usually the lexicographical order is used.

The number of all basic commutators on a set $X = \{x_1, x_2, \ldots x_n\}$ of length d is denoted by $l_d(n)$. Thanks to [16] and denoting Möbius function by $\mu(m)$, we have

$$l_d(n) = \sum_{m|n} \mu(m) d^{\frac{n}{m}}.$$

2 Main results

In this part we intend to give an explicit structure for the 2-nilpotent multiplier of a direct product of two Lie algebras in term of their 2-nilpotent multipliers. This not only generalizes [2, Theorem 1] for the Schur multiplier of a direct sum of two Lie algebras but also gives the structure of $\mathcal{M}^{(2)}(L_1 \oplus L_2)$. Here we give some notations which are needed.

Let L_1 and L_2 be two Lie algebras, and assume that

$$0 \longrightarrow R_1 \longrightarrow F_1 \xrightarrow{\phi_1} L_1 \longrightarrow 0,$$

and

$$0 \longrightarrow R_2 \longrightarrow F_2 \xrightarrow{\phi_2} L_2 \longrightarrow 0$$

be free presentations of L_1 and L_2, respectively; where F_i's are free Lie algebra on some sets. The techniques we use here are based on the concept of free products of Lie algebras and the expansion of an element of a free Lie algebra in terms of basic commutators. For a description on these two, see [16] and [17]. But a brief description may be useful. Every element of a free Lie algebra can be expressed as a sum of some basic commutators. For the elements of a free product of two free Lie algebras also we have a similar expression except that the basic commutators are on the union of the bases of the two free Lie algebras taken to form the free product. It is easy to see that the free product of two free Lie algebras is actually a free Lie algebra on the union of the bases of the two free Lie algebras.

By the above explanations and notations, the following very useful lemma gives a free presentation for $L_1 \oplus L_2$ and will be used in our further investigation.

Lemma 2.1 *Let $F = F_1 * F_2$ be the free product of F_1 and F_2. Then*

$$0 \longrightarrow R \longrightarrow F \longrightarrow L_1 \oplus L_2 \longrightarrow 0$$

is a free presentation for $L_1 \oplus L_2$ in which $R = R_1 + R_2 + [F_2, F_1]$.

Proof The coproduct property of free product emphasizes that there is a homomorphism $\phi : F \longrightarrow L_1 \oplus L_2$ satisfying $\phi|_{F_i} = \tau_i \phi_i$ in which $\tau_i : L_i \longrightarrow L_1 \oplus L_2$ are the natural embeddings for $i = 1, 2$. Since the images of ϕ_1 and ϕ_2 commute in $L_1 \oplus L_2$, it is easy to see that $R_1 + R_2 + [F_2, F_1] \subseteq R = \ker\phi$.

For the converse it is enough to consider an arbitrary element of $\ker\phi$ as a sum of basic commutators on the generating sets of F_1 and F_2 and use the fact that ϕ is defined via ϕ_1 and ϕ_2. □

Using the above, lemma we can compute the 2-nilpotent multiplier of $L_1 \oplus L_2$ in terms of F_i's and R_i's as follows

$$\mathcal{M}^{(2)}(L_1 \oplus L_2) = \frac{R \cap F^3}{[R, F, F]} = \frac{(R_1 + R_2 + [F_2, F_1]) \cap (F_1 * F_2)^3}{[R_1 + R_2 + [F_2, F_1], F_1 * F_2, F_1 * F_2]}.$$

The next lemma shows $\mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2)$ is a direct summand of $\mathcal{M}^{(2)}(L_1 \oplus L_2)$.

Lemma 2.2 *Let L_1 and L_2 be two Lie algebras, then*

$$\mathcal{M}^{(2)}(L_1 \oplus L_2) \cong \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2) \oplus K$$

for some subalgebra K of $\mathcal{M}^{(2)}(L_1 \oplus L_2)$.

Proof Let $F = F_1 * F_2$, then the epimorphism $F \longrightarrow F_1 \times F_2$ induces an epimorphism

$$\alpha : \frac{R \cap F^3}{[R, F, F]} \longrightarrow \frac{R_1 \cap F_1^3}{[R_1, F_1, F_1]} \oplus \frac{R_2 \cap F_2^3}{[R_2, F_2, F_2]}.$$

On the other hand, the map

$$\beta : \frac{R_1 \cap F_1^3}{[R_1, F_1, F_1]} \oplus \frac{R_2 \cap F_2^3}{[R_2, F_2, F_2]} \longrightarrow \frac{R \cap F^3}{[R, F, F]}$$

defined by $(x_1 + [R_1, F_1, F_1], x_2 + [R_2, F_2, F_2]) \mapsto (x_1 + y_1) + [R_2, F_2, F_2]$ is a well-defined homomorphism. It is easy to see that β is a left inverse to α, so that the sequence

$$0 \longrightarrow K \longrightarrow \mathcal{M}^{(2)}(L_1 \oplus L_2) \longrightarrow \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2) \longrightarrow 0$$

splits and the result holds. □

The last lemma shows we may give the behavior of 2-nilpotent multiplier respect to direct sums by getting the structure of K. To do this we need some preliminaries as follows.

Lemma 2.3 *With the above notations and assumption*

$$F^3 = [F_2, F_1, F_1] + [F_2, F_1, F_2] + F_1^3 + F_2^3.$$

Proof Straightforward. □

The ideal $[F_2, F_1, F_1] + [F_2, F_1 + F_2]$ plays an essential role in computing K.

Theorem 2.4 *Let*

$$\alpha : \mathcal{M}^{(2)}(L_1 \oplus L_2) \longrightarrow \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2)$$

be the epimorphism defined in Lemma 2.2. Then

$$\ker\alpha \equiv [F_2, F_1, F_1] + [F_2, F_1, F_2] \pmod{[R, F, F]}.$$

Proof Considering the definition of α, it is trivial that

$$[F_2, F_1, F_1] + [F_2, F_1, F_2] \pmod{[R, F, F]} \subseteq \ker\alpha.$$

Conversely, let $w + [R, F, F] \in \ker\alpha$, by Lemma 2.3 $w = a + b + c$ where $a \in [F_2, F_1, F_1] + [F_2, F_1, F_2]$, $b \in F_1^3$ and $c \in F_2^3$. The definition of α implies $b \in [R_1, F_1, F_1]$ and $c \in [R_2, F_2, F_2]$, therefore $w \equiv a \pmod{[R, F, F]}$ which completes the proof. □

The next result of Shirshov [16] should be mentioned before the next theorem.

Theorem 2.5 *Let F be a free Lie algebra on a set X, then F^c/F^{c+1} is an abelian Lie algebra with the basis of all basic commutators on X of length c.*

The next step is presenting a generating set for $[F_2, F_1, F_1] + [F_2, F_1, F_2]$ in terms of the free generators of F_1 and F_2.

Theorem 2.6 *Let F_1 and F_2 be two free Lie algebras freely generated by X and Y, respectively, and $F = F_1 * F_2$. Then $[F_2, F_1, F_1] + [F_2, F_1, F_2] \pmod{F^4}$ is an abelian Lie algebra generated by all basic commutators of the form $[y_i, x_j, x_k]$ and $[y_r, x_s, y_t]$ where y_i, y_r, y_t and x_j, x_k, x_s are taken from Y and X, respectively.*

Proof First, we put an order on $X \cup Y$ by keeping the orders in X and Y and every element of X proceeds each element of Y. By Theorem 2.5, $F^3 \pmod{F^4}$ is the abelian Lie algebra generated by all basic commutators of weight 3 on $X \cup Y$. Now Lemma 2.3 completes the proof. □

For a Lie algebra L we denote by L^{ab} the factor Lie algebra $\frac{L}{L^2}$, which is always abelian. By $\otimes$ we mean the usual tensor product of algebras.

Multi linearity of the generating elements of $[F_2, F_1, F_1] + [F_2, F_1, F_2] \pmod{F^4}$ suggests a relationship between

$$[F_2, F_1, F_1] + [F_2, F_1, F_2] \text{ and } \left(L_2^{ab} \otimes L_1^{ab} \otimes L_1^{ab}\right) \oplus \left(L_2^{ab} \otimes L_1^{ab} \otimes L_2^{ab}\right).$$

The following theorem states this relationship.

Theorem 2.7 *With the previous notations and assumptions we have*

$$[F_2, F_1, F_1] + [F_2, F_1, F_2] \equiv \left(L_2^{ab} \otimes L_1^{ab} \otimes L_1^{ab}\right) \oplus \left(L_2^{ab} \otimes L_1^{ab} \otimes L_2^{ab}\right) \pmod{[R, F, F]}.$$

Proof Obviously, the assignment $[a, b, c] + [R, F, F] \mapsto \overline{a} \otimes \overline{b} \otimes \overline{c}$ determines a homomorphism. Conversely the defining relations of the tensor product show $\overline{a} \otimes \overline{b} \otimes \overline{c} \mapsto [a, b, c] + [R, F, F]$ also defines a homomorphism which is the converse of the previous one. □

Now the final theorem can be proved.

Theorem 2.8 *Let L_1 and L_2 be two Lie algebras, then*

$$\mathcal{M}^{(2)}(L_1 \oplus L_2) \cong \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2) \oplus \left(L_2^{ab} \otimes L_1^{ab} \otimes L_1^{ab}\right) \oplus \left(L_2^{ab} \otimes L_1^{ab} \otimes L_2^{ab}\right).$$

Proof Using Theorems 2.4 and 2.7 the result will follows immediately. □

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