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A new iterative model updating technique based on least squares minimal residual method using measured modal data

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ABSTRACT

Finite element (FE) models are useful for many applications in engineering community such as structural analysis, dynamic behavior prediction, structural condition assessment, and damage detection. In reality, the FE models often differ with the real structures due to significant discrepancies between test measurements and model predictions. This study is intended to propose an iterative model updating method to adjust the mass and stiffness matrices of the FE model by improving model updating formulations. Under dynamic discrepancy theory, the mass and stiffness orthogonality conditions are independently expanded to establish two model updating formulations. In the proposed iterative method, each of the improved equations is solved by a robust iterative technique named as least squares minimal residual (LSMR) to compute structural discrepancy matrices after transforming linear matrix systems of the model updating equations into linear vector systems. In the following, the mass and stiffness matrices are updated in the algorithm of the proposed iterative model updating method using the structural discrepancy matrices obtained from the LSMR technique. The major contributions of this article are to propose a new iterative finite element model updating method and introduce the novel and robust LSMR approach for the vibration-based applications. Another novelty is to enhance the two well-known model updating formulations for achieving more appropriate updating results. The efficiency and accuracy of the proposed methods are numerically verified by a simple planer truss and two shear building models. Results demonstrate that the proposed methods provide reliable estimates of finite element model updating using the measured modal data.

1. Introduction

For many problems of design, implementation, and operation of engineering systems, accurate finite element (FE) models are required to predict the dynamic behavior of the systems and assess their performances. However, there may be differences between the FE models and real structures resulting from inaccuracy in modeling and various uncertainties, including approximation of boundary conditions, incorrect initial assumptions of geometry and material properties, and limitations of modeling structural connections.

Finite element model updating is a valuable tool in engineering applications that aims to update or correct the FE model of a real structure in order for minimizing the difference between the test measurements and model predictions [1,2]. Thus,

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it is evident that finite element model updating is essentially a minimization problem, which means that one attempts to reduce discrepancies between the analytical and real models [3]. This concept can be used in many engineering applications associated with civil, mechanical, and aerospace communities for subsequent design and analysis [4,5], and structural condition assessment [6]. In addition, there is another motivation for updating the FE models that is an evaluation of health and integrity of the structure or structural health monitoring (SHM). The process of SHM is an implementing process to detect any probable damage in the structure [7]. In a certain case, the model updating problem is a model-based damage detection, in which a correct FE model is needed to perform an accurate damage identification process [8,9]. For such purposes, efficient methods associated with the finite element model updating problems are needed to update the inherent characteristics of the model such as the mass and stiffness matrices using the measured vibration data of the real structure.

Many analytical methods have been developed to achieve this aim using the fundamental concepts of structural dynamics such as eigenvalue problem and orthogonality conditions. On this basis, the finite element model updating methods can be broadly categorized into direct and iterative methods [10–12]. The direct approaches aim to update the inherent properties of the structure in one step such that the updated model can reproduce the similar vibration responses to the test measurements [13]. In this area, Baruch and Bar Itzhack [14] and Baruch [15] presented methods to correct the stiffness matrix based on measured mode shapes by minimizing a norm to use the positive definite symmetric mass matrix as the weighting matrix. Berman [16] applied a minimum-weighted Euclidean norm and Lagrange multipliers method to describe the change in the mass matrix required to satisfy the orthogonality condition. Berman and Nagy [17] proposed a direct method to improve the mass and stiffness matrices of an analytical model using normal modes and natural frequencies. In their method, a set of minimum changes in the matrices was identified, which force the eigensolutions to adopt with the measured vibration data. Kabe [18] adjusted the stiffness matrix provided that the percentage change in each stiffness coefficient was minimized. The method preserved the physical configuration of the analytical model and reproduced the modes used in the identification. Caesar and Peter [19] discussed two direct methods for updating the mass and stiffness matrices using the modal parameters obtained from the modal test. Wei [20] proposed a new direct method to indicate the uniqueness of the corrected stiffness matrix in a different way. In another research, Wei [21] proposed an analytical method to adjust both the analytical mass and stiffness matrices by the element correction method combined with Lagrange multiplier technique. Yaun et al. [22] proposed a direct method to estimate the mass and stiffness matrices of shear building models using modal test data, in which the only first two modes were applied. In a similar study, Chakraverty [23] improved their method and presented a technique for estimating the mass and stiffness matrices of shear building models using Holzer criteria along with other numerical methods.

More recently, Carvalho et al. [24] presented a new method regarding the model updating problem in an undamped model through the incomplete measured modal data without using any modal expansion or reduction techniques. Kanev et al. [25] experimentally investigated an approach to damage detection and localization based on finite element model updating that could simultaneously update the mass, damping, and stiffness matrices at the same time preserving their connectivity, symmetry and positive definiteness. Moreno et al. [26] showed that the finite element model updating can be formulated as an optimization problem. They updated a symmetric second-order FE model, which could reproduce a given set of desirable eigenvalues and eigenvectors. Mukhopadhyay et al. [27] investigated the physical parameter identification with input–output measurements as well as the problem of mass normalized mode shape expansion developed for the shear-type structural systems. Their methodology does not require any *a priori* mass and stiffness information.

In other research, Yang and Chen [28] proposed a direct finite element model updating method to correct the mass and stiffness matrices of civil engineering systems in order for reproducing modal frequencies obtained by the updated structures. Lee and Eun [29] modified the mass and stiffness matrices of numerical FE models through the minimization of different cost functions proposed by other researchers using Moore-Penrose inverse method. In addition, Lee and Eun [30] presented model updating equations to simultaneously update the stiffness and mass matrices by minimizing a cost function in the satisfaction of dynamic orthogonality constraints and eigenvalue function. Lee et al. [31] introduced model updating methods to estimate changes in the mass and stiffness matrices using the measured modal data and compute alterations in the stiffness matrix through measured static displacements. As another study, Lee et al. [32] updated physical parameter matrices of damped dynamic systems by minimizing a cost function expressed as the sum of the norms of the difference between the updated and analytical matrices. Chen and Maung [33] proposed a regularized method on the basis of dynamic perturbation theory to update the structural parameters of analytical models using the measured incomplete modal data.

In the iterative finite element model updating methods, the inherent properties of the structure or structural matrices of the FE model are updated in several steps to make sure that there are inconsiderable discrepancies in the test measurements and model predictions. Li and Hong [34] presented a new iterative approach with a model reduction technique to update the structural properties of the FE model. Their method reduced undesirable effects of the model reduction process in the model updating problems by adding a correction term to the iterative algorithm. Yaun and Liu [35] mathematically investigated the model updating problem by some efficient iterative methods to adjust the mass and stiffness matrices in the undamped dynamic systems. In another study, Yaun and Liu [36] proposed an iterative method for the finite element model updating problem in damped structural systems. Gang et al. [37] proposed a new iterative method for updating the values of mass and stiffness matrices using incomplete frequency response function (FRF) data. In their method, a modified discrepancy vector between the analytical and experimental FRF data was used to construct a linear sensitivity updating equation system.

Although many model updating methods have been proposed in the direct and iterative manners, some of them need inverse solutions to solve the model updating problems or minimize the cost functions. Using the inverse problem may lead to erroneous results in updating process when the measured modal data are incomplete. On the other hand, most of the proposed updating methods using the dynamic orthogonality conditions and eigenvalue function are direct approaches. There is no doubt that these conditions directly reflect the relationship between the inherent properties of the structure and the dynamic characteristics such as modal data. Thus, it is necessary to improve such methods in an iterative manner for achieving appropriate updating results. In addition, these methods normally require the model reduction or modal expansion techniques for solving the updating problems in the presence of incomplete modal data.

The main objective of this paper is to propose a new iterative method by improving two model updating formulations proposed by Yang and Chen [28]. Under the dynamic discrepancy theory, the mass and stiffness orthogonality conditions are independently expanded to enhance their methods and establish new model updating equations. To solve the improved formulas, a novel iterative technique named as least square minimal residual (LSMR) is introduced that can be used as an influential approach to vibration-based applications without applying the inverse problem. On this basis, the linear matrix systems of the improved formulations are converted into the linear vector systems by Kronecker product and LSMR method is applied to compute the structural discrepancy matrices. In the following, the mass and stiffness matrices of the FE model are updated by the proposed iterative method with the aid of structural discrepancy matrices obtained from LSMR technique. The performance and capability of the proposed methods are numerically demonstrated by a simple planner truss as a comparative study and two shear building models. Numerical results of the comparative study show that the methods presented in this article give more accurate results than the other model updating approaches. Furthermore, it can be observed that the proposed methods are proper for updating the structural matrices of the shear building models by the measured modal data.

The remainder of the paper is organized as follows. Section 2 presents the improved model updating formulas on the basis of the dynamic discrepancy theory. Section 3 expresses the algorithm of LSMR technique in order that solving the improved model updating equations. Section 4 proposes the iterative method for updating the mass and stiffness matrices. Section 5 verifies the proposed methods by a numerical planner truss and two shear building models. Section 6 gives the concluding remarks.

2. Model updating formulas

2.1. Improving a model updating formulation for the mass matrix

The mass orthogonality condition is one of the important equations used in many purposes such as structural parameter estimation, finite element model updating, damage detection, and mode normalization. Recently, Yang and Chen [28] presented a new direct model updating method based on the mass orthogonality condition for updating the mass matrix with the aid of the first few measured modes. The objective of this section is to improve their methodology for achieving more precise and reliable results in updating the mass matrix. The general form of the mass orthogonality equation for a structure with n degrees of freedom (DOFs) and m measured modes is given by:

$$\mathbf{\Phi}_x^T \mathbf{M}_x \mathbf{\Phi}_x = \mathbf{I}, \tag{1}$$

where $\mathbf{\Phi}_\mathrm{x} \in \Re^{n\times m}$ is the measured mode shape matrix (modal matrix); $\mathbf{M}_\mathrm{x} \in \Re^{n\times n}$ denotes the experimental mass matrix belonging to the real structure, and $\mathbf{I} \in \Re^{m\times m}$ is the identity matrix. Assume that the modal matrix is directly obtained from the modal test and the experimental mass matrix should be estimated by the mass orthogonality equation. On the basis of the dynamic discrepancy theory, the difference between the mass matrices in the experimental and analytical models yields a useful matrix for the finite element model updating problem, called mass discrepancy matrix, in the following form:

$$\triangle\mathbf{M} = \mathbf{M}_x - \mathbf{M}_a, \tag{2}$$

where $\mathbf{M}_\mathrm{a} \in \Re^{n\times n}$ represents the mass matrix of the analytical model. While a FE model cannot accurately simulate a realistic model for the real structure, its dynamic responses differ from the test measurements. In a similar way, the modal and frequency discrepancy matrices are formulated as:

$$\triangle\mathbf{\Phi} = \mathbf{\Phi}_x - \mathbf{\Phi}_a, \tag{3}$$

$$\triangle\mathbf{\Lambda} = \mathbf{\Lambda}_x - \mathbf{\Lambda}_a, \tag{4}$$

where $\mathbf{\Phi}_\mathrm{a} \in \Re^{n\times m}$ denotes the mode shape matrix of the analytical model; $\mathbf{\Lambda}_\mathrm{x}$ and $\mathbf{\Lambda}_\mathrm{a}$ are the diagonal matrices of the modal frequency in the experimental and analytical models, respectively. According to Eqs. (2)–(4), one can realize that the mass and modal matrices of the experimental model are simply calculated using their discrepancy matrices in the following equations:

$$\mathbf{M}_x = \mathbf{M}_a + \triangle\mathbf{M}, \tag{5}$$

$$\mathbf{\Phi}_x = \mathbf{\Phi}_a + \triangle\mathbf{\Phi}. \tag{6}$$

Replacing Eqs. (5) and (6) into Eq. (1), one can write:

$$(\boldsymbol{\Phi}_a + \Delta\boldsymbol{\Phi})^T(\mathbf{M}_a + \Delta\mathbf{M})(\boldsymbol{\Phi}_a + \Delta\boldsymbol{\Phi}) = \mathbf{I}, \tag{7}$$

which can be expanded to yield:

$$\begin{aligned}&\left(\boldsymbol{\Phi}_a^T\mathbf{M}_a\boldsymbol{\Phi}_a\right) + \left(\boldsymbol{\Phi}_a^T\Delta\mathbf{M}\boldsymbol{\Phi}_a\right) + \left(\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\boldsymbol{\Phi}_a\right) + \left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{M}\boldsymbol{\Phi}_a\right)\\ &+\left(\boldsymbol{\Phi}_a^T\mathbf{M}_a\Delta\boldsymbol{\Phi}\right) + \left(\boldsymbol{\Phi}_a^T\Delta\mathbf{M}\Delta\boldsymbol{\Phi}\right) + \left(\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\Delta\boldsymbol{\Phi}\right) + \left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{M}\Delta\boldsymbol{\Phi}\right) = \mathbf{I}.\end{aligned} \tag{8}$$

In [28], one can neglect the higher order terms of Eq. (8) such as $\Delta\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}_{\mathrm{a}}\Delta\boldsymbol{\Phi}$ and $\Delta\boldsymbol{\Phi}^{\mathrm{T}}\Delta\mathbf{M}\Delta\boldsymbol{\Phi}$, whereas mathematically speaking, it is not allowable to neglect these expressions. In this study, therefore, an entire form of the expanded mass orthogonality condition is employed to propose an efficient model updating formulation for the mass matrix. Arranging Eq. (8) in order that determining the unknown mass discrepancy matrix $\Delta\mathbf{M}$, one can rewrite:

$$\begin{aligned}&\left(\boldsymbol{\Phi}_a^T\Delta\mathbf{M}\boldsymbol{\Phi}_a\right) + \left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{M}\boldsymbol{\Phi}_a\right) + \left(\boldsymbol{\Phi}_a^T\Delta\mathbf{M}\Delta\boldsymbol{\Phi}\right) + \left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{M}\Delta\boldsymbol{\Phi}\right)\\ &= \left(\mathbf{I} - \boldsymbol{\Phi}_a^T\mathbf{M}_a\boldsymbol{\Phi}_a\right) - \left(\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\boldsymbol{\Phi}_a\right) - \left(\boldsymbol{\Phi}_a^T\mathbf{M}_a\Delta\boldsymbol{\Phi}\right) - \left(\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\Delta\boldsymbol{\Phi}\right).\end{aligned} \tag{9}$$

In this equation, the term $\mathbf{I} - \boldsymbol{\Phi}_a^T\mathbf{M}_a\boldsymbol{\Phi}_a$ is always equal to zero and can be eliminated. Rearranging the above equation, one can obtain:

$$(\boldsymbol{\Phi}_a + \Delta\boldsymbol{\Phi})^T\Delta\mathbf{M}(\boldsymbol{\Phi}_a + \Delta\boldsymbol{\Phi}) = -\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\Delta\boldsymbol{\Phi} - \Delta\boldsymbol{\Phi}^T\mathbf{M}_a\boldsymbol{\Phi}_a - \boldsymbol{\Phi}_a^T\mathbf{M}_a\Delta\boldsymbol{\Phi}. \tag{10}$$

The mode shape expressions at the left-hand side of Eq. (10) are indicative of the experimental mode shape matrix. Thus:

$$\boldsymbol{\Phi}_x^T\Delta\mathbf{M}\boldsymbol{\Phi}_x = -\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\Delta\boldsymbol{\Phi} - \Delta\boldsymbol{\Phi}^T\mathbf{M}_a\boldsymbol{\Phi}_a - \boldsymbol{\Phi}_a^T\mathbf{M}_a\Delta\boldsymbol{\Phi}. \tag{11}$$

For the sake of simplicity, the terms of the right-hand side of Eq. (11) are compressed into a matrix namely mass coefficient matrix as follows:

$$\mathbf{C}_m = -\left(\Delta\boldsymbol{\Phi}^T\mathbf{M}_a\Delta\boldsymbol{\Phi} + \Delta\boldsymbol{\Phi}^T\mathbf{M}_a\boldsymbol{\Phi}_a + \boldsymbol{\Phi}_a^T\mathbf{M}_a\Delta\boldsymbol{\Phi}\right). \tag{12}$$

Therefore, the improved model updating equation for the mass matrix is rewritten in accordance with the following equation:

$$\boldsymbol{\Phi}_x^T\Delta\mathbf{M}\boldsymbol{\Phi}_x = \mathbf{C}_m. \tag{13}$$

The main aim is to determine the unknown mass discrepancy matrix by solving this equation through a direct or an iterative mathematical method. Although the form of this equation is similar to some existing methods, there are several evidences that confirm the novelty and improvement of the improved model updating formulation. An advantage of the improved formulation is concerned with the mass coefficient matrix ($\mathbf{C}_{\mathrm{m}}$), which is one of the underlying parts of the improved equation. By comparing the form of this matrix with the corresponding one in the reference [28], one can indicate that the improved formulation makes more precise updating results.

2.2. Improving a model updating formulation for the stiffness matrix

In this section, the stiffness orthogonality condition is used to update the stiffness matrix on the basis of the dynamic discrepancy theory. A key benefit of this condition is the capability of updating the stiffness matrix without applying the mass matrix. Accordingly, the stiffness matrix is independently updated by the measured modal data and the analytical stiffness matrix. Similar to the previous section, Yang and Chen's method [28] is improved to attain more accurate and precise results in the process of updating the stiffness matrix. For a dynamic system with n DOFs, the stiffness orthogonality condition concerning the experimental model is expressed as:

$$\boldsymbol{\Phi}_x^T\mathbf{K}_x\boldsymbol{\Phi}_x = \boldsymbol{\Lambda}_x, \tag{14}$$

where $\mathbf{K}_{\mathrm{x}} \in \Re^{n\times n}$ is the stiffness matrix of the experimental model. Based on the dynamic discrepancy theory, a matrix named as stiffness discrepancy matrix is defined indicating the difference between the stiffness matrices in the experimental and analytical models in the following form:

$$\Delta\mathbf{K} = \mathbf{K}_x - \mathbf{K}_a. \tag{15}$$

In the same manner, the stiffness and modal frequency matrices of the experimental model can be expressed as:

$$\mathbf{K}_x = \mathbf{K}_a + \Delta\mathbf{K}, \tag{16}$$

$$\boldsymbol{\Lambda}_x = \boldsymbol{\Lambda}_a + \Delta\boldsymbol{\Lambda}. \tag{17}$$

Substituting Eqs. (6), (16) and (17) into the stiffness orthogonality condition, one can write:

$$(\boldsymbol{\Phi}_a + \Delta\boldsymbol{\Phi})^T(\mathbf{K}_a + \Delta\mathbf{K})(\boldsymbol{\Phi}_a + \Delta\boldsymbol{\Phi}) = \boldsymbol{\Lambda}_a + \Delta\boldsymbol{\Lambda}. \tag{18}$$

By expanding this equation, yield:

$$\begin{aligned}&\left(\boldsymbol{\Phi}_a^T\mathbf{K}_a\boldsymbol{\Phi}_a\right)+\left(\boldsymbol{\Phi}_a^T\Delta\mathbf{K}\boldsymbol{\Phi}_a\right)+\left(\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\boldsymbol{\Phi}_a\right)+\left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{K}\boldsymbol{\Phi}_a\right)\\&+\left(\boldsymbol{\Phi}_a^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right)+\left(\boldsymbol{\Phi}_a^T\Delta\mathbf{K}\Delta\boldsymbol{\Phi}\right)+\left(\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right)+\left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{K}\Delta\boldsymbol{\Phi}\right)=\boldsymbol{\Lambda}_a+\Delta\boldsymbol{\Lambda}.\end{aligned} \tag{19}$$

Based on the proposed method in [28], it is possible to neglect the higher order terms of Eq. (19) such as $\Delta\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}_{\mathrm{a}}\Delta\boldsymbol{\Phi}$ and $\Delta\boldsymbol{\Phi}^{\mathrm{T}}\Delta\mathbf{K}\Delta\boldsymbol{\Phi}$ due to inconsiderable amounts in these terms compared with the other expressions. As explained in the previous section, this process is not reasonable and correct in the mathematical sense. Furthermore, the elimination of these terms may not provide trustworthy results in the process of updating the stiffness matrix. To overcome this problem, the complete form of Eq. (19) is applied to establish a new formulation for updating the stiffness matrix. Rearranging the above equation on the basis of the stiffness discrepancy matrix leads to:

$$\begin{aligned}&\left(\boldsymbol{\Phi}_a^T\Delta\mathbf{K}\boldsymbol{\Phi}_a\right)+\left(\boldsymbol{\Phi}_a^T\Delta\mathbf{K}\Delta\boldsymbol{\Phi}\right)+\left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{K}\boldsymbol{\Phi}_a\right)+\left(\Delta\boldsymbol{\Phi}^T\Delta\mathbf{K}\Delta\boldsymbol{\Phi}\right)\\&=\left(\boldsymbol{\Lambda}_a-\boldsymbol{\Phi}_a^T\mathbf{K}_a\boldsymbol{\Phi}_a\right)+\Delta\boldsymbol{\Lambda}-\left(\boldsymbol{\Phi}_a^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right)-\left(\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\boldsymbol{\Phi}_a\right)-\left(\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right).\end{aligned} \tag{20}$$

The first expression of the right-hand side of Eq. (20) is always equal to zero; therefore, the above equation can be written in the following form:

$$(\boldsymbol{\Phi}_a+\Delta\boldsymbol{\Phi})^T\Delta\mathbf{K}(\boldsymbol{\Phi}_a+\Delta\boldsymbol{\Phi})=\Delta\boldsymbol{\Lambda}-\left(\boldsymbol{\Phi}_a^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right)-\left(\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\boldsymbol{\Phi}_a\right)-\left(\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right). \tag{21}$$

According to Eq. (6), the mode shape expressions at the left-hand side of Eq. (21) are associated with the measured mode shape matrix. Thus:

$$\boldsymbol{\Phi}_x^T\Delta\mathbf{K}\boldsymbol{\Phi}_x=\Delta\boldsymbol{\Lambda}-\left(\boldsymbol{\Phi}_a^T\mathbf{K}_a\Delta\boldsymbol{\Phi}+\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\boldsymbol{\Phi}_a+\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right). \tag{22}$$

As can be observed, the expressions at the right-hand side of Eq. (22) consist of the stiffness and mode shape matrices of the analytical model along with the discrepancy modal matrices. Most of them are known; therefore, it is desirable to compact these terms within a matrix called stiffness coefficient matrix as follows:

$$\mathbf{C}_k=\Delta\boldsymbol{\Lambda}-\left(\boldsymbol{\Phi}_a^T\mathbf{K}_a\Delta\boldsymbol{\Phi}+\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\boldsymbol{\Phi}_a+\Delta\boldsymbol{\Phi}^T\mathbf{K}_a\Delta\boldsymbol{\Phi}\right). \tag{23}$$

Consequently, the improved model updating equation for the stiffness matrix can be presented in the following form:

$$\boldsymbol{\Phi}_x^T\Delta\mathbf{K}\boldsymbol{\Phi}_x=\mathbf{C}_k. \tag{24}$$

The main aim of this equation is to determine the unknown stiffness discrepancy matrix by mathematical techniques that may be a direct or an iterative method. Similar to the preceding section, the significant benefit of Eq. (24) is to provide a more complete form of the stiffness coefficient matrix ($\mathbf{C}_{\mathrm{k}}$) in comparison with the corresponding one in [28]. This is due to improving the structure of model updating formulation, applying the experimental modal matrix instead of using the analytical one at the left-hand side of the updating equation, and the lack of neglecting the higher order terms in the expanded stiffness orthogonality condition. Another benefit of the improved equation is that the stiffness matrix is separately updated without using the mass matrix or other additional information.

3. Least squares minimal residual method

The improved model updating formulas are linear matrix systems in the mathematical sense. These equations are similar to the common type of the linear matrix system such as $\mathbf{CYD}=\mathbf{E}$, in which $\mathbf{Y}$ is an n-by-n symmetric unknown matrix representing the discrepancy mass or stiffness matrices in each of the model updating procedures. The known information in this system includes $\mathbf{C}$ and $\mathbf{D}$ as n-by-m rectangular matrices implying the measured mode shape matrices, and $\mathbf{E}$ as an m-by-m matrix, which denotes the mass or stiffness coefficient matrices in the model updating methods.

To attain a faster convergence rate and solve the model updating formulas by LSMR method, the linear matrix equation $\mathbf{CYD}=\mathbf{E}$ transforms into the linear system in the vector form as $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}=\mathbf{C}\otimes\mathbf{D}$ is an m^2-by-n^2 rectangular matrix computed through Kronecker product. Moreover, $\mathbf{x}$ and $\mathbf{b}$ are vectors with n^2 and m^2 elements that are constructed by vectorizations of the matrices $\mathbf{Y}$ and $\mathbf{E}$, respectively.

In linear algebra and matrix theory, the vectorization of a matrix is a linear transformation that converts a matrix into a column vector. The vectorization of the n-by-n matrix $\mathbf{Y}$, for example, is a column vector with n^2 elements obtained by stacking the columns of the matrix $\mathbf{Y}$ on top of one another and denotes by $vec(\mathbf{Y})$. As a result, the improved model updating formulas in the matrix form $\mathbf{CYD}=\mathbf{E}$ can be expressed as the vector style $\mathbf{Ax}=\mathbf{b}$ in the following equations:

$$\left(\boldsymbol{\Phi}_x^T\otimes\boldsymbol{\Phi}_x^T\right)vec(\Delta\mathbf{M})=vec(\mathbf{C}_m), \tag{25}$$

$$\left(\boldsymbol{\Phi}_x^T\otimes\boldsymbol{\Phi}_x^T\right)vec(\Delta\mathbf{K})=vec(\mathbf{C}_k), \tag{26}$$

in which $\mathbf{A}=\boldsymbol{\Phi}_x^T\otimes\boldsymbol{\Phi}_x^T$. In the process of mass updating, $\mathbf{b}=vec(\mathbf{C}_{\mathrm{m}})$ and $\mathbf{x}=vec(\Delta\mathbf{M})$. In contrast, the vectors $\mathbf{b}$ and $\mathbf{x}$ in the process of updating the stiffness matrix are expressed as $vec(\mathbf{C}_{\mathrm{k}})$ and $vec(\Delta\mathbf{K})$, respectively.

Once the linear matrix systems of the model updating formulas have been converted into the linear vector systems, the LSMR method is applied to solve these equations in an iterative manner. This method is one of the Krylov subspace approaches that makes an attempt to solve the linear vector systems such as $\mathbf{Ax} = \mathbf{b}$, when the coefficient matrix $\mathbf{A}$ is large, rectangular and sparse with some singular values close to zero.

For the majority of the vibration-based applications with large and sparse coefficient matrices, the LSMR method can be used as an influential and applicable approach due to the avoidance of using the inverse problem for solving the linear systems. A high convergence rate is another property of this method, which makes it as a capable approach for applying to the large engineering systems. Another advantage of the LSMR method is that the coefficient matrix of the linear system, $\mathbf{A}$, can be a square matrix ($n = m$) or a rectangular matrix ($n > m$ or $n < m$). This specification of the LSMR method leads to a preference for solving the linear vector systems compared with the current iterative techniques such as conjugate gradients squared (CGS), generalized minimum residual (GMRES), minimum residual (MINRES), and preconditioned conjugate gradients (PCG) since all of them need a square coefficient matrix to solve the system of linear equation.

The LSMR method is analytically equivalent to MINRES approach applied to the normal equation $(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{x}_{\mathrm{k}} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$ in such a way that the quantities $\|\mathbf{A}^{\mathrm{T}}\mathbf{r}_{\mathrm{k}}\|$ monotonically decrease, where $\mathbf{r}_{\mathrm{k}} = \mathbf{b} - \mathbf{Ax}_{\mathrm{k}}$ is the residual vector in the kth iteration [38]. In fact, $\mathbf{x}_{\mathrm{k}}$ is an approximate solution for the equation $\mathbf{Ax} = \mathbf{b}$ obtaining from the LSMR algorithm in the kth step by applying the condition $min\|\mathbf{A}^{\mathrm{T}}\mathbf{r}_{\mathrm{k}}\|$. In addition, the LSMR method is similar to the well-known LSQR method [39]. Although both of them have the same structures and ultimately converge to similar points, it is better and safer to use the LSMR algorithm to early terminate the iterations [38]. This method uses an algorithm of Golub and Kahan [40], which is expressed as a bidiagonalization procedure to reduce a lower bidiagonal form as follows:

$$\left.\begin{array}{l} \boldsymbol{\beta}_1\mathbf{u}_1 = \mathbf{b}, \quad \boldsymbol{\alpha}_1\mathbf{v}_1 = \mathbf{A}^{\mathrm{T}}\mathbf{u}_1 \\ \boldsymbol{\beta}_{\mathrm{k}+1}\mathbf{u}_{\mathrm{k}+1} = \mathbf{A}\mathbf{v}_{\mathrm{k}} - \boldsymbol{\alpha}_{\mathrm{k}}\mathbf{u}_{\mathrm{k}} \\ \boldsymbol{\alpha}_{\mathrm{k}+1}\mathbf{v}_{\mathrm{k}+1} = \mathbf{A}^{T}\mathbf{u}_{\mathrm{k}+1} - \boldsymbol{\beta}_{\mathrm{k}+1}\mathbf{v}_{\mathrm{k}} \end{array}\right\}, \quad \mathrm{k} = 1, 2, \ldots \tag{27}$$

where β_1 and α_1 are the scalar values gained by the calculation of l_2-norm concerning the right-hand side of the bidiagonalization conditions as $\beta_1 = \|\mathbf{b}\|_2$ and $\alpha_1 = \|\mathbf{A}^{\mathrm{T}}\mathbf{u}_1\|_2$. In the first iteration, $\mathbf{u}_1$ and $\mathbf{v}_1$ are initial vectors that are determined as:

$$\mathbf{u}_1 = \frac{\mathbf{b}}{\boldsymbol{\beta}_1} = \frac{\mathbf{b}}{\|\mathbf{b}\|_2}, \tag{28}$$

$$\mathbf{v}_1 = \frac{\mathbf{A}^T\mathbf{u}_1}{\boldsymbol{\alpha}_1} = \frac{\mathbf{A}^T\mathbf{u}_1}{\|\mathbf{A}^T\mathbf{u}_1\|_2}. \tag{29}$$

Similar to the first step of the bidiagonalization process, one can write:

$$\boldsymbol{\beta}_{k+1} = \|\mathbf{A}\mathbf{v}_k - \boldsymbol{\alpha}_k\mathbf{u}_k\|_2, \tag{30}$$

$$\mathbf{u}_{k+1} = \frac{\mathbf{A}\mathbf{v}_k - \boldsymbol{\alpha}_k\mathbf{u}_k}{\boldsymbol{\beta}_{k+1}} = \frac{\mathbf{A}\mathbf{v}_k - \boldsymbol{\alpha}_k\mathbf{u}_k}{\|\mathbf{A}\mathbf{v}_k - \boldsymbol{\alpha}_k\mathbf{u}_k\|_2}, \tag{31}$$

and

$$\boldsymbol{\alpha}_{k+1} = \left\|\mathbf{A}^T\mathbf{u}_{k+1} - \boldsymbol{\beta}_{k+1}\mathbf{v}_k\right\|_2, \tag{32}$$

$$\mathbf{v}_{k+1} = \frac{\mathbf{A}^T\mathbf{u}_{k+1} - \boldsymbol{\beta}_{k+1}\mathbf{v}_k}{\boldsymbol{\alpha}_{k+1}} = \frac{\mathbf{A}^T\mathbf{u}_{k+1} - \boldsymbol{\beta}_{k+1}\mathbf{v}_k}{\left\|\mathbf{A}^T\mathbf{u}_{k+1} - \boldsymbol{\beta}_{k+1}\mathbf{v}_k\right\|_2}. \tag{33}$$

Note that these quantities are computed so that $\|\mathbf{u}_{\mathrm{k}}\|_2 = \|\mathbf{v}_{\mathrm{k}}\|_2 = 1$. With the definitions:

$$\begin{array}{l} \mathbf{U}_k = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k], \\ \mathbf{V}_k = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k], \end{array} \qquad \mathbf{H}_k = \begin{bmatrix} \boldsymbol{\alpha}_1 & & & \\ \boldsymbol{\beta}_1 & \boldsymbol{\alpha}_2 & & \\ & \boldsymbol{\beta}_2 & \ddots & \\ & \ddots & \ddots & \boldsymbol{\alpha}_k \\ & & & \boldsymbol{\beta}_{k+1} \end{bmatrix}. \tag{34}$$

The recurrence Eq. (27) can be written as:

$$\begin{array}{l} \mathbf{U}_{k+1}\left(\boldsymbol{\beta}_1\mathbf{z}_1\right) = \mathbf{b}, \\ \mathbf{A}\mathbf{V}_k = \mathbf{U}_{k+1}\mathbf{H}_k, \\ \mathbf{A}^T\mathbf{U}_{k+1} = \mathbf{V}_k\mathbf{H}_k^T + \boldsymbol{\alpha}_{k+1}\mathbf{v}_{k+1}\mathbf{z}_{k+1}^T. \end{array} \tag{35}$$

where $\mathbf{z}_1$ and $\mathbf{z}_{\mathrm{k}+1}$ are the first and last columns of the identity matrix $\mathbf{I}$, respectively. Using the bidiagonalization conditions, the algorithm of LSMR can be categorized in Table 1 in the following form.

Table 1
The algorithm of LSMR method [38].

Step 1: Initialize
$\boldsymbol{\beta}_1\mathbf{u}_1 = \mathbf{b},\quad \boldsymbol{\alpha}_1\mathbf{v}_1 = \mathbf{A}^T\mathbf{u}_1,\quad \bar{\boldsymbol{\alpha}}_1 = \boldsymbol{\alpha}_1,\quad \bar{\boldsymbol{\gamma}}_1 = \boldsymbol{\alpha}_1\boldsymbol{\beta}_1,\quad \boldsymbol{\rho}_1 = 1,\quad \bar{\boldsymbol{\rho}}_1 = 1$
$\bar{\boldsymbol{c}}_0 = 1,\quad \bar{\boldsymbol{s}}_0 = 1,\quad \mathbf{f}_1 = \mathbf{v}_1,\quad \bar{\mathbf{f}}_0 = 0,\quad \mathbf{x}_0 = 0.$
Step 2: For k=1,2,..., repeat steps 3–6 as follows
Step 3: Continue the bidiagonalization
$\boldsymbol{\beta}_{k+1}\mathbf{u}_{k+1} = \mathbf{A}\mathbf{v}_k - \boldsymbol{\alpha}_k\mathbf{u}_k$ and $\boldsymbol{\alpha}_{k+1}\mathbf{v}_{k+1} = \mathbf{A}^T\mathbf{u}_{k+1} - \boldsymbol{\beta}_{k+1}\mathbf{v}_k$
Step 4:
$\rho_k = \sqrt{(\bar{\alpha}_k^2 + \beta_{k+1}^2)},\quad c_k = \bar{\alpha}_k/\rho_k,\quad s_k = \beta_{k+1}/\rho_k,$
$\theta_{k+1} = s_k\alpha_{k+1},\quad \bar{\alpha}_{k+1} = c_k\alpha_{k+1}$
Step 5:
$\bar{\theta}_k = \bar{s}_{k-1}\rho_k,\quad \bar{\rho}_k = \sqrt{(\bar{c}_{k-1}\rho_k)^2 + \theta_{k+1}^2},$
$\bar{c}_k = \bar{c}_{k-1}\rho_k/\bar{\rho}_k,\quad \bar{s}_k = \theta_{k+1}/\bar{\rho}_k,$
$\gamma_k = \bar{c}_k\bar{\gamma}_k,\quad \bar{\gamma}_{k+1} = -\bar{s}_k\bar{\gamma}_k$
Step 6: Update $\mathbf{f}$, $\bar{\mathbf{f}}$ and $\mathbf{x}$
$\bar{\mathbf{f}}_k = \mathbf{f}_k - (\frac{\bar{\boldsymbol{\theta}}_k\cdot\boldsymbol{\rho}_k}{\boldsymbol{\rho}_{k-1}\cdot\bar{\boldsymbol{\rho}}_{k-1}})\bar{\mathbf{f}}_{k-1}$
$\mathbf{x}_k = \mathbf{x}_{k-1} + (\frac{\boldsymbol{\gamma}_k}{\boldsymbol{\rho}_k\cdot\bar{\boldsymbol{\rho}}_k})\bar{\mathbf{f}}_k$
$\mathbf{f}_{k+1} = \mathbf{v}_{k+1} - (\frac{\boldsymbol{\theta}_{k+1}}{\boldsymbol{\rho}_k})\mathbf{f}_k$

Table 2
The first two natural frequencies (rad/s) of the experimental and analytical models.

Experimental model		Analytical model	
Mode 1	Mode 2	Mode 1	Mode 2
240.88	467.95	240.32	478.71

Applying the bidiagonalization process, the LSMR method makes an iterative algorithm, which is used to compute the unique solution of vector $\mathbf{x}$ in the kth iteration so that $\|\mathbf{A}^T\mathbf{r}_k\|$ monotonically decrease. This method can improve the solution of the direct least-squares method using the iterative and bidiagonalization processes.

After computing the vector $\mathbf{x}_k$ in the kth iteration, the mass and stiffness discrepancy matrices are determined by transforming the vector $\mathbf{x}$ into the matrix style. In this regard, the first n rows of the vectors $vec(\triangle\mathbf{M})$ and $vec(\triangle\mathbf{K})$ insert into the first columns of the mass and stiffness discrepancy matrices. This process continues to constitute n-by-n matrices indicating the structural discrepancy matrices $\triangle\mathbf{M}$ and $\triangle\mathbf{K}$.

4. A new iterative method for updating the mass and stiffness matrices

In order to have precise and reliable updating results, it is necessary to use an iterative algorithm to adjust the structural parameters of the FE model rather than using a direct method. Even though the LSMR method can remarkably assist us in solving the model updating equations, it is only a mathematical tool and may be insufficient for achieving the correct structural matrices.

The proposed iterative method in this section sets out to update the inherent structural characteristics of the FE model by their discrepancy matrices obtained from the LSMR technique. Actually, this method makes an iterative-iterative algorithm in such a way that the first iterative process refers to the calculation of the discrepancy matrices by the iterative algorithm of LSMR method and the second one pertains to the proposed iterative method for updating the mass and stiffness matrices.

Given $\triangle\mathbf{M}$ and $\triangle\mathbf{K}$ as the mass and stiffness discrepancy matrices obtained by the LSMR algorithm, the structural matrices of the FE model are updated in an iterative manner as follows:

$$\mathbf{M}_u^{(i)} = \mathbf{M}_a^{(i)} + \triangle\mathbf{M}^{(i)}, \tag{36}$$

$$\mathbf{K}_u^{(i)} = \mathbf{K}_a^{(i)} + \triangle\mathbf{K}^{(i)}, \tag{37}$$

where i = 2,3, …, n_u represents the number of iterations in the proposed iterative method. Moreover:

$$\mathbf{M}_a^{(i)} = \boldsymbol{\beta}_{\boldsymbol{m}}\mathbf{M}_a^{(i-1)}, \tag{38}$$

$$\mathbf{K}_a^{(i)} = \boldsymbol{\beta}_{\boldsymbol{s}}\mathbf{K}_a^{(i-1)}. \tag{39}$$

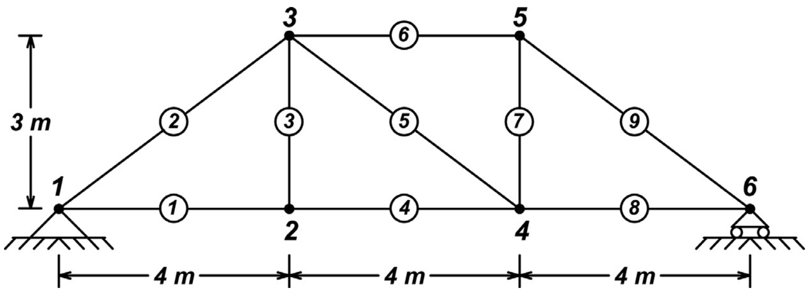

Fig. 1. The planner truss structure [30].

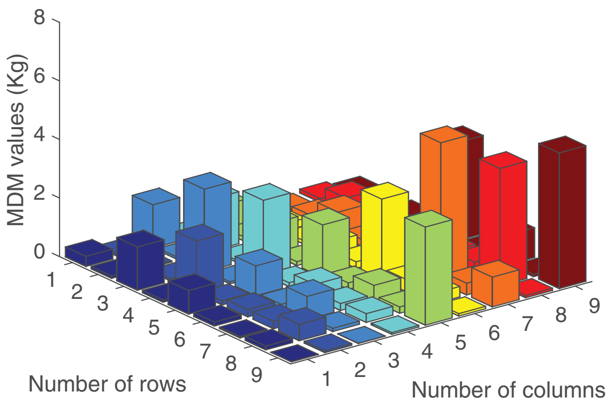

Fig. 2. The absolute values of the mass discrepancy matrix (MDM) using the proposed iterative finite element model updating method.

In these equations, β_m and β_s are the scalar amounts that denote the correction terms for updating the mass and stiffness matrices, respectively. For the process of mass updating, the correction parameter, β_m, is given by:

$$\boldsymbol{\beta_m} = \frac{\max\limits_{j} \sum\limits_{l=1}^{n} \left| vec\left(\mathbf{M}_u^{(i-1)}\right)_{jl} \right|}{\max\limits_{j} \sum\limits_{l=1}^{n} \left| vec\left(\mathbf{M}_a^{(i-1)}\right)_{jl} \right|}. \tag{40}$$

Similarly, in the process of updating the stiffness matrix, the correction parameter, β_s, is expressed as:

$$\boldsymbol{\beta_s} = \frac{\max\limits_{j} \sum\limits_{l=1}^{n} \left| vec\left(\mathbf{K}_u^{(i-1)}\right)_{jl} \right|}{\max\limits_{j} \sum\limits_{l=1}^{n} \left| vec\left(\mathbf{K}_a^{(i-1)}\right)_{jl} \right|}. \tag{41}$$

It is significant to note that the modal parameters of the analytical model should be determined in each of the iterations using the eigenvalue function. Due to improving the structural matrices of the analytical or FE model by Eqs. (38) and (39), it is essential to compute $\mathbf{C}_m$ and $\mathbf{C}_k$ in each iteration of the proposed iterative finite element model updating method. For this purpose, the structural matrices as well as the updated modal parameters of the analytical model are inserted into Eqs. (12) and (23).

For i = 2, in addition, the expressions associated with the (i − 1) iteration represent the first selection of the structural parameters for the model updating problems. For example, $\mathbf{M}_a{}^{(1)}$ and $\mathbf{K}_a{}^{(1)}$ mean the first assumptions of the mass and stiffness matrices of the analytical model. In this case, $\Delta\mathbf{M}^{(1)}$ and $\Delta\mathbf{K}^{(1)}$ are the first computations of the structural discrepancy matrices considering the first assumptions of the mass and stiffness matrices in the analytical model.

As another note, each iterative method requires a stopping criterion to terminate the iterations. In practice, the number of iterations of an iterative method is concerned with stopping criteria that are often finite. In this study, the stopping criterion is based on the l_2-norm of the difference between the updated matrices in the (i) and (i − 1) iterations in the

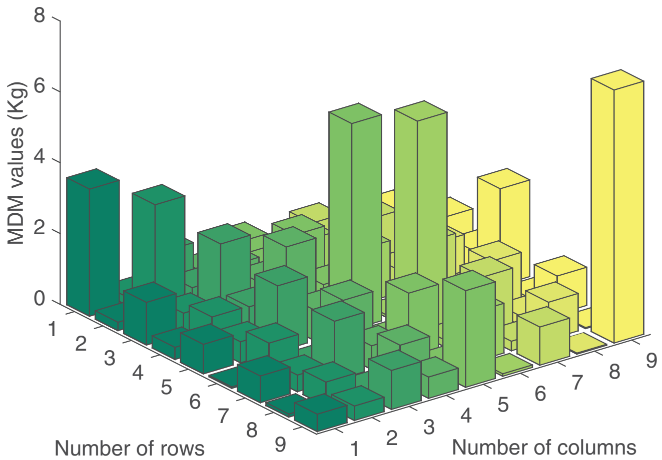

Fig. 3. The absolute values of the mass discrepancy matrix (MDM) using Yang and Chen's method.

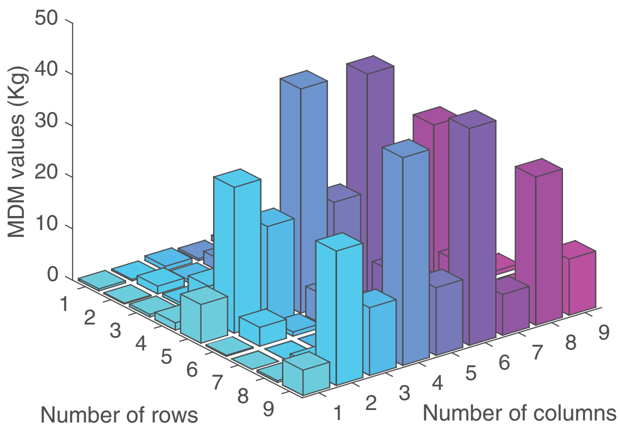

Fig. 4. The absolute values of the mass discrepancy matrix (MDM) using Lee and Eun's method.

following forms:

$$\boldsymbol{R_m} = \left\| \mathbf{M}_u^{(i)} - \mathbf{M}_u^{(i-1)} \right\|_2 \leq \boldsymbol{e_m}, \tag{42}$$

$$\boldsymbol{R_s} = \left\| \mathbf{K}_u^{(i)} - \mathbf{K}_u^{(i-1)} \right\|_2 \leq \boldsymbol{e_s}, \tag{43}$$

where e_m and e_s are the user-supplied threshold values for the stopping criteria of the mass and stiffness updating processes. These criteria are useful tools for applying to the practical applications since their parameters belong to the updated model.

Based on the proposed methods, the model updating procedures are begun by establishing the mass and stiffness matrices of the analytical model, $\mathbf{M}_\mathrm{a}$ and $\mathbf{K}_\mathrm{a}$, and determining the analytical modal parameters, $\boldsymbol{\Phi}_\mathrm{a}$ and $\boldsymbol{\Lambda}_\mathrm{a}$. It is assumed that the measured modal parameters obtained from the modal tests, $\boldsymbol{\Phi}_\mathrm{x}$ and $\boldsymbol{\Lambda}_\mathrm{x,}$ are available. The LSMR method is applied to solve the improved model updating formulations and determine the mass and stiffness discrepancy matrices, $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$, in each iteration of the proposed iterative finite element model updating method. In the following, the mass and stiffness matrices of the analytical model are updated in an iterative manner using Eqs. (36) and (37).

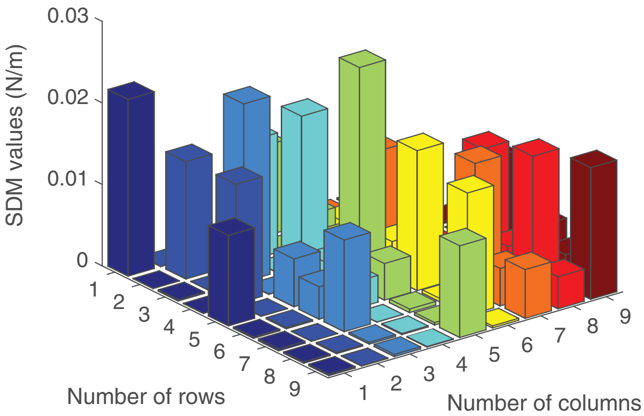

Fig. 5. The absolute values of the stiffness discrepancy matrix (SDM) using the proposed iterative finite element model updating method.

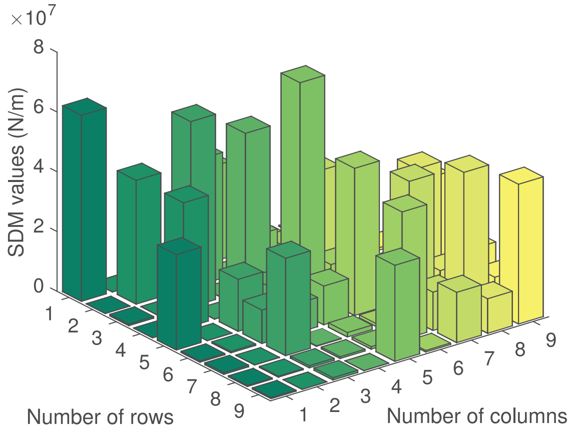

Fig. 6. The absolute values of the stiffness discrepancy matrix (SDM) using Yang and Chen's method.

5. Applications

5.1. A numerical planner truss

When a method is improved, a question arises how can to understand the level of improvement. Thus, in order to demonstrate the performance and capability of the proposed methods, a simple planner truss numerically constructed by Lee and Eun [30] is employed to compare updating results obtained by the proposed methods with their approach as well as Yang and Chen's method [28]. As Fig. 1 shows, the total length, and height of the truss are 12 m and 3 m, respectively. The material properties of this model include the modulus of elasticity 200 GPa and material density 7860 kg/m^3. The FE model of truss consists of 6 nodes, 9 elements, and 9 degrees of freedom (DOFs) in such a way that each node possesses two translation DOFs in the global coordinate.

To introduce modeling errors regarding the model updating problem, the cross sections of the third and eighth members reduce 20% and 30%, respectively. This procedure alters the structural characteristics of the FE model and provides a realistic simulation to use in the practical problems. In this study, the initial FE model is taken as the experimental or real structure and the truss with the modeling errors is considered as the analytical model.

After modeling the experimental and analytical models of the truss structure, the modal data such as the natural frequencies and mode shapes are computed by the eigenvalue problem. The first two natural frequencies and the first two mode shapes at all degrees of freedom are applied to both experimental and analytical models [30]. Even though complete sets of the modal parameters have not been used in updating procedures, they cannot make a realistic simulation of the incompleteness conditions. This is because of applying all of the modal displacements at all the degrees of freedom. In practice, the number of measurements is less than the degree of freedom of the model. The example presented considers incompleteness only in terms of the available number of natural frequencies, whereas the first two vectors of the mode

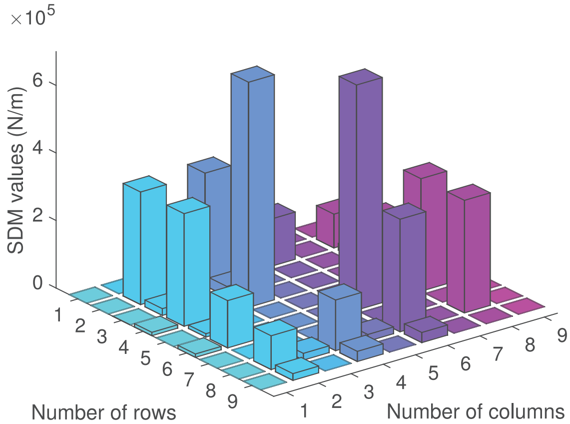

Fig. 7. The absolute values of the stiffness discrepancy matrix (SDM) using Lee and Eun's method.

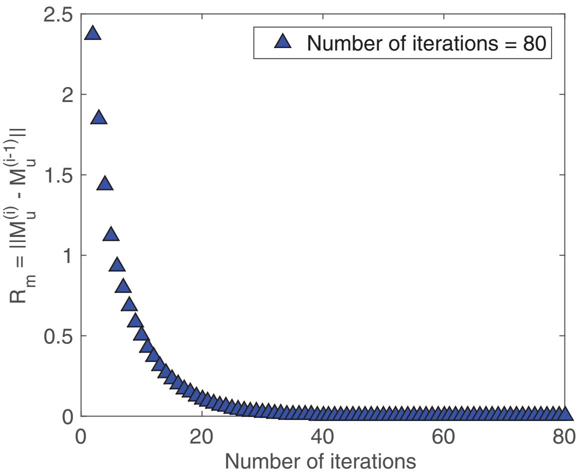

Fig. 8. The number of iterations in the process of updating the mass matrix.

Table 3
The structural characteristics of the shear building models.

Model	Story no.	Mass (kg)	Stiffness (kN/m^2)
5-story	1–2	3000	1200
	3–4	2000	1000
	5	1000	800
10-story	1–3	4000	1600
	4–6	3000	1300
	7–10	2000	1000

shapes are assumed to be available at all the degrees of freedom. As an example, Table 2 indicates the first and second circular natural frequencies of the experimental and analytical models.

To terminate the iterations in the proposed finite element model updating method, the user-threshold value e_m is 0.001 concerned with the process of updating the mass matrix. Because the components of the stiffness matrix are much more than the mass matrix, it is better to select a small threshold value as $e_s = 10^{-5}$.

One way to indicate updating results is to use the mass and stiffness discrepancy matrices, $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$, in the last iteration of the proposed iterative model updating method. These matrices compute the direct errors of the mass and stiffness matrices between the experimental and updated models. In the process of updating the mass matrix, for example, $\Delta\mathbf{M} = \mathbf{M}_{\mathrm{x}} - \mathbf{M}_{\mathrm{u}}$ represents the mass discrepancy matrix in the unit of kg, where $\mathbf{M}_{\mathrm{u}}$ is the updated mass matrix in the last iteration of the proposed iterative method. Figs. 2–4 indicate the results of updating the mass matrices obtained by the proposed methods in this article, Yang and Chen's approach, and Lee and Eun's method, respectively.

Fig. 2 shows the absolute values of the mass discrepancy matrix in the proposed iterative model updating method. The maximum quantity in this matrix is identical to 4.7906 kg that is equivalent to 5.7448% relative error, while most of the

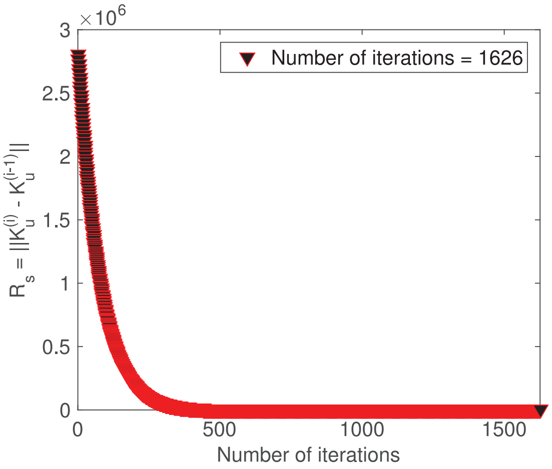

Fig. 9. The number of iterations in the process of updating the stiffness matrix.

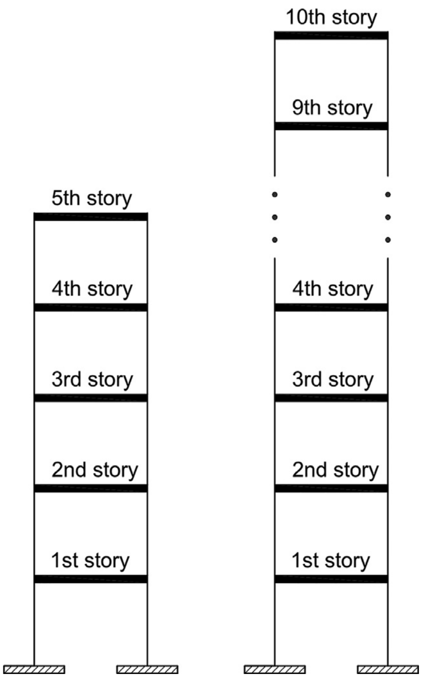

Fig. 10. The shear building models.

elements in the mass discrepancy matrix are less than 1 kg with the relative errors less than 2%. This means that there is a good similarity between the mass matrices of the experimental and updated models. Thus, it can be deduced that the proposed methods in this study are accurately able to update the mass matrix in the continuous dynamic systems.

Fig. 3 illustrates the absolute quantities of the mass discrepancy matrix in Yang and Chen's method. It is obvious that the maximum value of this matrix is roughly close to 8 kg, equivalent to 10% relative error, and the other ones are less than 4 kg. Although the results of updating the mass matrix in their method are approximately similar to the corresponding values in Fig. 2, it can be argued that the results of updating process gained by the proposed iterative method are more precise.

Fig. 4 shows the absolute values of the mass discrepancy matrix in Lee and Eun's method. As can be observed, there are large amounts at some elements of this matrix so that most of them are near 40 kg, which implies 30% relative error. This observation leads to the conclusion that the method of Lee and Eun does not have enough accuracy and precision for updating the mass matrix.

After comparing the results of updating the mass matrix, the stiffness discrepancy matrix is used to compute the direct errors in the elements of stiffness matrices of the experimental and updated models as $\Delta\mathbf{K} = \mathbf{K}_x - \mathbf{K}_u$ in the unit of N/m, where $\mathbf{K}_u$ is the updated stiffness matrix in the last iteration of the proposed iterative model updating method.

Table 4
Modeling errors in the shear building models.

Error no.	Model	Location	Mass	Stiffness
1	5-story	1st	–	−0.25
		2nd	0.3	–
		3rd	–	−0.4
		4th	0.2	–
		5th	−0.25	–
2	10-story	1st	–	−0.3
		2nd	0.3	−0.25
		4th	–	−0.4
		6th	0.5	–
		7th	–	−0.2
		8th	−0.25	–
		10th	0.25	–

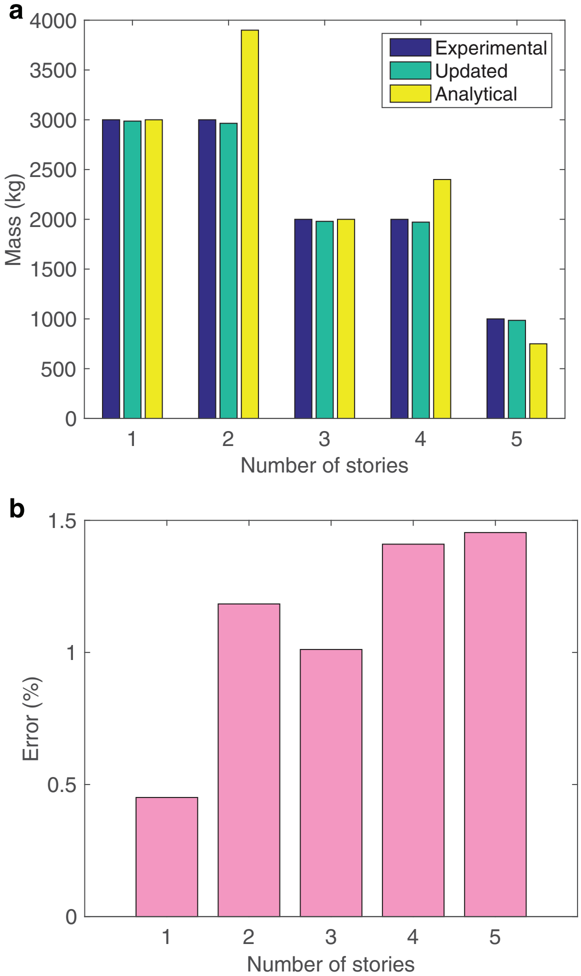

Fig. 11. Updating the mass matrix in the 5-story shear building model: (a) the values of diagonal elements in the mass matrices of the experimental, updated, and analytical models, (b) the relative errors at the diagonal elements of the updated mass matrix.

Fig. 5 indicates the absolute values of the stiffness discrepancy matrix obtained from the proposed iterative model updating method. It is clear from the figure that all elements of this matrix are approximately equal to zero since the maximum value of the stiffness discrepancy matrix is 0.03 N/m. This means that there is no difference between the updated and experimental stiffness matrices. Therefore, the proposed iterative model updating method is potentially capable of adjusting the stiffness matrix.

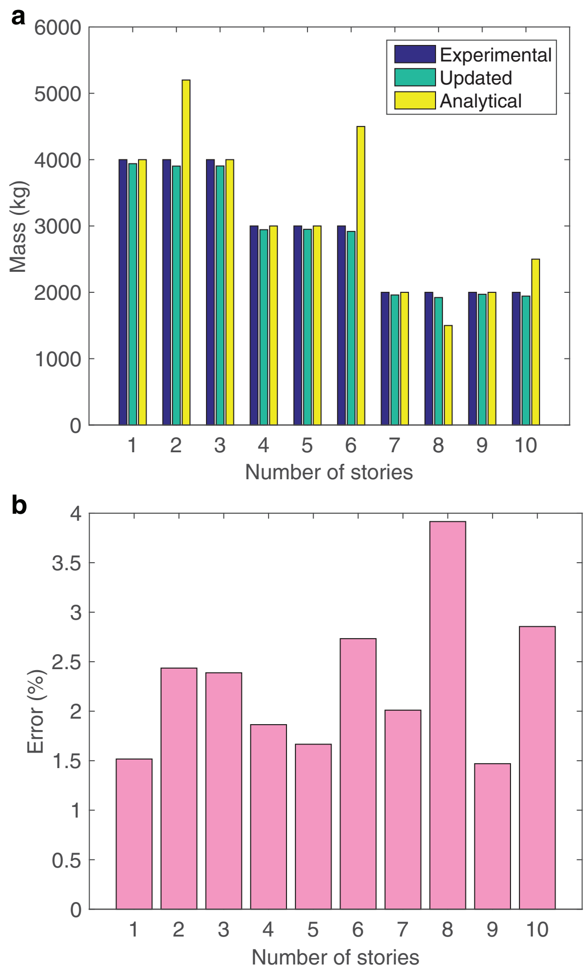

Fig. 12. Updating the mass matrix in the 10-story shear building model: (a) the values of diagonal elements in the mass matrices of the experimental, updated, and analytical models, (b) the relative errors at the diagonal elements of the updated mass matrix.

In contract, Figs. 6 and 7 show the results of the stiffness discrepancy matrix using Yang and Chen's approach and Lee and Eun's method, respectively. From these figures one can clearly realize that there are substantial errors in both methods, particularly in Yang and Chen's method. In other words, it is necessary to improve their methods to achieve more appropriate updating results, which has been performed in this study. Even though the results of Lee and Eun's method are better than the corresponding ones in the method of Yang and Chen, there are still unreliable values in the stiffness discrepancy matrix that should be improved.

Eventually, the comparative results demonstrate that the proposed methods including the improved model updating formulations, the LSMR technique, and the iterative finite element model updating method can provide better and more trustworthy updating results in comparison to the other model updating methods.

Figs. 8 and 9 suggest the number of iterations in the processes of updating the mass and stiffness matrices, respectively. As these figures appear, it can be observed that the quantities of R_m and R_s reduce with increasing the iterations in both model updating problems. These mean that the proposed iterative algorithm and the correction parameters, β_m and β_s, improve updating results in each iteration. On the other hand, the model updating problems are approximately terminated in values near zero; that is, the mass and stiffness matrices are updated with insubstantial computational errors.

Fig. 8 clearly demonstrates that updating the mass matrix needs a few iterations, whereas it is obvious from Fig. 9 that there are a large number of iterations in the process of updating the stiffness matrix. There are several reasons for this issue that one can point out the simplicity of mass matrix, choosing the threshold value e_s in a strict manner, and using both types of the modal parameters including the mode shape and natural frequency in the stiffness coefficient matrix $\mathbf{C}_\mathrm{k}$. According to Eqs. (12) and (23), the stiffness coefficient matrix uses both the modal and frequency matrices, whereas the mass coefficient matrix $\mathbf{C}_\mathrm{m}$ only utilizes the mode shape matrix. Another note is that the conditions of the measured modal parameters affect the model updating results and the number of iterations. In other words, these data play significant roles in the model updating problems so that further measured modes lead to better results with a few iterations.

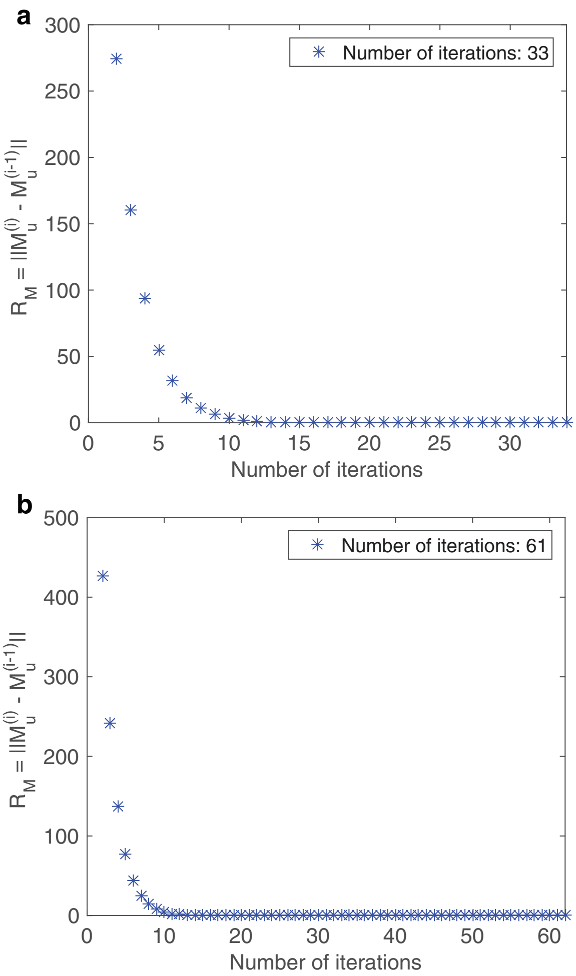

Fig. 13. The number of iterations in updating the mass matrix: (a) the 5-story shear building model, (b) the 10-story shear building model.

5.2. Numerical shear building models

For further verifications, two numerical shear building models as the discrete dynamic systems are constructed as shown in Fig. 10. Assume that each floor of the models is equipped with one sensor to simulate the measurement of all modal displacements in each measured mode. Table 3 appears the initial structural characteristics of the models including the mass and stiffness at each story.

Based on modeling the discrete dynamic systems in structural dynamics, the matrices of mass and stiffness are simply constructed to build FE models for the shear building structures. On this moment, it is assumed that these models are equivalent to the experimental or real models of the shear buildings. Subsequently, several modeling errors are applied to the real shear building structures for constructing their analytical models as presented in Table 4. Notice that the negative signs in some modeling errors are indicative of the reduction factors.

The modal parameters of the analytical and experimental models are simply determined using the eigenvalue problem. For real applications, the only first 2 and 5 natural frequencies and mode shapes are considered as the measured modal parameters in the 5 and 10-story shear building models, respectively. In a similar manner to the previous numerical example, since the modal displacements of all the degrees of freedom are utilized in updating procedures, they cannot deal with the incompleteness conditions of the modal data despite using incomplete natural frequencies. After modeling the shear buildings and determining the modal data, the analytical mass and stiffness matrices of the shear building models are updated based on the proposed methods.

One approach to indicate the results of model updating problems in the shear building models is to observe the values of diagonal elements of the mass and stiffness matrices. In most of the discrete dynamic systems, these elements consist of the significant structural characteristics. Figs. 11 and 12 illustrate the results of updating process concerning the mass matrix in the shear building models.

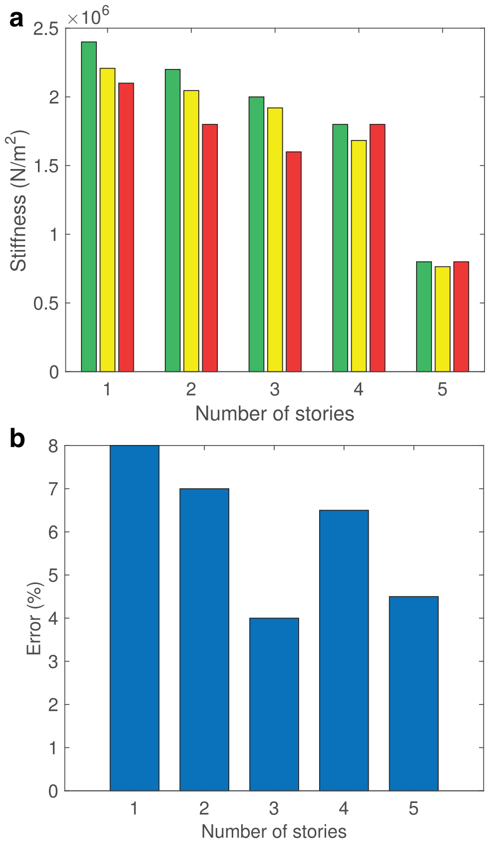

Fig. 14. Updating the stiffness matrix in the 5-story shear building model: (a) the values of diagonal elements in the stiffness matrices of the experimental, updated, and analytical models, (b) the relative errors at the diagonal elements of the updated stiffness matrix.

Fig. 11(a) shows the comparison of the values of diagonal elements in the mass matrices of the experimental ($\mathbf{M}_x$), updated ($\mathbf{M}_u$), and analytical ($\mathbf{M}_a$) models. From this figure, it can be perceived that the analytical mass matrix have correctly been updated resulting from the presence of insubstantial differences between the diagonal elements of mass matrix in the experimental and updated models. Fig. 11(b) indicates the relative errors at the diagonal elements of the updated mass matrix. The small relative errors at these elements confirm that the analytical mass matrix of the 5-story shear building model has been adjusted accurately.

On the other hand, Fig. 12(a) displays the diagonal elements of the mass matrices in the experimental, updated, and analytical 10-story shear building models. The observations in this figure demonstrate that the quantities of the diagonal elements of the mass matrix in the updated model are roughly identical to the corresponding values in the experimental model. Moreover, Fig. 12(b) illustrates the relative errors between the diagonal elements of the mass matrix in the updated and experimental models. From this figure, the inconsiderable relative errors verify the accuracy of updating the mass matrix.

The number of iterations in updating the mass matrix of the shear building models is shown in Fig. 13. As can be seen, the analytical mass matrix has been updated in a few iterations since the values of R_m are close to zero in both shear building models. It is worthwhile remaking that the threshold value e_m for updating the mass matrix in both models is 0.001. All the obtained results in Figs. 11–13 lead to the conclusion that the proposed methods, particularly the iterative finite element model updating method, influentially enable us to update the mass matrix of the shear building models.

In a similar manner, the values of diagonal elements of the stiffness matrices are used to evaluate the results of model updating problem. Fig. 14(a) shows the comparison of such quantities in the experimental, updated and analytical models of the 5-story shear building. It can be realized from this figure that there are insubstantial differences between the stiffness matrices of the experimental and updated models. Fig. 14(b) illustrates the relative errors at the diagonal elements of the stiffness matrices in these models. The small error amounts prove that the analytical stiffness matrix of the 5-story shear building model has been updated correctly.

The results of updating the stiffness matrix for the 10-story shear building model are shown in Fig. 15. As this figure reveals, the differences and the relative errors in the values of diagonal elements of the stiffness matrices are small in the experimental and updated models. More precisely, it is obvious from Fig. 15(b) that the relative errors in the stiffness

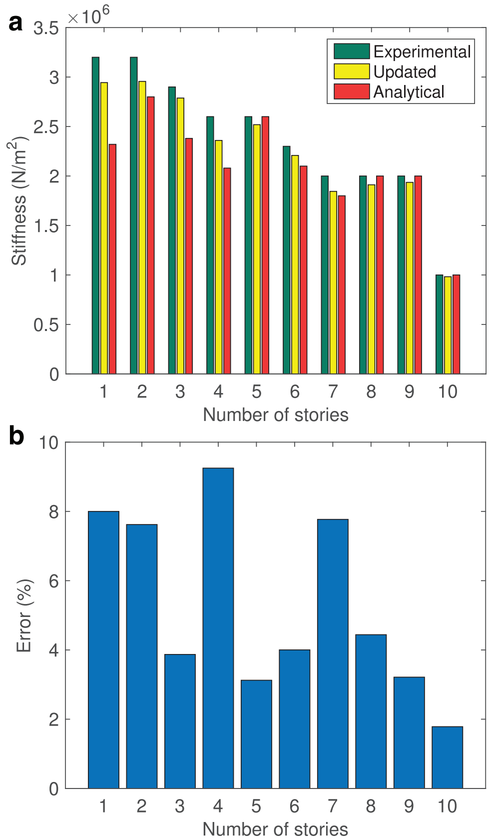

Fig. 15. Updating the stiffness matrix in the 10-story shear building model: (a) the values of diagonal elements in the stiffness matrices of the experimental, updated, and analytical models, (b) the relative errors at the diagonal elements of the updated stiffness matrix.

matrices are approximately less than 8%, with the exception of story 4. Such observations demonstrate the accuracy of updating the stiffness matrix in the 10-story shear building.

In addition, Fig. 16 shows the number of iterations in the process of updating the stiffness matrix in both shear building models. For this process, the value of user-threshold e_s is 0.001. Comparing the results of Figs. 13 and 16, one can infer that the process of updating the mass matrix requires much fewer iterations in comparison with updating the stiffness matrix in the shear building models. This is the same result, which has been achieved in the model updating problems concerning the numerical planner truss.

6. Concluding remarks

In this study, a new iterative model updating method was proposed to correct the mass and stiffness matrices of the FE model. This method relies on improving the two model updating formulas based on the dynamic discrepancy theory. In these formulations, the mass and stiffness orthogonality conditions were independently expanded to establish new model updating equations. The iterative LSMR method was applied to solve the improved model updating equations and determine the mass and stiffness discrepancy matrices in each iteration of the proposed iterative finite element model updating method. In the following, the accuracy and performance of the proposed methods were verified by a numerical planner truss and two numerical shear building models.

Based on the numerical examples, the following conclusions are drawn. (1) The model updating equations presented in this study were successfully improved resulting from more accurate results than Yang and Chen's formulations, particularly in updating the stiffness matrix. (2) The proposed iterative model updating method can precisely update the structural matrices of the FE models. (3) The comparative study on the planner truss showed that the proposed methods in this article provide more reliable and precise results, particularly in the procedure of updating the stiffness matrix. (4) In this model, the updated mass matrix was approximately similar to the corresponding one in Yang and Chen's method, whereas the results of Lee and Eun's method indicated the large errors in some elements of the mass discrepancy matrix. (5) In the comparative study, it was seen that the stiffness discrepancy matrix in the methods of Yang-Chen and Lee-Eun give unreliable updating results since there were several substantial errors in the elements of stiffness discrepancy matrices. (6) In the shear building models, the mass and stiffness matrices have accurately been updated due to the small errors at the

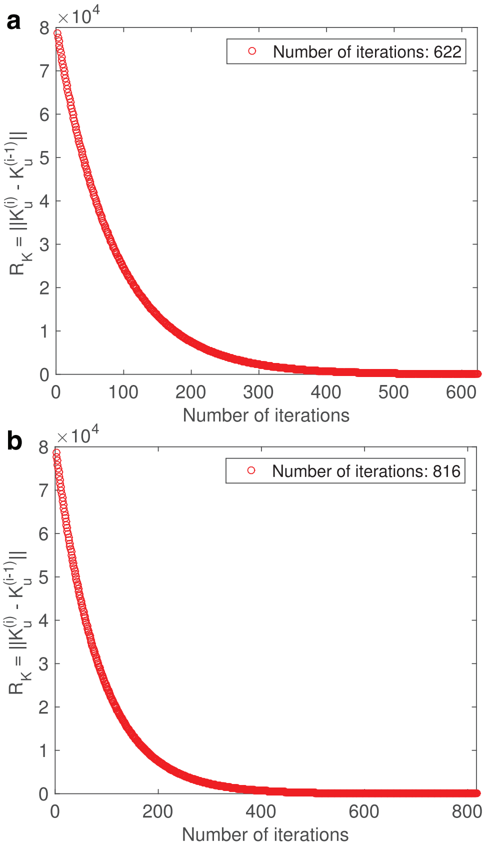

Fig. 16. The number of iterations in updating the stiffness matrix: (a) the 5-story shear building model, (b) the 10-story shear building model.

diagonal elements of these matrices. (7) The process of updating the mass matrix needed a few iterations, whereas the number of iterations in updating the stiffness matrix was much more than the mass updating problem.

Although the obtained results in this study are reliable and accurate, there are some limitations that can be improved in further studies. The first limitation is related to measuring all modal displacements in each measured mode that make the proposed methods impractical for engineering systems. However, several model reduction and modal expansion techniques can be applied to address this problem. The second limitation pertains to the lack of using noisy incomplete modal data in the model updating problems. In reality, the modal parameters are contaminated by noise; therefore, it would be interested in evaluating the proposed methods by applying noise in the incomplete modal data. Eventually, in the third limitation, it is important to verify the proposed methods at least by an experimental example.

Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.apm.2016.07.015.

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