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Variance Approximation of Stress-Strength Reliability Estimator in Burr XII Distribution

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Abstract

One of the most fundamental issues in estimation theory about accuracy of an unbiased estimator is computing or approximating its variance. Very often, the variance has a complicated form or cannot be computed explicitly. In this paper, we find the MLE and UMVUE of stress-strength reliability measure of form $R = P(Y < X)$, when X and Y have Burr XII distribution. Also, in this distribution, we obtain the general form of Bhattacharyya matrix and then by using Bhattacharyya lower bounds, we approximate the variance of any unbiased estimator of R.

Keywords: Stress-Strength model, Burr XII distribution, Bhattachrayya lower bound, Cramer-Rao lower bound.

Mathematics Subject Classification (2010): 65C20, 65C60.

1 Introduction

The problem of estimating $R = P(Y < X)$ arises in the context of mechanical reliability of a system with strength X and stress Y , the reliability, R, is chosen as a measure of system reliability. In a stress strength model, the system fails if and only if, at any time, the applied stress is greater than its strength. This model, first considered by Birnbaum (1956), is commonly used in many engineering applications, such as civil, mechanical and aerospace. Recently, a number of papers have dealt with the stress strength reliability problem. Several distributions have been used in the literature as failure models. For references see the book by Kotz et al. (2003) or the articles by Church and Harris (1970), Chao (1982), Surles and Padgett (2001), Raqab and Kundu (2005) and Kundu and Gupta (2005).

The Burr-XII distribution, which was originally derived by Burr (1942) and received more attention by the researchers due to its broad applications in different fields including the area of reliability, failure time modeling and acceptance sampling plan. Reader can find the applications in various fields from Ali Mosa and Jaheen (2002) and Burr (1942). The two parameters Burr-XII distribution has the following density function

$$f(x) = \alpha\theta x^{\alpha-1}(1 + x^{\alpha})^{-(\theta+1)}; \quad for \quad x > 0,$$

and the distribution function

$$F(x) = 1 - (1 + x^{\alpha})^{-\theta}; \quad for \quad x > 0.$$

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Here $\alpha > 0$ and $\theta > 0$ are the shape parameters respectively. It is important to note that when $\alpha = 1$, the Burr-XII reduces to the log-logistic distribution.

The Burr XII is a unimodal distribution and has a non-monotone hazard function, which can accommodate many shapes of it. Thus, the use of this distribution as a failure model is appropriate and useful in applied statistics, especially in survival analysis and actuarial studies.

Ahmad et al. (2011) and Surles and Padgett (2001) considered the estimation of $P[Y < X]$, where X and Y are Burr Type X random variables. Panahi and Asadi (2010) compared different methods of estimating R when Y and X both follow Burr Type XII distribution with parameters (θ_1, α) and (θ_2, α), respectively.

A lower bound for the variance of an estimator is one of the fundamental things in the estimation theory because it gives us an idea about the accuracy of an estimator. When the variance has complicated form and we can not compute it, by lower bounds, we can approximate it. Up to now, many studies have been done for the lower bound of the variance of an unbiased estimator of the parameter. The well-known lower bounds are Cramer-Rao, Bhattacharyya, Hammersley-Chapman-Robins, Kshirsagar and Koike.

The evaluation of variance of R's estimator in Burr-XII distribution is impossible or very hard. So, we can approximate it by some lower bounds.

In this paper, according to usefulness and wide applications of the Burr XII distribution and importance of finding and approximating a lower bound for the variance of the estimator of $R = P(Y < X)$, we first introduce the most sharper bounds which is the Bhattacharyya bound under regularity conditions. Then, we construct the general form of the Bhattacharyya matrice which is used in its inequality. Also, we evaluate and compare different Bhattacharyya bounds for the variance of estimator of R in Burr XII distribution.

2 The Stress-Strength reliability measure

Let X and Y are two independent Burr Type XII random variables with parameters θ_1, α and θ_2, α respectively. Therefore

$$
\begin{aligned}
R = P(Y < X) &= \int_0^\infty \int_0^x f_X(x) f_Y(y)\,dy\,dx \\
&= \int_0^\infty F_Y(x) f_Y(x)\,dx \\
&= \frac{\theta_2}{\theta_1 + \theta_2}.
\end{aligned}
$$

Now we try to find the MLE of R. We consider the estimation of R when α is known. Without loss of generality, we can assume that $\alpha = 1$. Let $X_1, \ldots, X_n$ is a random sample from $BurrXII(\theta_1, 1)$ and $Y_1, \ldots, Y_m$ is a random sample from $BurrXII(\theta_2, 1)$. Then we have:

$$
L(\theta_1, \theta_2) = n \ln(\theta_1) + m \ln(\theta_2) - (\theta_1 + 1) \sum_{i=1}^{n} \ln(1 + x_i) - (\theta_2 + 1) \sum_{j=1}^{m} \ln(1 + y_j).
$$

Differentiating partially with respect to θ_1 and θ_2 and setting the results equal to zero we get two non-linear equations as follow:

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= \frac{n}{\theta_1} - \sum_{i=1}^{n} \ln(1 + x_i) = 0, \\
\frac{\partial L}{\partial \theta_2} &= \frac{m}{\theta_2} - \sum_{j=1}^{m} \ln(1 + y_j) = 0,
\end{aligned}
$$

and we obtain,

$$
\begin{aligned}
\widehat{\theta}_1 &= \frac{n}{\sum_{i=1}^{n} \ln(1 + x_i)}, \\
\widehat{\theta}_2 &= \frac{m}{\sum_{j=1}^{m} \ln(1 + y_j)}.
\end{aligned}
$$

Therefore the MLE of R is given by,

$$\widehat{R} = \frac{\widehat{\theta_2}}{\widehat{\theta_1} + \widehat{\theta_2}}.$$

In the following we obtain the UMVUE of R. When the common parameter is known, $(\sum_{i=1}^{n} \ln(1 + x_i), \sum_{j=1}^{m} \ln(1 + y_j))$ is a jointly sufficient statistic for (θ_1, θ_2). Therefore using the results of Tong [12, 13], the UMVUE of R in Bur-XII is as follow:

$$\widetilde{R} = \begin{cases} 1 - \sum_{i=0}^{n-1} (-1)^i \frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!} \left(\frac{T_2}{T_1}\right)^i & T_2 < T_1, \\ 1 - \sum_{i=0}^{m-1} (-1)^i \frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!} \left(\frac{T_1}{T_2}\right)^i & T_1 < T_2, \end{cases}$$

where $T_1 = \sum_{j=1}^{m} \ln(1 + y_j)$ and $T_2 = \sum_{i=1}^{n} \ln(1 + x_i)$. The variance of $\widetilde{R}$ has not closed form and has not been considered precisely yet. We show that, one can approximate the variance by lower bounds of higher orders.

3 Bhattacharyya bound

Bhattacharyya (1946, 1947) obtained a generalized form of the Cramer-Rao inequality which is related to the Bhattacharyya matrix. The Bhattacharyya matrix is the covariance matrix of the random vector,

$$\frac{1}{f(X|\theta)} (f^{(1)}(X|\theta), f^{(2)}(X|\theta), \ldots, f^{(k)}(X|\theta)),$$

where $f^{(j)}(.|\theta)$ is the j^{th} derivative of the probability density function $f(.|\theta)$ w.r.t. the parameter θ. The covariance matrix of the above random vector is referred to as the $k \times k$ Bhattacharyya matrix and k is the order of it. It is clear that $(1,1)^{th}$ element of the Bhattacharyya matrix is the Fisher information.

Under some regularity conditions, the Bhattacharyya bound for any unbiased estimator of the $g(\theta)$ is defined as follows,

$$Var_\theta(T(X)) \geq \mathbf{J}_\theta \mathbf{W}^{-1} \mathbf{J}_\theta^t := B_k(\theta), \tag{1}$$

where t refers to the transpose, $\mathbf{J}_\theta = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(k)}(\theta))$, $g^{(j)}(\theta) = \partial^j g(\theta)/\partial\theta^j$ for $j = 1, 2, \ldots, k$ and $\mathbf{W}^{-1}$ is the inverse of the Bhattacharyya matrix, where

$$\mathbf{W} = (W_{rs}) = \left(Cov_\theta \left\{ \frac{f^{(r)}(X|\theta)}{f(X|\theta)}, \frac{f^{(s)}(X|\theta)}{f(X|\theta)} \right\} \right),$$

such that $E_\theta(\frac{f^{(r)}(X|\theta)}{f(X|\theta)}) = 0$ for $r, s = 1, 2, \ldots, k$.

If we substitute $k = 1$ in (1), then it indeed reduces to the Cramer-Rao inequality. By using the properties of the multiple correlation coefficient, it is easy to show that as the order of the Bhattacharyya matrix (k) increases, the Bhattacharyya bound becomes sharper.

Shanbhag (1972, 1979) characterized the natural exponential family with quadratic variance function (NEF-QVF) via diagonality of the Bhattacharyya matrix, and also showed that for this family, the Bhattacharyya matrix of any order exists and is diagonal. One can see more details and information about Bhattacharyya bound in the papers such as, Blight and Rao (1974), Tanaka and Akahira (2003), Tanaka (2003, 2006), Mohtashami Borzadaran (2006), Khorashadizadeh and Mohtashami Borzadaran (2007), Mohtashami Borzadaran et al. (2010) and Nayeban et al. (2013).

In this paper, we consider θ as unknown parameter as an example. Similar results can be obtained when α is unknown or furthermore in multiparameter case when both parameters are unknown.

By some mathematical computation, it is easy to see that the term $\frac{f^{(r)}(X|\theta)}{f(X|\theta)}$ in Burr XII is as follow,

$$\frac{f^{(r)}(X|\theta)}{f(X|\theta)} = \begin{cases} \frac{1-\ln(1+X^\alpha)^\theta}{\theta}; & r = 1 \\ \frac{(-1)^r}{\theta} [\ln(1 + X^\alpha)^\theta - r] \ln(1 + X^\alpha)^{r-1}; & r = 2, 3, \ldots \end{cases}$$

So, using above equation, we obtained the general form of the 5×5 Bhattacharyya matrix in Burr XII as follow,

$$\mathbf{W} = \begin{pmatrix} \frac{1}{\theta^2} & \frac{-2}{\theta^3} & \frac{6}{\theta^4} & \frac{-24}{\theta^5} & \frac{120}{\theta^6} \\ & \frac{8}{\theta^4} & \frac{-36}{\theta^5} & \frac{192}{\theta^6} & \frac{-1200}{\theta^7} \\ & & \frac{216}{\theta^6} & \frac{-1440}{\theta^7} & \frac{10800}{\theta^8} \\ & & & \frac{11520}{\theta^8} & \frac{-100800}{\theta^9} \\ & & & & \frac{1008000}{\theta^{10}} \end{pmatrix}. \tag{2}$$

As an example for W_{11}, the $(1,1)^{th}$ element of the matrix, we have,

$$\begin{aligned} W_{11} &= E\left(\frac{f^{(1)}(X|\theta)}{f(X|\theta)} \cdot \frac{f^{(1)}(X|\theta)}{f(X|\theta)}\right) \\ &= \int_0^\infty \frac{\alpha x^{\alpha-1}}{\theta(1+x^\alpha)^{\theta+1}} (\ln(1+x^\alpha)^\theta - 1)^2 dx \\ &= -\frac{[1+x^\alpha][1+\theta^2 \ln(1+x^\alpha)^2]}{\theta^2(1+x^\alpha)^{\theta+1}}|_0^\infty \\ &= \frac{1}{\theta^2}, \end{aligned}$$

or for W_{24} ,

$$\begin{aligned} W_{24} &= E\left(\frac{f^{(2)}(X|\theta)}{f(X|\theta)} \cdot \frac{f^{(4)}(X|\theta)}{f(X|\theta)}\right) \\ &= \int_0^\infty \frac{\alpha x^{\alpha-1} \ln(1+x^\alpha)^4 (\ln(1+x^\alpha)^\theta - 2)(\ln(1+x^\alpha)^\theta - 4)}{\theta(1+x^\alpha)^{\theta+1}} dx \\ &= \frac{192}{\theta^6}. \end{aligned}$$

4 Numerical Studies

In this section, using the Bhattacharyya lower bounds of different orders, we study the convergence of bounds and approximating the variance of any unbiased estimator of R in Burr XII distribution.

For the remainder of this paper, we will consider the case where θ_2 is known and will, without loss of generality, take $\theta_2 = 1$. In fact we want to find lower bounds for variance of any unbiased estimator of $g(\theta) = \frac{1}{1+\theta}$ in Burr XII distribution. Using the Bhattacharyya matrix of form (2), we get the first four orders of Bhattacharyya lower bounds as following:

$$\begin{aligned} B_1(\theta) &= \frac{\theta^2}{(1+\theta)^4} \\ B_2(\theta) &= \frac{\theta^2(\theta^2+2\theta+2)}{(1+\theta)^6} \\ B_3(\theta) &= \frac{\theta^2(\theta^4+4\theta^3+7\theta^2+6\theta+3)}{(1+\theta)^8} \\ B_4(\theta) &= \frac{\theta^2(\theta^6+6\theta^5+16\theta^4+24\theta^3+22\theta^2+12\theta+4)}{(1+\theta)^{10}} \end{aligned}$$

The general form of Bhattacharyya lower bounds can be stated by,

$$B_i(\theta) = \frac{\theta^2}{(1+\theta)^{2(i+1)}} \sum_{r=1}^{2i-1} c_{i,r} \theta^{r-1}; \quad for \quad i = 1, 2, \ldots,$$

where $c_{i,r}$ is a constant value depending on order of the bound (i.e i) and r. Figure 1 and Table 1 show the Bhattacharyya lower bounds for variance of any unbiased estimator of R in Burr XII distribution.

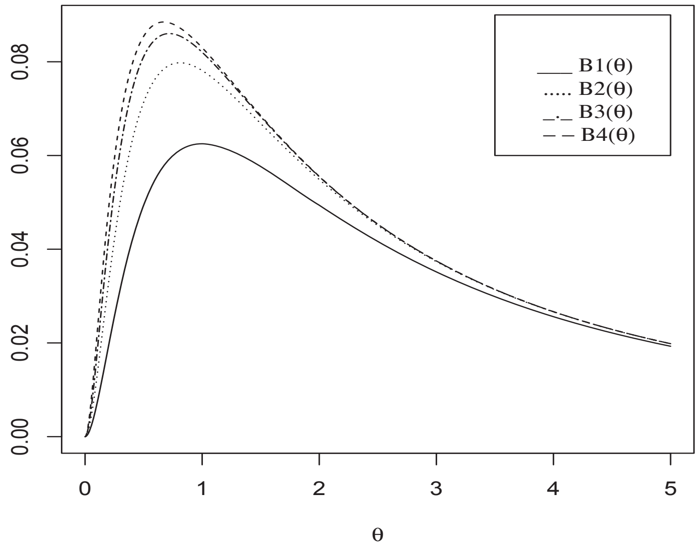

Figure 1: First four Bhattacharyya lower bounds of variance of any unbiased estimator of R in Burr XII distribution with parameters θ and $\alpha = 1$.

It is seen that, for any θ, as the order of bounds are increased the bounds get sharpers and converge to a specified point.

Also, it can be conclude that the variance of unbiased estimator of R is increasing function with respect to $\theta < 1$ and decreasing with respect to $\theta \geq 1$. Furthermore, for large values of θ, the differences of lower bounds are negligible.

Table 1: First four Bhattacharyya lower bounds of variance of any unbiased estimator of R in Burr XII distribution with parameters θ and $\alpha = 1$.

θ	$B_1(\theta)$ = Cramer-Rao	$B_2(\theta)$	$B_3(\theta)$	$B_4(\theta)$
0.1	0.006830	0.012474	0.017139	0.020995
0.5	0.049382	0.071330	0.081085	0.085420
1	0.062500	0.078125	0.082031	0.083007
3	0.035156	0.037353	0.037490	0.037499
5	0.019290	0.019825	0.019840	0.019841
10	0.006830	0.006886	0.006887	0.006887
20	0.002056	0.002061	0.002061	0.002061

Conclusion

In this paper, first we consider the MLE and UMVUE of $R = P(Y < X)$ when Y and X both follow Burr XII distribution with parameters $(\theta_1, 1)$ and $(\theta_2, 1)$, respectively. Then the general form of Bhattacharyya matrix for Burr XII distribution is obtained and in special case of $\theta_2 = 1$, first four Bhattahcaryya lower bounds for variance of any unbiased estimator of R are computed.

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