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THE GENERALISED NON-COMMUTING GRAPH OF A FINITE GROUP

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Abstract

In this paper we define the generalised non-commuting graph $\Gamma_{(H,K)}$, where H and K are two subgroups of a non-abelian group G. Take $(H \cup K) \setminus (C_H(K) \cup C_K(H))$ as the vertices of the graph and two distinct vertices x and y join, whenever x or y is in H and $[x, y] \neq 1$. We obtain diameter and girth of this graph. Also, we discuss the dominating set and planarity of $\Gamma_{(H,K)}$. Moreover, we try to find a connection between $\Gamma_{(H,K)}$ and the relative commutativity degree of two subgroups $d(H, K)$. Furthermore, we prove that if $\Gamma_{(H,G)} \cong \Gamma_{(K,G)}$, then $\Gamma_H \cong \Gamma_K$. And finally we introduce a special case when subgroup K is equal to the non-abelian group G.

Key words: commutativity degree, non-abelian group, non-commuting graph

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1. Introduction and preliminaries. The study of algebraic structures, by using the properties of graphs, is an exciting research topic in the last twenty years. This fact leads to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and investigation of algebraic properties of ring or group using the associated graph, for instance see [1,2].

A simple graph Γ_G is associated with a group G, whose vertex set is $G \setminus Z(G)$ and the edge set is all pairs (x, y), where x and y are distinct non-central elements such that $[x, y] = x^{-1}y^{-1}xy \neq 1$. The non-commuting graph of G was introduced by Erdős. He asked whether there is a finite bound for the cardinalities of cliques in Γ_G, if Γ_G has no infinite clique. This problem was posed by NEUMANN in [3] and a positive answer was given to Erdös's question. Later, many similar researches about this graph have been done by authors which some of them related to the work given by Neumann in [3]. Of course, there are some other ways to construct a graph associated with a given group or semigroup. We may refer to the works of Bertram et al. [4], Grunewald et al. [5], Moghadamfar et al. [6], and Williams [7], or recent papers on the relative non-commuting graph, Erfanian et al. [8].

In the next section, we introduce the generalised non-commuting graph $\Gamma_{(H,K)}$. We state some of the basic graph theoretical properties of $\Gamma_{(H,K)}$ which are mostly new or a generalisation of some results in [8]. For instance, determining diameter, dominating set, domination number and planarity of the graph. The third section aims to state a connection between the generalised non-commuting graph and the commutativity degree. We will present a formula for the number of edges of $\Gamma_{(H,K)}$ in terms of $d(H)$ and $d(H,K)$. Moreover, we observe that the generalised non-commuting star graph exists, although in [8] we see there is no relative non-commuting star graph. We also present some conditions under which we have generalised non-commuting complete bipartite and bipartite graph. In the last section, we explain some properties of $\Gamma_{(H,K)}$, where $K = G$.

2. The generalised non-commuting graphs. In this section, we define the generalised non-commuting graph for any non-abelian group G and subgroups H, K.

Definition 2.1. *Let H and K be subgroups of non-abelian group G. We associate a graph $\Gamma_{(H,K)}$ with the subgroups H and K as follows: Take $(H \cup K) \setminus (C_H(K) \cup C_K(H))$ as the vertices of the graph and two distinct vertices x and y are adjacent, whenever x or y is in H and $[x, y] \neq 1$. We call it the generalised non-commuting graph of subgroups H and K of G.*

It is easy to see that if $K = G$ then the generalised non-commuting graph $\Gamma_{(H,K)}$ coincides with the relative non-commuting graph $\Gamma_{(H,G)}$ (see [8]). If $H, K = G$ then the generalised non-commuting graph $\Gamma_{(H,K)}$ is the non-commuting graph Γ_G (see [9]). Thus we discuss the generalised non-commuting graph such that it does not coincide with the relative non-commuting graph or non-commuting graph unless we mentioned it in the text. Let us start with the following result about the degree of the vertices. The proof is straightforward so we omit it.

Proposition 2.2. *Suppose $\Gamma_{(H,K)}$ is the generalised non-commuting graph of the non-abelian group G and its subgroups H and K. Then*

(i) *If $x \in H \setminus (K \cup C_H(K))$, then* $\deg(x) = |H \cup K| - |C_H(x) \cup C_K(x) \cup C_H(K)|$.

(ii) $\deg(x) = |H \cup K| - |C_H(x) \cup C_K(x)|$ *for $x \in H \cap K$.*

(iii) *Finally, if $x \in K \setminus (H \cup C_K(H))$, then* $\deg(x) = |H| - |C_H(x) \cup C_{H\cap K}(H)|$.

Recall that the diameter of the graph is the greatest distance between any pair of vertices and the girth of the graph is the length of the shortest cycle.

Theorem 2.3. *For non-abelian group G and its subgroups H, K with trivial centre,* $\mathrm{diam}(\Gamma_{(H,K)}) \leq 3$. *Moreover,* $\mathrm{girth}(\Gamma_{(H,K)}) \leq 4$.

Proof. Let x be a vertex of $\Gamma_{(H,K)}$. If $H \subseteq C_K(x)$, then $x \in C_K(H)$ or $x = 1$ which is a contradiction. Therefore $H \nsubseteq C_K(x)$ and similarly $K \nsubseteq C_K(x)$. Suppose x and y are non-adjacent vertices. Now consider the following three cases:

Case 1. Suppose $x, y \in K$. There exist vertices $h_1, h_2 \in H$ such that $[x, h_1] \neq 1$ and $[y, h_2] \neq 1$. If x joins h_2 or y joins h_1, then $d(x, y) = 2$. Assume this does not happen. Therefore, the non-central element h_1h_2 is adjacent to x and y.

Case 2. Let $x \in H$ and $y \in K$. Since $x \notin C_H(K)$ and $y \notin C_K(H)$, there exist $k \in K$ and $h \in H$ such that $[x, k] \neq 1$ and $[y, h] \neq 1$. If x joins h, then $d(x, y) = 2$. Assume they are not adjacent, and k joins to h, then $d(x, y) = 3$. If x is not adjacent to h and k does not join h, then $xh \in H$ exists such that $[xh, k] \neq 1$ and $[xh, y] \neq 1$. Thus $d(x, y) = 3$.

Case 3. If $x, y \in H$, then there exist $k_1, k_2 \in K$ such that $[x, k_1] \neq 1$, $[y, k_2] \neq 1$. If x joins k_2 or y meets k_1 then $d(x, y) = 2$. Otherwise, the non-central element k_1k_2 is adjacent to x and y so $d(x, y) = 2$. Consequently, we can say that $\mathrm{diam}(\Gamma_{(H,K)}) \leq 3$.

By similar argument, if $x \in K$ and $y \in H$, then there exist $h \in H$ and $k \in K$, such that $[x, h] \neq 1$ and $[y, k] \neq 1$. If y joins h, then there is a triangle of the form $\{x, h, y\}$. Assume y does not join h and h and k are adjacent. Thus there is a cycle of the form $\{x, y, k, h\}$. Now suppose y and h are not adjacent and h does not joins k. So there exist $xy \in K$, $[xy, h] \neq 1$ and $[xy, y] \neq 1$. Hence there is a cycle of the form $\{x, y, xy, h\}$ and $\mathrm{girth}(\Gamma_{(H,K)}) \leq 4$. □

Now, let us start discussion about the dominating sets of generalised non-commuting graphs. A subset of the graph is a dominating set if every vertex which is not in the subset is adjacent to at least one member of the subset. We should note that the following three propositions are a generalisation of some results in [8].

Proposition 2.4. *Let H, K be subgroups of non-abelian group G. If $x \in H$ and $\{x\}$ is a dominating set for $\Gamma_{(H,K)}$, then $C_H(K) \cap C_K(H) = 1$, $x^2 = 1$ and $C_H(x) = \langle x \rangle$, or $\langle x, y \rangle$, where $y \in C_H(K)$ and $xy \in C_K(H)$.*

Proof. Suppose $1 \neq z \in C_H(K) \cap C_K(H)$. Thus $[z, h] = 1$ and $[z, k] = 1$ for all $h \in H$ and $k \in K$. It is clear that $zx \in H$ is a vertex and does not join x, which is a contradiction. Now assume $x^2 \neq 1$. Therefore x^{-1} is a vertex which is not adjacent to x, which is a contradiction. If $t \in C_H(x)$ and $t \notin \{1, x\}$, then the vertex t is not adjacent to x, which is a contradiction. □

Proposition 2.5. *Let H, K be subgroups of non-abelian group G and $S \subseteq V(\Gamma_{(H,K)})$. Then S is a dominating set for $\Gamma_{(H,K)}$ if and only if $C_K(S) \cup C_H(S) \subseteq C_K(H) \cup C_H(K) \cup S$.*

Proof. Suppose that S is a dominating set. If t is a vertex such that

$t \in C_K(S) \cup C_H(S)$, then $t \in C_H(S)$ or $t \in C_K(S)$ which implies $[t, S] = 1$. As S is a dominating set, $t \in S$. If t is not a vertex, then $t \in C_K(H) \cup C_H(K)$. Conversely, we suppose on the contrary that S is not a dominating set. Thus there is a vertex $t \notin S$ and it is not adjacent to any element of S. Therefore $[t, S] = 1$ and so $t \in C_K(S) \cup C_H(S) \subseteq C_K(H) \cup C_H(K) \cup S$. Hence $t \in S$, which is a contradiction. $\square$

Proposition 2.6. *Let H, K be subgroups of non-abelian group G, $X = \{h_1, \ldots, h_n\}$ and $Y = \{k_1, \ldots, k_l\}$ are generating sets for H and K, respectively, such that $h_i h_j, k_s k_t \notin C_H(K) \cup C_K(H)$ for $1 \leq i < j \leq n$ and $1 \leq s < t \leq l$. If $X \cap C_H(K) = \{h_{m+1}, \ldots, h_n\}$ and $Y \cap C_K(H) = \{k_{s+1}, \ldots, k_l\}$, then*

$$S = \{h_1, \ldots, h_m, k_1, \ldots, k_s\} \cup \{h_1 h_{m+1}, \ldots, h_1 h_n, k_1 k_{s+1}, \ldots, k_1 k_l\}$$

is a dominating set for $\Gamma_{(H,K)}$.

Proof. Let t be a vertex which does not belong to S. Consider the following two cases:

Case 1. If $t \in H$, then there exists an element $k \in K$ such that $k = k_{i_1}^{\alpha_1} \cdots k_{i_m}^{\alpha_l}$, $k_{i_j} \in Y$, α_i are integers with $[t, k] \neq 1$. Thus $[t, k_{i_j}] \neq 1$. If $1 \leq i_j \leq s$, then t joins $k_{i_j} \in S$ as required. If k_{i_j} is not a member of S, then $k_{i_1} k_{i_j} \in S$ is adjacent to t.

Case 2. If $t \in K$, then there exists an element $h \in H$ such that $h = h_{i_1}^{\beta_1} \cdots h_{i_m}^{\beta_n}$, $h_{i_j} \in X$, β_i are integers with $[t, h] \neq 1$. Thus $[t, h_{i_j}] \neq 1$ for some $1 \leq i_j \leq n$. If $h_{i_j} \in S$, then the result is clear. If h_{i_j} does not belong to S, then $h_{i_1} h_{i_j} \in S$ and joins t. $\square$

In graph theory an independent set is a set of vertices in a graph, no two of which are adjacent. It is clear that $V(\Gamma_{(H,K)}) \setminus H$ is an independent set for $\Gamma_{(H,K)}$.

3. The generalised non-commuting graphs and $d(H, K)$. For any finite group G, the commutativity degree of G, denoted by $d(G)$ is the probability that two randomly chosen elements of G commute with each other [10]. It can be defined as the following ratio:

$$d(G) = \frac{1}{|G|^2} |\{(x, y) \in G \times G : [x, y] = 1\}|.$$

Similarly, if H and K are two subgroups of G, then the generalised commutativity degree of H, K in G is defined as follows:

$$d(H, K) = \frac{1}{|H||K|} |\{(h, k) \in H \times K : [h, k] = 1\}|.$$

It is clear that if one of H or K is a central subgroup of G, then $d(H, K) = 1$ (see [11]). In this section, we present a formula for the number of edges of the generalised non-commuting graph $\Gamma_{(H,K)}$. Consequently we will give an upper bound for $|E(\Gamma_{(H,K)})|$.

Proposition 3.1. *Let H, K be subgroups of non-abelian group G. Then the number of edges for the generalised non-commuting graph is obtained by*

(1)

$$|E(\Gamma_{(H,K)})| = |H||K|(1 - d(H,K)) + \frac{|H|^2}{2}(1 - d(H)) - \frac{|H \cap K|^2}{2}(1 - d(H \cap K)).$$

Proof. It is clear that the number of edges with two ends in H is computed by $(|H|^2/2)(1 - d(H))$. Furthermore, the number of edges with one end in H and another in K is $|H||K| - |H||K|d(H,K)$. Finally, we should eliminate the edges that have been calculated twice by $(|H \cap K|^2/2)(1 - d(H \cap K))$, which implies the assertion. □

Example 3.2. In this example we compute the number of edges for some certain groups.

(i) Suppose $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1} \rangle$ is the dihedral group of order 8, $H = \langle ab \rangle$ and $K = \langle b \rangle$ are two subgroups of D_8. Obviously $V(\Gamma_{(H,K)}) = \{ab, b\}$, $d(H) = 1$, $d(H,K) = 3/4$, $|E(\Gamma_{(H,K)})| = 1$ and $\Gamma_{(H,K)}$ is K_2.

(ii) Let $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ be the symmetric group of order 6, $H = \{e, (1\ 2)\}$ and $K = \{e, (1\ 3)\}$ be subgroups of S_3. It is clear that again $\Gamma_{(H,K)} \cong K_2$.

Corollary 3.3. *Let $\Gamma_{(H,K)}$ be a generalised non-commuting graph. Then*

$$|E(\Gamma_{(H,K)})| \leq |H|(|K| + \frac{3}{16}|H| - 1) - |C_H(K)|(|K| - 1).$$

Proof. By using Theorem 2.1 in [8] and [10] the assertion is clear. □

Now, we recall the star graph as a tree on n vertices in which one vertex is of degree $n - 1$ and the others are of degree 1.

Example 3.4. Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ be the dihedral group of order $2n$, $H = \langle a \rangle$ and $K = \langle b \rangle$. Then $\Gamma_{(H,K)}$ is a star graph. If n is an even number, then $V(\Gamma_{(H,K)}) = n - 1$, $\deg(a^i) = 1$, $i \neq \frac{n}{2}$, $1 \leq i \leq n - 1$ and $\deg(b) = n - 2$. Therefore, $\Gamma_{(H,K)}$ is a star graph. Moreover, $d(H,K) = (n + 2)/2n$ and by Proposition 3.1 or by the fact $\Gamma_{(H,K)}$ is a star graph it follows that $|E(\Gamma_{(H,K)})| = n - 2$. If n is an odd number, then $V(\Gamma_{(H,K)}) = n$. Furthermore, $\deg(a^i) = 1$, $1 \leq i \leq n - 1$ and $\deg(b) = n - 1$. Hence $\Gamma_{(H,K)}$ is a star graph. We deduce $d(H,K) = (n + 1)/2n$ and so $|E(\Gamma_{(H,K)})| = n - 1$.

Since there is no edge between vertices of $\Gamma_{(H,K)}$ which belongs to K, $\Gamma_{(H,K)}$ is not complete in general. $\Gamma_{(H,K)}$ is a complete graph if and only if $|H| = |K| = 2$ and generators of H and K do not commute.

If H is an abelian group, obviously $\Gamma_{(H,K)}$ is bipartite. Clearly, if H is an abelian subgroup of G and $C_K(x) = 1$ for all $x \in H \setminus \{1\}$, then $\Gamma_{(H,K)}$ is complete bipartite.

Example 3.5. In this example we present groups such that their associated generalised non-commuting graphs are complete bipartite or regular.

(i) $H = \{e, (1\ 3\ 4), (1\ 4\ 3)\}$ and $K = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ are two subgroups of alternating group A_4. It is clear that $V(\Gamma_{H,K}) = \{(1\ 3\ 4), (1\ 4\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Moreover $d(H) = 1$, $d(H, K) = 1/2$ and so by Proposition 3.1 $|E(\Gamma_{(H,K)})| = 6$. Thus $\Gamma_{(H,K)}$ is a complete bipartite graph.

(ii) Let $D_{12} = \langle a, b : a^6 = b^2 = 1, a^b = a^{-1}\rangle$ be dihedral group of order 12, $H = \{1, a^2, a^4, ab, a^3b, a^5b\}$ and $K = \{1, b\}$ be its subgroups. By a simple computation we have $d(H) = 20/36$, $d(H, K) = 8/12$ and so Proposition 3.1 implies $|E(\Gamma_{(H,K)})| = 12$. Moreover, $\Gamma_{H,K}$ is a 4-regular graph, i.e. each vertex has 4 neighbours.

4. The special case $K = G$. Now, let us start to discuss $\Gamma_{(H,K)}$ in special cases. In this section we denoted $V_{G\setminus H} = V(\Gamma_{(H,G)}) \cap (G \setminus H)$ and $V_H = V(\Gamma_{(H,G)}) \cap H$.

Lemma 4.1. *Let G be a finite non-abelian group. Then $\Gamma_{(H,G)}$ is empty graph if and only if H is abelian subgroup of G.*

Theorem 4.2. *Let G be a non-abelian group and $\Gamma_{(H,G)}$ be a non-empty graph. Then $\Gamma_{(H,G)}$ is connected and* $\mathrm{diam}(\Gamma_{(H,G)}) = 2$. *Also,* $\mathrm{girth}(\Gamma_{(H,G)}) = 3$.

Proof. Let x, y be vertices of $\Gamma_{(H,G)}$ which are not adjacent. Consider the following three cases.

Case 1: Let $x, y \in V_{G\setminus H}$ be not adjacent. First we claim that x, y are not isolated vertices. Let x be an isolated vertex. Then $H \setminus Z(H) \subseteq C_G(x)$. Therefore, $H = \langle H \setminus Z(H)\rangle \subseteq C_G(x)$ and $x \in C_G(H)$, which is a contradiction. Hence x is not a single vertex, as claimed.

Thus we can consider a vertex $h_1 \in H \setminus (Z(H) \cup C_G(x))$ such that x joins to h_1. By similar argument there exists a vertex $h_2 \in H \setminus (C_H(h_1) \cup C_G(y))$ such that h_2 joins to y. If either $[h_1, y] \neq 1$ or $[h_2, x] \neq 1$, then $d(x, y) = 2$. Otherwise $[h_1, y] = [h_2, x] = 1$ and there exists the vertex h_1h_2 which is adjacent to x and y. So $d(x, y) = 2$.

Case 2: Suppose $x \in V_H$ and $y \in V_{G\setminus H}$ such that x and y are not-adjacent in $\Gamma_{(H,G)}$. Assume that $h' \in H \setminus (C_G(y) \cup C_H(x))$, then h' is adjacent to x and y so $d(x, y) = 2$.

Case 3: Consider $x, y \in V_H$ are not-adjacent. Then there exists a vertex $h \in V_H$ which is adjacent to x and y so $d(x, y) = 2$. Consequently, we can say that the $\mathrm{diam}(\Gamma_{(H,G)}) = 2$.

Now, since Γ_H is an induced subgraph of $\Gamma_{(H,G)}$ and has girth 3 (see [9]), so the girth of $\Gamma_{(H,G)}$ is also 3. □

Proposition 4.3. *Let G be a finite non-abelian group and $\Gamma_{(H,G)}$ be a non-empty regular graph. Then $G = H$ is nilpotent of class at most 3 and also*

$G = P \times A$, where A is an abelian group and P is a p-group (p is a prime) and furthermore Γ_P is a regular graph.

Proof. By [9], it is enough to show that $H = G$. Let $H \neq G$ and $\{g, h\}$ be an edge in $\Gamma_{(H,G)}$ such that $g \in V_{G\setminus H}$ and $h \in V_H$. Since $\Gamma_{(H,G)}$ is a regular graph, then

$$\begin{aligned}
\deg(h) &= \deg(g)\\
\Longrightarrow |G| - |C_G(h)| &= |H| - |C_G(H) \cup C_H(g)|\\
&= |H| - |C_G(H)| - |C_H(g)| + |C_G(H) \cap C_H(g)|\\
\Longrightarrow |G| + |C_G(H)| + |C_H(g)| &= |H| + |C_G(h)| + |C_G(H) \cap C_H(g)|\\
&\leq |H| + |C_G(h)| + |C_H(g)|\\
\Longrightarrow |G| + |C_G(H)| &\leq |H| + |C_G(h)| \leq \frac{|G|}{2} + \frac{|G|}{2}\\
\Longrightarrow |G| + |C_G(H)| &\leq |G|,
\end{aligned}$$

which is a contradiction. Therefore $G = H$. □

Theorem 4.4. *Let H_1, H_2 be subgroups of non-abelian group G such that $\Gamma_{(H_1,G)} \cong \Gamma_{(H_2,G)}$. Then $\Gamma_{H_1} \cong \Gamma_{H_2}$.*

Proof. First we claim that the vertices of H_1 are mapped to the set of vertices in H_2. Assume that there exists $h \in V_{H_1}$ such that $\phi(h) \in V_{G\setminus H_2}$, where ϕ is a graph isomorphism. Thus $\deg(h) = \deg(\phi(h))$ and we have

$$|G \setminus C_G(h)| = |H_2| \setminus |C_G(H_2) \cup C_{H_2}(\phi(h))| < |H_2|.$$

But $|G \setminus C_G(h)| \geq |G|/2$, so $|H_2| > |G|/2$, which is a contradiction. Therefore, we have that vertices of H_1 are mapped to the set of vertices of H_2.

On the other hand, vertices of $G \setminus H_1$ are mapped to the set of vertices of $G \setminus H_2$. Hence restriction of ϕ to $H_1 \setminus Z(H_1)$ is a graph isomorphism from Γ_{H_1} to Γ_{H_2}, as claimed. □

Theorem 4.5. *Let H be a non-abelian subgroup of G such that $\Gamma_{(H,G)} \cong \Gamma_S$ for some non-abelian finite simple group S. Then $H = G \cong S$.*

Proof. By [12], it is enough to show that $H = G$. Let $H \neq G$ and $g \in V_{G\setminus H}$. Assume that $\phi : V_{\Gamma_S} \to V_{\Gamma_{(H,G)}}$ is a graph isomorphism such that $\phi(\alpha) = g$, where $\alpha \in V_{\Gamma_S}$. Thus $\deg(\alpha) = \deg(g)$ and we have

$$|S| - |C_S(\alpha)| = |H| - |C_G(H) \cup C_H(g)|. \tag{2}$$

But $|G| - |C_G(H)| = |S| - 1$, so $|G| - |C_G(H)| + 1 = |S|$ and by (2) we have

$$\begin{aligned}
|G| - |C_G(H)| + 1 - |C_S(\alpha)| &= |H| - |C_G(H)| - |C_H(g)| + |C_G(H) \cap C_H(g)|\\
|G| - |C_S(\alpha)| + 1 &= |H| - |C_H(g)| + |C_G(H) \cap C_H(g)| \leq \frac{|G|}{2}
\end{aligned}$$

$$\Longrightarrow |C_S(\alpha)| - 1 \geq \frac{|G|}{2} \geq \frac{|S|}{2},$$

which is a contradiction. Hence $H = G$, as required. $\square$

Theorem 4.6. *Let H be a non-abelian subgroup of G such that $\Gamma_{(H,G)} \cong \Gamma_{S_n}$. Then $H = G \cong S_n$.*

Proof. By similar argument of previous Theorem we can show that $H = G$. Hence by main Theorem of [9], $H = G \cong S_n$. $\square$

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