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Perfect groups and normal subgroups related to an automorphism

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Abstract In this paper, we introduce the concept of α-normal subgroup of a finite group G, where α is an automorphism of G. We also introduce the concept of absolute normal subgroup and investigate all absolute normal subgroups of some groups. Furthermore, we define a group G α-perfect if the α-commutator subgroup of G coincides with G. We prove that for every finite abelian group G, there exists a finite abelian group H and $\alpha \in Aut(H)$ such that $D_\alpha(H) = G$.

Keywords Normal subgroup · Perfect group · α-Perfect group · α-Normal subgroup

Mathematics Subject Classification Primary 20F12; Secondary 20D45

1 Introduction

For any element $g \in G$ and automorphism $\alpha \in Aut(G)$ the element $[g, \alpha] = g^{-1}g^{\alpha}$ is said to be an autocommutator of g and α. The absolute center of group G, denoted by $L(G)$, is a characteristic subgroup where is defined as $\{g \in G : g^{\alpha} = g \ \ \forall \alpha \in Aut(G)\}$. Also, the autocommutator subgroup is $K(G) = \langle g^{-1}g^{\alpha} : g \in G, \alpha \in Aut(G)\rangle$. Assume that $G/L(G)$ is a finite group, can we conclude finiteness of the autocommutator subgroup $K(G)$? This is a question that has been answered by Hegarty

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(see [4]). In [3], he also proved that if K is a finite group, then there are only finitely many finite groups G for which $G/L(G) \cong K$. Lots of authors studied the absolute center and the autocommutator subgroup of a group that we refer to [5,6] for more information.

Recently, Barzegar et al. [1] introduced a generalization of the center and commutator subgroup of a group. The α-center of a group G is the subgroup $Z^{\alpha}(G) = \{y : [x, y]_{\alpha} = x^{-1}y^{-1}xy^{\alpha} = 1 \ \forall x \in G\}$ and an α-commutator subgroup of G is $D_{\alpha}(G) = \langle x^{-1}y^{-1}xy^{\alpha} : x, y \in G \rangle$ that $\alpha \in Aut(G)$. If α is a non-identity automorphism then there exists an element such that $1 \neq x^{-1}x^{\alpha} \in D_{\alpha}(G)$. Thus, $D_{\alpha}(G)$ is a non-trivial subgroup of G. They also introduced α-nilpotent and α-solvable groups and tried to verify which properties of the ordinary nilpotent and solvable groups are valid.

Put $Z_0^{\alpha}(G) = \{1\}$ and $Z_1^{\alpha}(G) = Z^{\alpha}(G)$. Then inductively define

$$Z^{\alpha}\left(\frac{G}{Z_{i-1}^{\alpha}(G)}\right) = \frac{Z_i^{\alpha}(G)}{Z_{i-1}^{\alpha}(G)} \quad \text{for every} \quad i \geq 1.$$

For a group G, the normal series $\{1\} = Z_0^{\alpha}(G) \trianglelefteq Z_1^{\alpha}(G) \trianglelefteq Z_2^{\alpha}(G) \trianglelefteq \ldots$ is called the upper central α-series of G. By $\overline{\alpha}$ we mean the automorphism of group G/N such that $(gN)^{\overline{\alpha}} = g^{\alpha}N$ where $\alpha \in Aut(G)$ and N is a normal subgroup of G that is invariant under α. It has been proved that a group G is α-nilpotent if and only if there is a positive integer s such that $Z_s^{\alpha}(G) = G$. Inductively, they introduced the subgroup $D_{\alpha}^i(G)$ as $D_{\alpha}^1(G) = D_{\alpha}(G)$ and $D_{\alpha}^i(G) = D_{\alpha}(D_{\alpha}^{i-1}(G))$ for all $i \geq 2$ and proved that, a group G is α-solvable if and only if there exists a positive integer r such that $D_{\alpha}^r(G) = \{1\}$. They focused on the definition of α-nilpotent groups and studied some properties of them. Furthermore, they proved a generalization of Schur's theorem and showed that the finiteness of $G/Z^{\alpha}(G)$ implies the finiteness of $D_{\alpha}(G)$. For more details see [1].

In this paper, we introduce some new concepts related to the α-commutator subgroup of a group. If G is a group and $\alpha \in Aut(G)$, we say that a subgroup H of G is α-normal in G if $x^{-1}hx^{\alpha} \in H$ for all $x \in G$, $h \in H$. We may show that an α-normal subgroup is normal but the converse is not valid in general. We give the definition of absolute normal subgroup and study all absolute normal subgroups of a finite abelian group. In addition, we study groups such that all subgroups are α-normal for a fixed automorphism α. In the last section, we give the definition of α-perfect group and present results about some classes of groups that are perfect with respect to their automorphisms. Also, we prove that for a given finite abelian group G there is a finite abelian group H and an automorphism $\alpha \in Aut(H)$ such that $D_{\alpha}(H) = G$.

2 Normal subgroups with respect to an automorphism

In this section, if G is a group and $\alpha \in Aut(G)$, we define an α-normal subgroup of G and study these subgroups in some classes of finite groups. We also classify finite groups such that all their subgroups are normal with respect to a fixed automorphism.

Definition 2.1 Let G be a finite group and $H \leq G$. We define the α-normalizer subgroup of H in G as follows

$$N_G^{\alpha}(H) = \{x \in G : [h, x]_{\alpha} \in H \; \forall \, h \in H\}.$$

The set $N_G^{\alpha}(H)$ is a subgroup, because if $x, y \in N_G^{\alpha}(H)$ then $x^{-1}hx^{\alpha}$ and $y^{-1}hy^{\alpha} \in H$ for every $h \in H$. If we put $x^{-1}hx^{\alpha} = h_1$, then $y^{-1}h_1y^{\alpha} \in H$ and we have $(xy)^{-1}h(xy)^{\alpha} = y^{-1}x^{-1}hx^{\alpha}y^{\alpha} = y^{-1}h_1y^{\alpha} \in H$. Since G is finite then $N_G^{\alpha}(H)$ is a subgroup of G.

Definition 2.2 A subgroup H is called an α-normal subgroup of G, and we denote it by $H \overset{\alpha}{\trianglelefteq} G$, whenever for every $h \in H$ and $x \in G$ we can conclude that $x^{-1}hx^{\alpha} \in H$.

Clearly $H \overset{\alpha}{\trianglelefteq} G$ if and only if $N_G^{\alpha}(H) = G$. Indeed, $N_G^{\alpha}(H)$ is the greatest subgroup of G such that H is α-normal in it. It is easy to see that if $H \overset{\alpha}{\trianglelefteq} G$ then H is invariant under α. If $D_{\alpha}(G) \leq H$ then H is an α-normal subgroup, in particular $D_{\alpha}(G)$ is α-normal. Now, we may compare normality and α-normality of a subgroup H in a group G.

Lemma 2.3 *Let $H \leq G$, then $N_G^{\alpha}(H) \leq N_G(H)$.*

Proof Assume that $x \in N_G^{\alpha}(H)$, then $x^{-1}x^{\alpha} \in H$. Now if $h \in H$ then $x^{-1}hx^{\alpha} = x^{-1}hxx^{-1}x^{\alpha} \in H$ implies that $x^{-1}hx \in H$ and $x \in N_G(H)$. □

As an immediate consequence of the previous lemma we can see that if $H \overset{\alpha}{\trianglelefteq} G$ then H is a normal subgroup. Now, we may ask the following question: If H is a normal subgroup of G is there a non-identity automorphism α of G such that H is α-normal? The answer will be positive if H is a non-central normal subgroup of G. Because it is normal with respect to a non-identity inner automorphism α_h for some element $h \in H \setminus Z(G)$. But if we consider the subgroup of order 2 of a cyclic group of even order more than 4, then it is α-normal if and only if α is the identity automorphism. We know that in a finite nilpotent group every maximal subgroup is normal. Now, the following theorem is about the relation between α-nilpotency and α-normality for some automorphism α in a group G.

Theorem 2.4 *If G is α-nilpotent and $H \lneq G$ then $H \lneq N_G^{\alpha}(H)$, in particular every maximal subgroup of G is α-normal.*

Proof Since G is α-nilpotent, then there is an integer n such that $Z_1^{\alpha}(G) \leq \cdots \leq Z_n^{\alpha}(G) = G$. Put i the greatest integer such that $Z_i^{\alpha}(G) \subseteq H \subsetneq G$. There is an element $a \in Z_{i+1}^{\alpha}(G) \setminus H$, so $aZ_i^{\alpha}(G) \in Z^{\alpha}(G/Z_i^{\alpha}(G))$ and $[g, a]_{\alpha} Z_i^{\alpha}(G) = Z_i^{\alpha}(G)$ for every $g \in G$. Now, let $h \in H$, then $[h, a]_{\alpha} \in H$ and this implies that $a \in N_G^{\alpha}(H)$ and hence $H \leq N_G^{\alpha}(H)$. □

Definition 2.5 A subgroup H of group G is called absolute normal if $H \overset{\alpha}{\trianglelefteq} G$ for all automorphisms $\alpha \in Aut(G)$.

We are interested to know all absolute normal subgroups of a group. Here, we just investigate abelian groups and we show that if a finite abelian group contains a proper absolute normal subgroup then it should be a group of even order. Obviously, an absolute normal subgroup of G is a characteristic subgroup in G. But, as we said, a subgroup of order 2 of a cyclic group of even order more than 4, is a characteristic subgroup which is not absolute normal.

Proposition 2.6 *A characteristic subgroup H of index 2 in G is always absolute normal.*

Proof Assume that $G = H \cup yH$ for some $y \in G \setminus H$. Then $y^{\alpha} \notin H$ and so $g^{-1}hg^{\alpha} \in H$ for every $h \in H$ and $\alpha \in Aut(G)$. Hence, H is absolute normal. □

Lemma 2.7 *Let G be a finite abelian group that possesses a proper absolute normal subgroup. Then the order of G should be even.*

Proof Assume that H is a proper absolute normal subgroup of group G then $x^{\alpha} - x \in H$ for all $x \in G$ and all automorphisms $\alpha \in Aut(G)$. Consider α by $x^{\alpha} = -x$ then $2x \in H$ for all $x \in G$ and so $\langle 2x \rangle \leq H$. If the order of G is an odd number then $H = G$ which is a contradiction. Hence, G is a group of even order. □

The structure of autocommutator subgroup of some classes of groups has been investigated by some authors, for instance see [2]. Now, it is possible to find all absolute normal subgroups in a finite abelian group.

Theorem 2.8 *Let G be a finite abelian group. Then there is a proper absolute normal subgroup H if and only if $G = O \times \langle b \rangle \times V$ with O of odd order, $|b| = 2^n$, $exp(V) \leq 2^{n-1}$, and in this case $H = K(G)$.*

Proof If H is a proper absolute normal subgroup in G, then the autocommutator subgroup $K(G) = \langle g^{\alpha} - g : g \in G, \alpha \in Aut(G) \rangle$ is contained in H. But we know the structure of $K(G)$, by a result of [2]. Let $G = O \times B$ with O of odd order and B 2-group of exponent 2^n, then if there are at least two independent elements in B of order 2^n, then $K(G) = G$. So, assume that $B = \langle b \rangle \times V$ with $|b| = 2^n$, $exp(V) \leq 2^{n-1}$, then $K(G) = O \times G_{2^{n-1}}$ where $G_{2^{n-1}} = \{x \in G : 2^{n-1}x = 0\}$. Hence, the proof is completed. □

We may emphasize on the cyclic case in the previous theorem by the following lemma.

Lemma 2.9 *Assume that $G \cong \mathbb{Z}_{2m}$ where $(2, m) = 1$. Then a proper subgroup H of G is absolute normal in G if and only if $H = 2G$.*

Example 2.10 Assume that H is a proper absolute normal subgroup of $D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle$. We know that automorphisms of D_{2n} are $\phi_{i,j}$ that $1 \leq i \leq n-1$, $(i, n) = 1$ and $0 \leq j \leq n-1$ and $a^{\phi_{i,j}} = a^i$, $b^{\phi_{i,j}} = ba^j$. Since $bb^{\phi_{i,j}} \in H$ for all automorphisms $\phi_{i,j}$, then if we consider automorphism $\phi_{1,1}$ then $a \in H$. It is easy to see that the subgroup $\langle a : a^n = 1 \rangle$ is an absolute normal subgroup of D_{2n}. Hence, $\langle a \rangle$ is the only proper absolute normal subgroup of this group.

Suppose that G is a group and α is a non-identity automorphism of G. If all subgroups of G are α-normal, then G is a Dedekind group. Hence, we are going to investigate Dedekind groups which all subgroups are α-normal.

Theorem 2.11 *Let G be a finite abelian group, α a non-identity automorphism of G. Every subgroup of G is α-normal in G if and only if G is a cyclic p-group (where p is a prime number), and $x^\alpha = (p^{n-1}+1)x$ for every $x \in G$, where $|G| = p^n$.*

Proof If $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$ is a non-cyclic abelian group and subgroups $H = \{(a,0) : a \in \mathbb{Z}_p\}$ and $K = \{(0,b) : b \in \mathbb{Z}_q\}$ are α-normal. Then α should be the identity automorphism because if $(x', y') - (x, y) \in H \cap K$ then $x' = x$ and $y' = y$ where $(x', y') = (x, y)^\alpha$. Therefore, let $x^\alpha = ax$ be an automorphism of cyclic group G, where $(a, |G|) = 1$. Assume that p and q are two different prime divisors of $|G|$ and H, K are subgroups of order p and q respectively. Since $H \cap K$ are α-normal then $a - 1 \in H$ and K which is a contradiction. Thus G is a p-group and $G \cong \mathbb{Z}_{p^n}$ for some $n \in \mathbb{N}$. Let $H = \langle p^{n-1} \rangle$ then there is an integer $1 \le k \le p-1$ such that $a = kp^{n-1} + 1$. Hence, all subgroups of $\mathbb{Z}_{p^n}$ are α-normal if and only if $a = kp^{n-1} + 1$. $\square$

Now, if G is the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order, then, by a similar argument in previous theorem we can see that all subgroups of G are α-normal if and only if α is the identity automorphism. But if $G = \langle x, y : x^4 = 1,\ x^2 = y^2,\ yxy^{-1} = x^{-1} \rangle$, the quaternion group of order 8, then all subgroups are α-normal that $x^\alpha = x$ and $y^\alpha = yx^2$. Hence, all subgroups of a Dedekind group are normal with respect to a fixed non-identity automorphism if and only if either G is a cyclic p-group or the quaternion group of order 8.

3 Generalized perfect groups

In this section, we give the definition of α-perfect group and we find the α-commutator subgroup of some special groups.

Definition 3.1 Let G be a group and $\alpha \in Aut(G)$, then G is called α-perfect if $D_\alpha(G) = G$.

If G is a finite group and α is a fixed-point-free automorphism of G then for different elements x and y we have $x^{-1}x^\alpha \neq y^{-1}y^\alpha$. So, $D_\alpha(G) = G$ and G is an α-perfect group. The following lemma gives a relation between concepts of perfect groups and normal subgroups with respect to an automorphism.

Lemma 3.2 *Assume that G is an abelian group. Then G is α-perfect if and only if it dose not posses a proper α-normal subgroup.*

Proof Let H be an α-normal subgroup of G, then $x^\alpha - x \in H$ for all $x \in G$. Thus, $D_\alpha(G) = \langle x^\alpha - x : x \in G \rangle$ is a subgroup of H. It is obvious that if $D_\alpha(G) \le H$ then $H \trianglelefteq^{\alpha} G$. Now, if G is α-perfect then H is not α-normal for every proper subgroup

H. Conversely, if G dose not contain a proper α-normal subgroup then $D_\alpha(G) = G$, because $D_\alpha(G) \overset{\alpha}{\trianglelefteq} G$. Hence, G is α-perfect. □

The fact that an α-nilpotent group is α-solvable can help us to prove the following proposition.

Proposition 3.3 *If a group G is an α-nilpotent group, then it is not α-perfect.*

Proof If G is α-perfect, then $D_\alpha(G) = G$ and $D_\alpha^n(G) = G$ for all $n \in \mathbb{N}$. Hence G is not α-solvable and it is a contradiction. □

As an immediate consequence of Proposition 3.3, an absolute nilpotent group G, a group that is α-nilpotent for every $\alpha \in Aut(G)$, is not α-perfect for all $\alpha \in Aut(G)$. For example, D_{2^r} is an absolute nilpotent group, so is not α-perfect for all $\alpha \in Aut(D_{2^r})$, for some positive integer r. We can show that D_{2n} is not α-perfect for all $\alpha \in Aut(D_{2n})$. Because, $D_{\phi_{i,j}}(D_{2n}) \leq \langle a : a^n = 1 \rangle$ which $\phi_{i,j}$ was defined in Example 2.10. Thus D_{2n} is not $\phi_{i,j}$-perfect.

Assume that $\alpha \in Aut(G)$ is an arbitrary automorphism, then we can see that $G' \leq D_\alpha(G)$. So, if G is a perfect group then so is α-perfect, for all automorphisms $\alpha \in Aut(G)$. Therefore, we may study non-perfect groups to know whether they are perfect with respect to a non-identity automorphism or not. Also, if $\alpha \in Inn(G)$ then $D_\alpha(G) = G'$ and α-perfectly is equal to perfectly. So we may investigate abelian groups, because they are non-perfect and they do not possess non-identity inner automorphisms.

The converse of Proposition 3.3 is not valid in general. For example S_n is not α-nilpotent for all $\alpha \in Aut(S_n)$ and also it is not α-perfect for all $\alpha \in Aut(S_n)$, because $D_\alpha(S_n) = S_n' = A_n$.

In the following lemma, we present equivalent condition for a finite cyclic group to be α-perfect for an automorphism α.

Lemma 3.4 *Assume $G \cong \mathbb{Z}_n$ and $\alpha \in Aut(G)$. Write $x^\alpha = ux$ (for every $x \in G$), where u is a positive integer $(u, n) = 1$. Then G is α-perfect if and only if $(u - 1, n) = 1$.*

Proof Clearly, $D_\alpha(G) = \{(u-1)x : x \in G\} = (u-1)\mathbb{Z}_n$, so $(u-1)\mathbb{Z}_n = \mathbb{Z}_n$ if and only if $(u-1, n) = 1$. Hence, the assertion is clear. □

By Lemma 3.4, $\mathbb{Z}_n$ is not α-perfect for all $\alpha \in Aut(\mathbb{Z}_n)$ whenever n is an even number. Now if $G \cong \mathbb{Z}_{p^r}$, p is an odd prime number and $u < p$ then $(u, p) = (u-1, p) = 1$. So, G is α-perfect which $x^\alpha = ux$ for all $x \in \mathbb{Z}_{p^r}$. If $r > 1$ then there is a non-identity automorphism $\alpha \in Aut(G)$ such that G is α-nilpotent, so is not α-perfect. Indeed, G is α-nilpotent that $x^\alpha = (p+1)x$ for all $x \in G$.

Proposition 3.5 *If $\alpha \in Aut(G)$ and $\beta \in Aut(H)$, then $D_{\alpha\times\beta}(G \times H) = D_\alpha(G) \times D_\beta(H)$.*

Proof It is straightforward. □

According to previous proposition, if G is α-perfect and H is β-perfect then $G \times H$ is $\alpha \times \beta$-perfect. Now, if G and H are two groups that $(|G|, |H|) = 1$ then $Aut(G \times H) =$

$Aut(G) \times Aut(H)$. Thus, if $G \times H$ is γ-perfect for some $\gamma \in Aut(G \times H)$ then there are automorphisms $\alpha \in Aut(G)$ and $\beta \in Aut(H)$ such that $\gamma = \alpha \times \beta$ and $D_{\alpha\times\beta}(G \times H) = D_\alpha(G) \times D_\beta(H)$. Hence, G is α-perfect and H is β-perfect.

Lemma 3.6 *If $N \trianglelefteq G$, $N^\alpha = N$ and G is α-perfect then G/N is $\bar{\alpha}$-perfect.*

Proof We can see that $D_{\bar{\alpha}}(G/N) = (D_\alpha(G)N)/N$ and since $D_\alpha(G) = G$ then the proof is completed. □

In the end of this section, we may prove a theorem that shows that every finite abelian group is a commutator subgroup of a finite abelian group with respect to an automorphism.

Theorem 3.7 *For every finite abelian group G there exists a finite abelian group H and an automorphism $\alpha \in Aut(H)$ such that $D_\alpha(H) = G$.*

Proof

Step 1. If G is a cyclic group of odd order then $D_\alpha(G) = G$ for some automorphism $\alpha \in Aut(G)$. Indeed, assume that $n = |G| = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t}$ for different primes p_i's, $1 \leq i \leq t$. Define $x^{\alpha_i} = a_i x$ where $a_i < p_i$ then $D_{\alpha_i}(\mathbb{Z}_{p_i^{r_i}}) = \mathbb{Z}_{p_i^{r_i}}$, $1 \leq i \leq t$. Now, if put $\alpha = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_t$ then $D_\alpha(\mathbb{Z}_n) = \mathbb{Z}_n$.

Step 2. If G is an abelian group of odd order then $G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$ for some positive integers n_i, $1 \leq i \leq s$. Then by step 1, there is $\alpha_i \in Aut(\mathbb{Z}_{n_i})$ such that $D_{\alpha_i}(\mathbb{Z}_{n_i}) = \mathbb{Z}_{n_i}$, $1 \leq i \leq s$. Hence $D_{\alpha_1\times\alpha_2\times\cdots\times\alpha_s}(G) = G$.

Step 3. Let $G = \mathbb{Z}_{2^n}$ and $x^\alpha = ux$ where $u = 2^n - 1$ then $D_\alpha(\mathbb{Z}_{2^n}) = (2^n - 2)\mathbb{Z}_{2^n} = \{2x : x \in \mathbb{Z}_{2^n}\}$. Now, if we define $H = \mathbb{Z}_{2^{n+1}}$ and $\alpha \in Aut(H)$ by argument $x^\alpha = (2^{n+1} - 1)x$ then $D_\alpha(H) \cong G$.

Step 4. If G is an arbitrary abelian group then it is enough to use previous steps and Proposition 3.5. □

References

1. Barzegar, R., Erfanian, A.: Nilpotency and solubility of groups relative to an automorphism. Caspian J. Math. Sci. **4**(2), 271–283 (2015)
2. Garrison, D., Kappe, L.-C., Yull, D.: Autocommutators and the autocommutator subgroup. Contemp. Math. **421**, 137–146 (2006)
3. Hegarty, P.: Autocommutator subgroups of finite groups. J. Algebra **190**(2), 556–562 (1997)
4. Hegarty, P.: The absolute centre of a group. J. Algebra **169**, 929–935 (1994)
5. Moghaddam, M.R.R., Safa, H., Mousavi, A.K.: Autocommutators and auto-bell groups. Bull. Korean Math. Soc. **51**(4), 923–931 (2014)
6. Naghshineh, M., Farrokhi, D.G.M., Moghaddam, M.R.R.: Autocommutator subgroups with cyclic outer automorphism group. Note Mat. **30**(2), 9–16 (2011)