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A modified Burzynski criterion for anisotropic pressure-dependent materials

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Abstract. In this paper the Burzynski criterion, which was introduced for isotropic pressure-dependent materials, is modified for anisotropic pressure-dependent materials in plane-stress condition. The modified criterion can be calibrated with 10 experimental data points such as tensile stress at 0°, 45° and 90°, compressive stress at 0° and 90° and *R*-values in tensile stress at 0°, 45° and 90° from rolling direction and also biaxial tensile stress and tensile *R*-value. To identify the anisotropic parameters an error function is set up through comparison of the predicted yield stresses and *R*-values with those from experiments. Then the Downhill simplex method is applied to solve 10 high-nonlinearity equations. Finally, considering Al 2008-T4 (BCC), Al 2090-T3 (FCC), AZ31 (HCP) and also Mg–0.5% Th alloy, Mg–4% Li alloy, pure textured magnesium, textured magnesium and Ti–4Al–1/4O_2, which are HCP materials with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ as case studies and comparing the results for the modified Burzynski criterion with experiments, it is shown that the Burzynski criterion is appropriate for pressure-dependent anisotropic materials with proper accuracy.

Keywords. Modified Burzynski criterion; Burzynski criterion; pressure-dependent anisotropic materials; Downhill simplex method.

1. Introduction

In new research works to accurately model the behaviour of materials, anisotropic behaviour along with pressure dependency is adopted. In the current study a criterion used for pressure-dependent isotropic materials is newly developed for anisotropic pressure-dependent materials and compared with other related criteria and experimental results. In the following the effects of anisotropy and pressure dependency on yielding of materials are reviewed briefly.

Spitzig and Richmond [1] presented experimental results on iron-based materials and aluminium and showed that there was no need for the pressure dependency of yielding to be associated with irreversible plastic dilatancy. Liu *et al* [2] presented a new pressure dependency criterion based on the criterion given by Hill for anisotropic solids and the criterion proposed by Drucker and Prager for soil that tensile and compressive strengths are widely different. Barlat *et al* [3] proposed a generalized yield description to account for binary aluminium–magnesium sheet samples that were fabricated by different processing paths to obtain different microstructures. It was subsequently shown that this yield function was suitable for description of the plastic behaviour of any aluminium alloy sheet. Yoon *et al* [4] implemented a non-quadratic yield function that simultaneously accounted for the anisotropy of uniaxial yield stresses and *R*-values in a finite-element code. Barlat *et al* [5] proposed a new plane stress yield function that described the anisotropic behaviour of sheet metals, in particular, aluminium alloy sheets. The anisotropy of the function was introduced in the formulation using two linear transformations on the Cauchy stress tensor. Stoughton and Yoon [6] proposed a non-associated flow rule based on a pressure-sensitive yield criterion with isotropic hardening that was consistent with the Spitzig and Richmond [1] results. The significance of their work was that the model distorted the shape of the yield function in tension and compression, fully accounting for the strength-differential effect (SDE). Hu and Wang [7] proposed a yield criterion with three independent invariants and deviatoric stress tensors. They considered strength-differential in tension and compression and predicted that yielding behaviours of many isotropic materials exhibit both the pressure and stress-state dependence. Hu [8] suggested a yield criterion derived with the use of invariants of the stress tensor to consider anisotropy. Anisotropic properties of the predicted yield surface were characterized by seven experimental data points obtained from three standard uniaxial-tension tests and one equibiaxial-tension test. Aretz [9] proposed a yield function that

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included eight anisotropy parameters, which could be fitted to experimental data. Furthermore, it was shown that the proposed yield function had nearly the same flexibility like that of Yld2000-2D. Burzynski [10] presented an energy-based hypothesis, called by the author as the hypothesis of variable volumetric-distortional limit energy. Lee *et al* [11] extended continuum plasticity models considering the unusual plastic behaviour of magnesium alloy sheet. Hardening law based on two-surface model was further extended to consider the general stress–strain response of metal sheets, including the Bauschinger effect, transient behaviour and the unusual asymmetry. Aretz [12] presented the issue of yield function convexity in the presence of a hydrostatic-pressure-sensitive yield stress. Hu and Wang [13] proposed a new theory for pressure-dependent materials, in which a corresponding constitutive model could be constructed and characterized experimentally via two steps, i.e., one related to the characterization of yielding behaviour of material, and the other to the plastic flow of material deformation. Huh *et al* [14] evaluated the accuracy of popular anisotropic yield functions based on the root-mean-square error (RMSE) of the yield stresses and *R*-values. The yield functions included were Hill48, Yld89, Yld91, Yld96, Yld2000-2d, BBC2000 and Yld2000-18p yield criteria. They concluded that the Yld2000-18 yield function was the best criterion to accurately describe the yield stress and *R*-value directionalities of sheet metals. Vadillo *et al* [15] formulated an implicit integration of elastic–plastic constitutive equations for the paraboloid case of Burzynski yield condition and the tangent operator consistent with the integration algorithm was developed. Taherizadeh *et al* [16] developed a generalized finite-element formulation of stress integration method for non-quadratic yield functions and potentials with mixed non-linear hardening under non-associated flow rule to analyse the anisotropic behaviour of sheet materials. Gao *et al* [17] described a plasticity model for isotropic materials, which was a function of the hydrostatic stress as well as the second and third invariants of the stress deviator. They presented its finite-element implementation, including integration of the constitutive equations using the backward Euler method and formulation of the consistent tangent moduli. Lou *et al* [18] proposed a simple approach to extend symmetric yield functions for the consideration of SDE in sheet metals. The SDE was coupled with symmetric yield functions by simply adding a weighted pressure term for anisotropic materials. This approach was applied to the symmetric Yld2000-2d and the yield function was modified to describe the anisotropic and asymmetric yielding of two aluminium alloys with small and strong SD effects. Yoon *et al* [19] proposed a general asymmetric yield function with dependence on the stress invariants for pressure-sensitive metals. The proposed function was transformed in the space of the stress, the von Mises stress and the normalized invariant to theoretically investigate the possible reason of SD effect. The yield function reasonably modeled the evolution of yield surfaces for a zirconium clock-rolled plate during in-plane and through-thickness compression. The yield function was also applied to describe the orthotropic behaviour of a face-centred cubic metal of AA2008-T4 and two hexagonal close-packed metals of high-purity a-titanium and AZ31 magnesium alloy.

In the current research, a pressure-dependent isotropic criterion, the Burzynski criterion, for isotropic materials presented by Burzynski [10], is developed to consider the anisotropy effects along with the pressure dependency in plane-stress condition called the 'modified Burzynski criterion'. The obtained results are compared with Hill 48, Yld2000-2d, Modified Yld2000-2d and experimental data points. It is shown that the presented criterion can be adopted with proper errors by comparing to experimental results for anisotropic pressure-dependent materials such as Al 2008-T4, Al 2090-T3 and AZ31 and also Mg–0.5% Th alloy, Mg–4% Li alloy, pure textured magnesium, textured magnesium and Ti–4Al–1/4O_2 with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

2. Burzynski criterion

The Burzynski criterion for three-dimensional states of stress- and pressure-dependent isotropic materials has the following form Vadillo *et al* [15]:

$$\Phi = A\sigma_e^2 + B\sigma_m^2 + C\sigma_m - 1 = 0 \tag{1}$$

where σ_e is the effective stress:

$$\sigma_e = \sqrt{\frac{3}{2}}\sqrt{s_{xx}^2 + s_{yy}^2 + s_{zz}^2 + 2s_{xy}^2 + 2s_{xz}^2 + 2s_{yz}^2} \tag{2}$$

in which s_{ij} is the deviatoric stress tensor and its components are

$$\begin{cases} s_{xx} = \dfrac{2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}}{3} \\ s_{yy} = \dfrac{2\sigma_{yy} - \sigma_{xx} - \sigma_{zz}}{3} \\ s_{zz} = \dfrac{2\sigma_{zz} - \sigma_{xx} - \sigma_{yy}}{3} \\ s_{xy} = \tau_{xy}, s_{xz} = \tau_{xz}, s_{yz} = \tau_{yz} \end{cases} \tag{3}$$

and σ_m is the hydrostatic stress:

$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}. \tag{4}$$

A–*C* can be determined by three experimental data points such as σ_Y^T (uniaxial tensile test), σ_Y^C (uniaxial compressive test) and τ^S (simple-shear test). For plane-stress problems, (i.e., $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$), Eq. (1) still takes its current form and the effective stress, considering $s_z = -s_x - s_y$, becomes

$$\sigma_e = \sqrt{3}\sqrt{s_x^2 + s_y^2 + s_x s_y + s_{xy}^2} \tag{5}$$

in which

$$\begin{cases} s_x = \dfrac{2\sigma_x - \sigma_y}{3} \\ s_y = \dfrac{2\sigma_y - \sigma_x}{3} \\ s_{xy} = \tau_{xy} \end{cases} \tag{6}$$

and the hydrostatic stress becomes

$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy}}{3}. \tag{7}$$

3. Modified Burzynski criterion for plane-stress problems

To develop the Burzynski criterion to consider anisotropic materials, a linear transformation L, for modified deviatoric stress in terms of stress components with five independent parameters can be defined as

$$\bar{s} = L\sigma \tag{8}$$

where

$$\begin{Bmatrix} \bar{s}_{xx} \\ \bar{s}_{yy} \\ \bar{s}_{xy} \end{Bmatrix} = \begin{bmatrix} L_{11} & L_{12} & 0 \\ L_{21} & L_{33} & 0 \\ 0 & 0 & L_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}. \tag{9}$$

Then L_{ij} can be defined in terms of α_i [18]:

$$\begin{Bmatrix} L_{11} \\ L_{12} \\ L_{21} \\ L_{22} \\ L_{66} \end{Bmatrix} = \frac{1}{9} \begin{bmatrix} -2 & 2 & 8 & -2 & 0 \\ 1 & -4 & -4 & 4 & 0 \\ 4 & -4 & -4 & 1 & 0 \\ -2 & 8 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix}. \tag{10}$$

It should be noted that $\alpha_i (i = 1, 2, 3, 4, 5)$ parameters can give the ability to consider anisotropy effects for deviatoric stress tensor. Therefore, the modified deviatoric stress can be expressed in terms of five independents parameters of α_i as

$$\begin{Bmatrix} \bar{s}_{xx} \\ \bar{s}_{yy} \\ \bar{s}_{xy} \end{Bmatrix} = \frac{1}{9} \begin{Bmatrix} 2(-\alpha_1 + \alpha_2 + 4\alpha_3 - \alpha_4)\sigma_{xx} + (\alpha_1 - 4\alpha_2 - 4\alpha_3 + 4\alpha_4)\sigma_{yy} \\ (4\alpha_1 - 4\alpha_2 - 4\alpha_3 + \alpha_4)\sigma_{xx} + 2(-\alpha_1 + 4\alpha_2 + \alpha_3 - \alpha_4)\sigma_{yy} \\ 9\alpha_5\tau_{xy} \end{Bmatrix}. \tag{11}$$

This idea has arisen from one of the linear transformation and the Yld2000-2d criterion as in Barlat *et al* [5]. In this case the modified effective stress can take the following form:

$$\bar{\sigma}_e = \sqrt{3}\sqrt{\bar{s}_{xx}^2 + \bar{s}_{yy}^2 + \bar{s}_{xx}\bar{s}_{yy} + \bar{s}_{xy}^2} \tag{12}$$

and to modify the hydrostatic stress the following form can be employed [18]:

$$\bar{\sigma}_m = \frac{\alpha_6\sigma_{xx} + \alpha_7\sigma_{yy}}{3}. \tag{13}$$

α_6 and α_7 give the ability to consider the anisotropy effects for hydrostatic stress. Finally the Burzynski criterion in Eq. (1) can be modified to consider the anisotropic material effects as

$$\bar{\Phi} = \alpha_8\bar{\sigma}_e^2 + \alpha_9\bar{\sigma}_m^2 + \alpha_{10}\bar{\sigma}_m - 1 = 0. \tag{14}$$

In Eq. (15), the parameter α_8 weights the modified effective deviatoric stress $(\bar{\sigma}_e)$, and the parameters α_9 and α_{10} weight the modified hydrostatic pressure $(\bar{\sigma}_m)$ in the modified Burzynski criterion. Inserting Eq. (12) into Eq. (13) and the result in Eq. (15) and also using Eq. (14), the following form for the modified Burzynski criterion is obtained in terms of stress components:

$$\begin{aligned} &\bar{\Phi}(\sigma_{xx}, \sigma_{yy}, \tau_{xy}) \\ &\left(\frac{1}{27} \left[\begin{aligned} &\left(2(-\alpha_1 + \alpha_2 + 4\alpha_3 - \alpha_4)\sigma_{xx} + (\alpha_1 - 4\alpha_2 - 4\alpha_3 + 4\alpha_4)\sigma_{yy}\right)^2 \\ &+\left((4\alpha_1 - 4\alpha_2 - 4\alpha_3 + \alpha_4)\sigma_{xx} + 2(-\alpha_1 + 4\alpha_2 + \alpha_3 - \alpha_4)\sigma_{yy}\right)^2 \\ &+\left(2(-\alpha_1 + \alpha_2 + 4\alpha_3 - \alpha_4)\sigma_{xx} + (\alpha_1 - 4\alpha_2 - 4\alpha_3 + 4\alpha_4)\sigma_{yy}\right) \\ &\times\left((4\alpha_1 - 4\alpha_2 - 4\alpha_3 + \alpha_4)\sigma_{xx} + 2(-\alpha_1 + 4\alpha_2 + \alpha_3 - \alpha_4)\sigma_{yy}\right) \\ &+\left(9\alpha_5\tau_{xy}\right)^2 \end{aligned} \right] \right) \\ &+ \alpha_9\left(\frac{\alpha_6\sigma_{xx} + \alpha_7\sigma_{yy}}{3}\right)^2 + \alpha_{10}\left(\frac{\alpha_6\sigma_{xx} + \alpha_7\sigma_{yy}}{3}\right) - 1 = 0. \end{aligned} \tag{15}$$

These 10 material parameters can be determined by 10 experimental data, which is explained in section 3. Furthermore, the first differentiation of the proposed criterion is useful for calibration. Therefore, from Eq. (16) this differentiation can be obtained as

$$\begin{cases} \dfrac{\partial\bar{\Phi}}{\partial\sigma_{xx}} = \alpha_8 \left[\begin{aligned} &2(-\alpha_1 + \alpha_2 + 4\alpha_3 - \alpha_4)\left((4\alpha_3 - \alpha_4)\sigma_{xx} + 2(-\alpha_3 + \alpha_4)\sigma_{yy}\right) \\ &+(4\alpha_1 - 4\alpha_2 - 4\alpha_3 + \alpha_4)\left(2(\alpha_1 - \alpha_2)\sigma_{xx} + (-\alpha_1 + 4\alpha_2)\sigma_{yy}\right) \end{aligned} \right] \\ +\dfrac{1}{3}\alpha_6\left[2\alpha_9\left(\dfrac{\alpha_6\sigma_{xx} + \alpha_7\sigma_{yy}}{3}\right) + \alpha_{10}\right] \\ \dfrac{\partial\bar{\Phi}}{\partial\sigma_{yy}} = \alpha_8 \left[\begin{aligned} &(\alpha_1 - 4\alpha_2 - 4\alpha_3 + 4\alpha_4)\left((4\alpha_3 - \alpha_4)\sigma_{xx} + 2(-\alpha_3 + \alpha_4)\sigma_{yy}\right) \\ &+2(-\alpha_1 + 4\alpha_2 + \alpha_3 - \alpha_4)\left(2(\alpha_1 - \alpha_2)\sigma_{xx} + (-\alpha_1 + 4\alpha_2)\sigma_{yy}\right) \end{aligned} \right] \\ \dfrac{1}{3}\alpha_7\left[2\alpha_9\left(\dfrac{\alpha_6\sigma_{xx} + \alpha_7\sigma_{yy}}{3}\right) + \alpha_{10}\right] \\ \dfrac{\partial\bar{\Phi}}{\partial\tau_{xy}} = 54\alpha_8\alpha_5^2\tau_{xy} \end{cases}. \tag{16}$$

4. Calibration of the modified Burzynski criterion

To calibrate the proposed criterion, 10 experimental data points are required [18], which consist of six data in stresses such as uniaxial tensile tests at 0°, 45° and 90° from rolling direction, the tensile biaxial stress test and uniaxial compressive test at 0° and 90° from rolling direction, four experimental tests on tensile plastic strain increment ratio R-values $\left(R = \frac{d\varepsilon_y^p}{d\varepsilon_z^p}\right)$ at 0°, 45° and 90° and also tensile biaxial R-value $\left(R = \frac{d\varepsilon_y^p}{d\varepsilon_x^p}\right)$. The effect of pressure dependency can be automatically satisfied because of existence of modified hydrostatic stress inherently. Hence, the proposed criterion can have the effect of anisotropy effect and pressure dependency simultaneously in a different way from that of Modified Yld2000-2d in Lou *et al* [18].

4.1 *Yield stress tests*

Because of pressure dependency of the proposed criterion, the behaviour of material is different in tension and compression and therefore the uniaxial experimental result in both tension and compression cases is required. For tensile yield stress tests in θ from the rolling direction it is considered that

$$\begin{cases} \sigma_{xx} = \sigma_\theta^T \cos^2\theta \\ \sigma_{yy} = \sigma_\theta^T \sin^2\theta \\ \tau_{xy} = \sigma_\theta^T \sin\theta\cos\theta \end{cases} \tag{17}$$

where θ is the angle from the rolling direction and σ_θ^T is the tensile yield stress in θ direction. Substituting these values in Eq. (16), a second-order equation in terms of σ_θ^T can be obtained as

$$A_\theta\left(\sigma_\theta^T\right)^2 + B_\theta\left(\sigma_\theta^T\right) - 1 = 0. \tag{18}$$

Considering the positive root of this equation, σ_θ^T can be found as

$$\sigma_\theta^T = \frac{-B_\theta + \sqrt{B_\theta^2 + 4A_\theta}}{2A_\theta} \tag{19}$$

in which

$$\begin{cases} A_\theta = \alpha_8\left[\frac{1}{27}\left(\begin{array}{r} \left[2(-\alpha_1+2\alpha_2+4\alpha_3-\alpha_4)\cos^2\theta + (\alpha_1-4\alpha_2-4\alpha_3+4\alpha_4)\sin^2\theta\right]^2 \\ +\left[(4\alpha_1-4\alpha_2-4\alpha_3+\alpha_4)\cos^2\theta + 2(-\alpha_1+4\alpha_2+\alpha_3-\alpha_4)\sin^2\theta\right]^2 \\ +\left[2(-\alpha_1+\alpha_2+4\alpha_3-\alpha_4)\cos^2\theta + (\alpha_1-4\alpha_2-4\alpha_3+4\alpha_4)\sin^2\theta\right] \\ \times\left[(4\alpha_1-4\alpha_2-4\alpha_3+\alpha_4)\cos^2\theta + 2(-\alpha_1+4\alpha_2+\alpha_3-\alpha_4)\sin^2\theta\right] \\ +\left[9\alpha_5\sin\theta\cos\theta\right]^2 \end{array}\right)\right] + \alpha_9\left(\frac{\alpha_6\cos^2\theta + \alpha_7\sin^2\theta}{3}\right)^2 \\ B_\theta = \alpha_{10}\left(\frac{\alpha_6\cos^2\theta + \alpha_7\sin^2\theta}{3}\right) \end{cases} . \tag{20}$$

For the compressive yield stress tests it is considered that

$$\begin{cases} \sigma_{xx} = -\sigma_\theta^C \cos^2\theta \\ \sigma_{yy} = -\sigma_\theta^C \sin^2\theta \\ \tau_{xy} = -\sigma_\theta^C \sin\theta\cos\theta \end{cases} . \tag{21}$$

With the same process, the following second-order equation for σ_θ^C is obtained:

$$A_\theta\left(\sigma_\theta^C\right)^2 - B_\theta\left(\sigma_\theta^C\right) - 1 = 0 \tag{22}$$

in which

$$\sigma_\theta^C = \frac{B_\theta + \sqrt{B_\theta^2 + 4A_\theta}}{2A_\theta}. \tag{23}$$

For balance biaxial yield stress test it is considered that

$$\begin{cases} \sigma_{xx} = \sigma_b^T \\ \sigma_{yy} = \sigma_b^T \\ \tau_{xy} = 0 \end{cases} . \tag{24}$$

By substituting these values in Eq. (16), a second-order equation in terms of σ_b^T can be obtained as

$$A_b\left(\sigma_b^T\right)^2 + B_b\left(\sigma_b^T\right) - 1 = 0 \tag{25}$$

and its positive root is

$$\sigma_b^T = \frac{-B_b + \sqrt{B_b^2 + 4A_b}}{2A_b} \tag{26}$$

where

$$\begin{cases} A_b = \alpha_8\left[\frac{1}{27}\left(\begin{array}{r} [-\alpha_1-2\alpha_2+4\alpha_3+2\alpha_4]^2 + [2\alpha_1+4\alpha_2-2\alpha_3-\alpha_4]^2 \\ +[-\alpha_1-2\alpha_2+4\alpha_3+2\alpha_4][2\alpha_1+4\alpha_2-2\alpha_3-\alpha_4] \end{array}\right)\right] + \alpha_9\left(\frac{\alpha_6+\alpha_7}{3}\right)^2 \\ B_b = \alpha_{10}\left(\frac{\alpha_6+\alpha_7}{3}\right) \end{cases} . \tag{27}$$

4.2 *R-value tests along with associated flow rule*

Although the presented criterion is pressure dependent, the associated flow rule for plane stress problems can be used as

$$\begin{cases} d\varepsilon_{xx}^{p} = d\lambda \dfrac{\partial \bar{\Phi}}{\partial \sigma_{xx}} \\ d\varepsilon_{yy}^{p} = d\lambda \dfrac{\partial \bar{\Phi}}{\partial \sigma_{yy}} \\ d\varepsilon_{xy}^{p} = d\lambda \dfrac{\partial \bar{\Phi}}{\partial \tau_{xy}} \end{cases} \tag{28}$$

The thickness strain is calculated by the assumption of incompressibility as

$$d\varepsilon_{zz}^{p} = -d\varepsilon_{xx}^{p} - d\varepsilon_{yy}^{p} \tag{29}$$

The R-value in θ-direction from rolling direction under tension is denoted as R_{θ}^{T}, which is calculated as follows:

$$\begin{aligned} R_{\theta}^{T} &= \frac{d\varepsilon_{yy}^{p}}{d\varepsilon_{zz}^{p}} = -\frac{d\varepsilon_{yy}^{p}}{d\varepsilon_{xx}^{p} + d\varepsilon_{yy}^{p}} \\ &= -\frac{\frac{\partial \bar{\Phi}}{\partial \sigma_{xx}} \sin^2\theta + \frac{\partial \bar{\Phi}}{\partial \sigma_{yy}} \cos^2\theta - \frac{\partial \bar{\Phi}}{\partial \sigma_{xy}} \sin\theta\cos\theta}{\frac{\partial \bar{\Phi}}{\partial \sigma_{xx}} + \frac{\partial \bar{\Phi}}{\partial \sigma_{yy}}}. \end{aligned} \tag{30}$$

The R-value in the balanced biaxial tension is defined as the ratio of the strain increments in transverse direction to that in rolling direction in the balanced biaxial tension using Eq. (24), which is obtained as

$$R_{b}^{T} = \frac{d\varepsilon_{yy}^{p}}{d\varepsilon_{xx}^{p}} = \frac{\frac{\partial \bar{\Phi}}{\partial \sigma_{yy}}}{\frac{\partial \bar{\Phi}}{\partial \sigma_{xx}}} \tag{31}$$

5. Parameter evaluation and RMSE of the yield stresses and *R*-values

Ten material constants denoted as $\alpha_i (i = 1 - 10)$ in the modified Burzynski yield function of Eq. (16) are calibrated using experimental data points for numerical analysis. These material constants are calculated by 10 experimental results including $\sigma_0^T, \sigma_{45}^T, \sigma_{90}^T, \sigma_b^T, \sigma_0^C, \sigma_{90}^C, R_0^T, R_{45}^T, R_{90}^T, R_b^T$ and then they are utilized to set up an error function as

$$\begin{aligned} E = {} & \left[\frac{(\sigma_0^T)_{\exp.}}{(\sigma_0^T)_{pred.}} - 1\right]^2 + \left[\frac{(\sigma_{45}^T)_{\exp.}}{(\sigma_{45}^T)_{pred.}} - 1\right]^2 + \left[\frac{(\sigma_{90}^T)_{\exp.}}{(\sigma_{90}^T)_{pred.}} - 1\right]^2 \\ & + \left[\frac{(\sigma_b^T)_{\exp.}}{(\sigma_b^T)_{pred.}} - 1\right]^2 + \left[\frac{(\sigma_0^C)_{\exp.}}{(\sigma_0^C)_{pred.}} - 1\right]^2 + \left[\frac{(\sigma_{90}^C)_{\exp.}}{(\sigma_{90}^C)_{pred.}} - 1\right]^2 \\ & + \left[\frac{(R_0^T)_{pred.}}{(R_0^T)_{exp.}} - 1\right]^2 + \left[\frac{(R_{45}^T)_{pred.}}{(R_{45}^T)_{exp.}} - 1\right]^2 \\ & + \left[\frac{(R_{90}^T)_{pred.}}{(R_{90}^T)_{exp.}} - 1\right]^2 + \left[\frac{(R_b^T)_{pred.}}{(R_b^T)_{exp.}} - 1\right]^2 \end{aligned} \tag{32}$$

The error function is minimized by the Downhill simplex method to identify the material parameters. The RMSEs of tensile and compressive yield stresses and also tensile R-values are computed as

The yield stresses and R-values are computed from experiments for Al 2008-T4 and Al 2090-T3 [18].

$$E_{\sigma}^{T} = \frac{1}{7}\left(\begin{aligned} & \left[\frac{(\sigma_0^T)_{exp.} - (\sigma_0^T)_{pred.}}{(\sigma_0^T)_{exp.}}\right]^2 + \left[\frac{(\sigma_{15}^T)_{exp.} - (\sigma_{15}^T)_{pred.}}{(\sigma_{15}^T)_{exp.}}\right]^2 + \left[\frac{(\sigma_{30}^T)_{exp.} - (\sigma_{30}^T)_{pred.}}{(\sigma_{30}^T)_{exp.}}\right]^2 \\ & + \left[\frac{(\sigma_{45}^T)_{exp.} - (\sigma_{45}^T)_{pred.}}{(\sigma_{45}^T)_{exp.}}\right]^2 + \left[\frac{(\sigma_{60}^T)_{exp.} - (\sigma_{60}^T)_{pred.}}{(\sigma_{60}^T)_{exp.}}\right]^2 + \left[\frac{(\sigma_{75}^T)_{exp.} - (\sigma_{75}^T)_{pred.}}{(\sigma_{75}^T)_{exp.}}\right]^2 \\ & + \left[\frac{(\sigma_{90}^T)_{exp.} - (\sigma_{90}^T)_{pred.}}{(\sigma_{90}^T)_{exp.}}\right]^2 \end{aligned}\right)^{\frac{1}{2}} \times 100 \tag{33}$$

$$E_{\sigma}^{C} = \frac{1}{7}\left(\begin{aligned} & \left[\frac{(\sigma_0^C)_{exp.} - (\sigma_0^C)_{pred.}}{(\sigma_0^C)_{exp.}}\right]^2 + \left[\frac{(\sigma_{15}^C)_{exp.} - (\sigma_{15}^C)_{pred.}}{(\sigma_{15}^C)_{exp.}}\right]^2 + \left[\frac{(\sigma_{30}^C)_{exp.} - (\sigma_{30}^C)_{pred.}}{(\sigma_{30}^C)_{exp.}}\right]^2 \\ & + \left[\frac{(\sigma_{45}^C)_{exp.} - (\sigma_{45}^C)_{pred.}}{(\sigma_{45}^C)_{exp.}}\right]^2 + \left[\frac{(\sigma_{60}^C)_{exp.} - (\sigma_{60}^C)_{pred.}}{(\sigma_{60}^C)_{exp.}}\right]^2 + \left[\frac{(\sigma_{75}^C)_{exp.} - (\sigma_{75}^C)_{pred.}}{(\sigma_{75}^C)_{exp.}}\right]^2 \\ & + \left[\frac{(\sigma_{90}^C)_{exp.} - (\sigma_{90}^C)_{pred.}}{(\sigma_{90}^C)_{exp.}}\right]^2 \end{aligned}\right)^{\frac{1}{2}} \times 100 \tag{34}$$

$$E_R^T=\frac{1}{7}\left(\begin{array}{l}\left[\frac{\left(R_0^T\right)_{exp.}-\left(R_0^T\right)_{pred.}}{\left(R_0^T\right)_{exp.}}\right]^2+\left[\frac{\left(R_{15}^T\right)_{exp.}-\left(R_{15}^T\right)_{pred.}}{\left(R_{15}^T\right)_{exp.}}\right]^2+\left[\frac{\left(R_{30}^T\right)_{exp.}-\left(R_{30}^T\right)_{pred.}}{\left(R_{30}^T\right)_{exp.}}\right]^2\\+\left[\frac{\left(R_{45}^T\right)_{exp.}-\left(R_{45}^T\right)_{pred.}}{\left(R_{45}^T\right)_{exp.}}\right]^2+\left[\frac{\left(R_{60}^T\right)_{exp.}-\left(R_{60}^T\right)_{pred.}}{\left(R_{60}^T\right)_{exp.}}\right]^2+\left[\frac{\left(R_{75}^T\right)_{exp.}-\left(R_{75}^T\right)_{pred.}}{\left(R_{75}^T\right)_{exp.}}\right]^2\\+\left[\frac{\left(R_{90}^T\right)_{exp.}-\left(R_{90}^T\right)_{pred.}}{\left(R_{90}^T\right)_{exp.}}\right]^2\end{array}\right)^{\frac{1}{2}}\times 100 \tag{35}$$

6. Results and discussion

The yield surfaces are constructed by Hill48, Yld2000-2d, Modified Yld2000-2d and presented modified Burzynski yield functions and compared with experimental results for Al2008-T4 (a BCC material) and Al 2090-T3 (a FCC material) in figures 1, 2, 3, 4, 5 and 6. The mechanical properties of these materials in different directions are available in table 1. Table 2 shows the α_i ($i = 1$–10) parameters computed by minimizing the error function of Eq. (32) using the Downhill simplex method. In Hill48 and Yld2000-2d, yield surfaces are symmetric with respect to the stress-free condition (pressure-independent yield

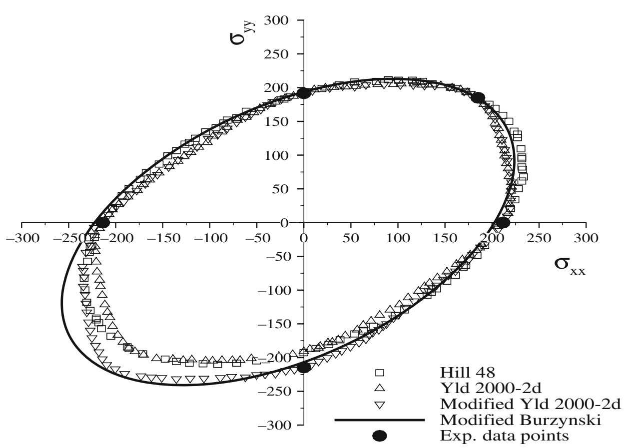

Figure 1. Comparison of the yield surfaces for Al2008-T4.

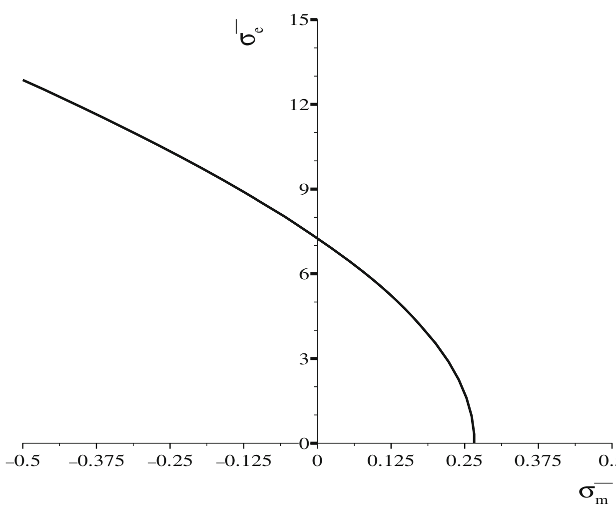

Figure 2. Modified Burzynski yield condition for Al 2008-T4 in $\bar{\sigma}_e - \bar{\sigma}_m$ plane, modified Burzynski–Torre paraboloid.

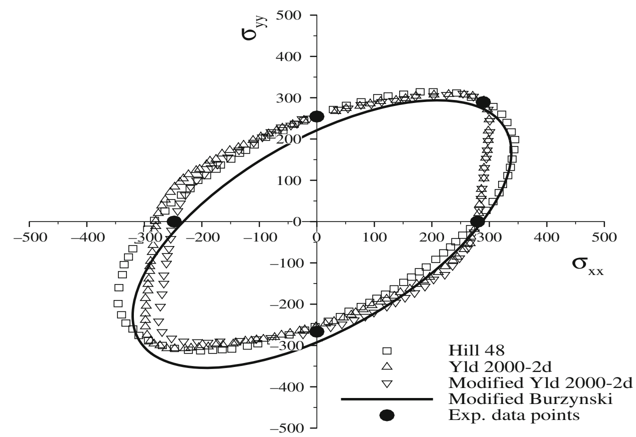

Figure 3. Comparison of the yield surfaces for Al2090-T3.

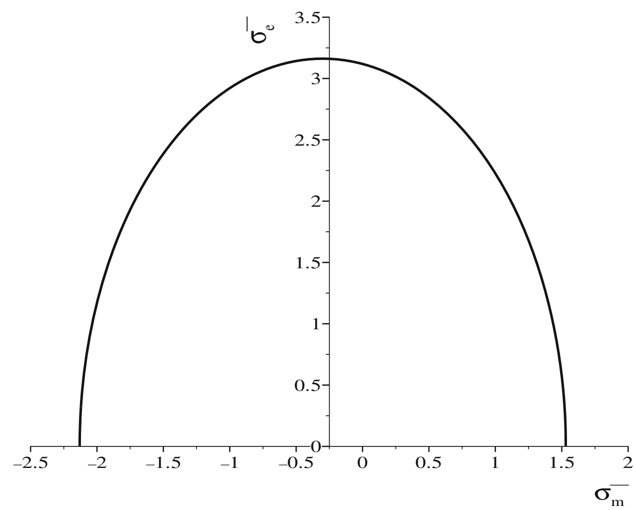

Figure 4. The Burzynski yield condition for Al 2090-T3 in $\bar{\sigma}_e - \bar{\sigma}_m$ plane and the modified Burzynski ellipsoid.

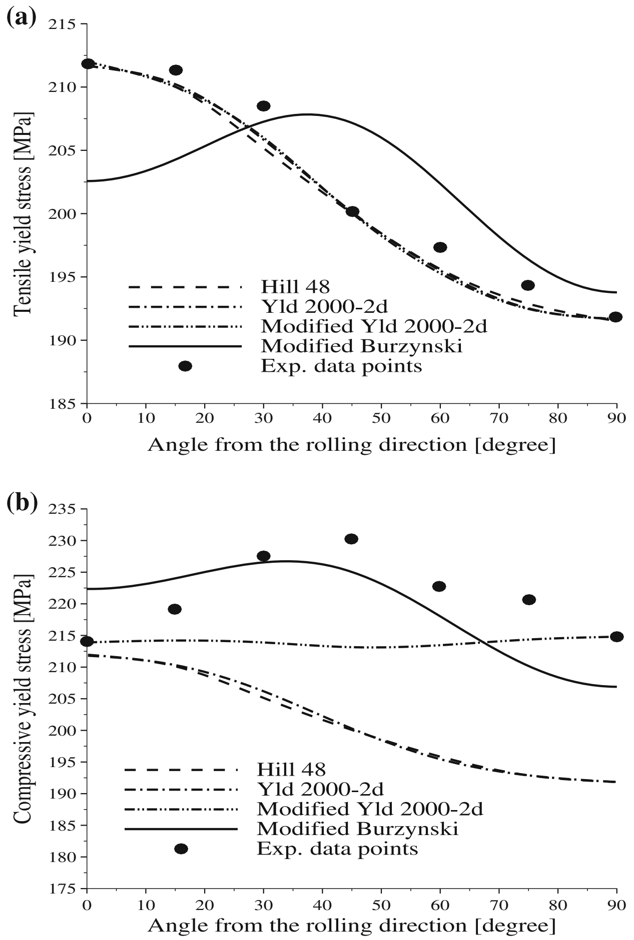

Figure 5. Comparison of the yield stress directionality for Al 2008-T4 for (**a**) uniaxial tensile yield stress and (**b**) uniaxial compressive yield stress.

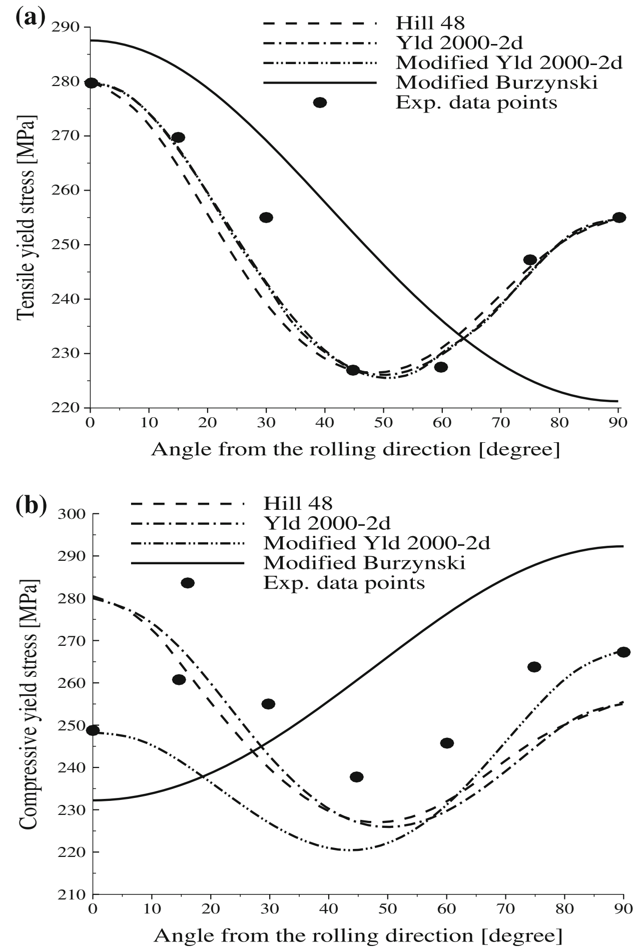

Figure 6. Comparison of the yield stress directionality for Al 2090-T3 for (**a**) uniaxial tensile yield stress and (**b**) uniaxial compressive yield stress.

criteria) but Modified Yld2000-2d and the presented modified Burzynski are asymmetric ones.

The Modified Yld2000-2d has been presented for anisotropic pressure-dependent materials newly by Lou *et al* [18] and introduced as a powerful criterion to predict experimental results rather than pressure-dependent criteria such as Hill 48 and Yld-2000-2d. Figure 1 shows that the presented criterion predicts the experimental results satisfactorily and it is also close to Modified Yld2000-2d especially in the third quadrant. It may be deduced that the presented criterion is quite successful in predicting the experimental results for Al 2008-T4 as a pressure-dependent anisotropic criterion.

Figure 2 shows the modified deviatoric stress versus modified hydrostatic pressure using parameters α_8, α_9 and α_{10} of table 2. For the pressure-independent materials, the modified deviatoric and modified hydrostatic stresses are independent of each other; however, one can observe their dependency for pressure-dependent ones. It is seen that for Al 2008-T4 a modified Burzynski–Torre paraboloid by Burzynski [10] and Vadillo *et al* [15] is obtained and it may be deduced that the presented modified Burzynski is a proper criterion in $\sigma_{xx} - \sigma_{yy}$ plane compared to the experimental results for Al 2008-T4 (a BCC material).

Figure 3 displays the situation of modified Burzynski criterion compared to other criteria for Al 2090-T3. The modified Burzynski can be a reasonable criterion in $\sigma_{xx} - \sigma_{yy}$ plane compared to the experimental results for Al 2090-T3 (a FCC material); however, it may be concluded that the presented modified Burzynski criterion can estimate the yield surface in $\sigma_{xx} - \sigma_{yy}$ plane for BCC materials more accurately rather than FCC materials when the modified deviatoric stress versus modified hydrostatic pressure has an ellipsoid shape (figure 4).

Figure 5 shows the variations of tensile and compressive yield stresses of four criteria versus the angle from the rolling direction compared to experimental results. It is seen that in predicting the tensile yield stress the Hill 48, Yld2000-2d and Modified Yld2000-2d are nearly the same and more accurate than the presented modified Burzynski criterion. However, the presented criterion is quite successful in predicting the compressive yield stress, which may be attributed to its dependency to hydrostatic pressure in its

Table 1. Experimental data points of Al 2008-T4 and Al 2090-T3 [18].

Material	σ_0^T	σ_{45}^T	σ_{90}^T	σ_b^T	σ_0^C	σ_{90}^C	R_0^T	R_{45}^T	R_{90}^T	R_b^T
Al 2008-T4	211.67	200.03	191.56	185.00	213.79	214.64	0.87	0.500	0.530	1.000
Al 2090-T3	279.62	226.77	254.45	289.40	248.02	266.48	0.210	1.580	0.690	0.670

Table 2. Material parameters in modified Burzynski criterion of Al 2008-T4 and Al 2090-T3.

Material	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
Al 2008-T4	−0.0137	0.0269	0.0290	0.0768	0.0326	−0.0047	−0.0035	0.0190	−0.6852	−0.2803
Al 2090-T3	0.0151	0.0006	−0.0027	−0.0141	0.0125	−0.0099	0.0132	0.1051	0.5711	0.2512

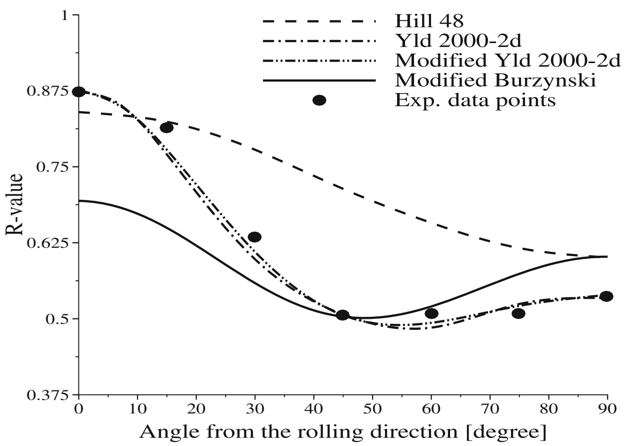

Figure 7. Comparison of the R-value directionality for Al 2008-T4.

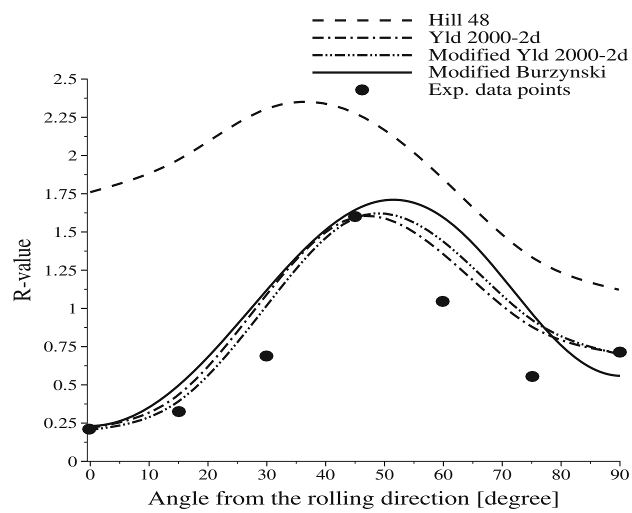

Figure 8. Comparison of the R-value directionality for Al 2090-T4.

mathematical presentation as in Eq. (15). But, the Hill 48 and Yld2000-2d are close to each other and compute the compressive yield stress far from the experimental results. Moreover, it is noted that the Modified Yld2000-2d predicts the compressive yield stress nearly independent of angle of rolling direction which cannot be true for a direction-dependent material criterion (figure 5). Figure 6 shows that the modified Burzynski has not enough accuracy to predict the yield and compressive yield stress for Al 2090-T3; however, as mentioned earlier the modified Burzynski can be used for BCC materials more accurately rather than FCC materials.

In figures 7 and 8, R-values versus angle from the rolling direction for Al 2008-T4 and Al 2090-T3 are shown. Compared to other criteria, the presented modified Burzynski is nearly well fitted with experimental results, especially for Al 2090-T3, in predicting R-values.

In table 3a, b the differences between the current results and experimental data are computed for two materials from Eqs. (33)–(35). As seen, the discrepancy of predicted compressive yield stress for Al 2008-T4 is about 1.16%, which is the lowest compared with other criteria. For the tensile yield stress; however, the discrepancy is about 1.03% which is higher than from other criteria but it is still small enough compared to experimental results. In addition, the discrepancy of obtained tensile R-value is about 5.19%, which is less than that of Hill criterion.

Table 3. Discrepancy of different criteria compared with experimental results for (a) Al 2008-T4, (b) Al 2090-T3 and (c) AZ31 (in percentage).

Criterion	E_σ^T	E_σ^C	E_R^T
(a)			
Hill 48	0.2929	3.7944	8.7271
Yld2000-2d	0.2732	3.7676	1.3801
Modified Yld2000-2d	0.2754	1.5768	1.0846
Modified Burzynski	1.0283	1.1636	5.197
(b)			
Hill 48	0.9553	2.5597	133.8507
Yld2000-2d	0.7666	2.5875	13.1056
Modified Yld2000-2d	0.7216	2.3752	12.7688
Modified Burzynski	3.037	3.33	18
(c)			
Modified Yld 2000-2d	0.1980	0.3896	
Modified Burzynski	0.3549	0.4375	

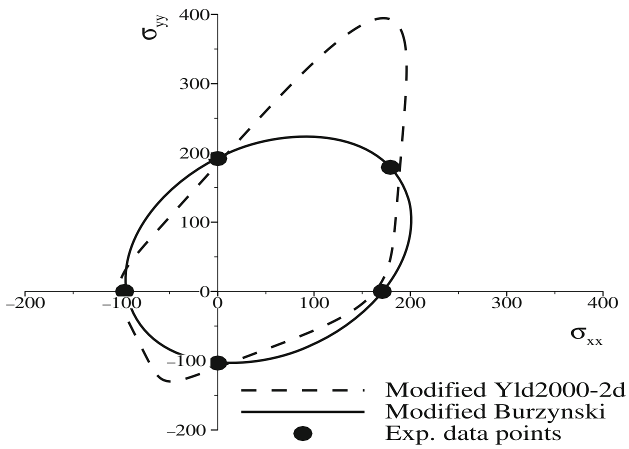

Figure 9. Comparison of the yield surfaces for AZ31.

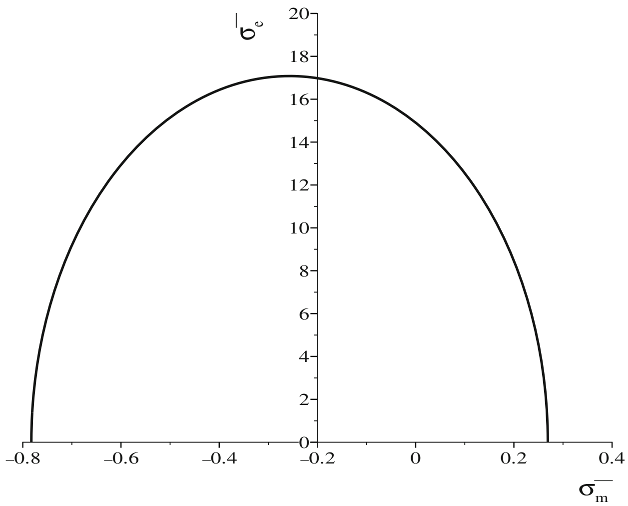

Figure 10. The Burzynski yield condition for AZ31 in $\bar{\sigma}_e - \bar{\sigma}_m$ plane and the modified Burzynski ellipsoid.

It is also seen that the differences of obtained results for Al 2090-T3 with experimental data are about 3.33% and 3.047% for compressive and tensile yield stresses, respectively. However, it is about 18% for *R*-values, which is still less than that of Hill 48 but more than those of Yld2000-2d and Modified Yld2000-2d (table 3a, b).

To check the proposed criterion for strong SDE, AZ31, which is a HCP material, at 3% plastic strain is selected for case study. Its experimental data points are utilized from Yoon *et al* [19]. Figure 9 shows the yield function for AZ31 in $\sigma_{xx} - \sigma_{yy}$ and as seen the modified Burzynski predicts experimental results with good accuracy and more realistically than Modified Yld2000-2d. In figure 10, $\bar{\sigma}_e - \bar{\sigma}_m$ is plotted for AZ31 and in figure 11 tensile and compressive yield stresses are compared with experimental results and as seen the Modified Burzynski is successful in predicting the mechanical behaviour of a HCP material like FCC and BCC materials.

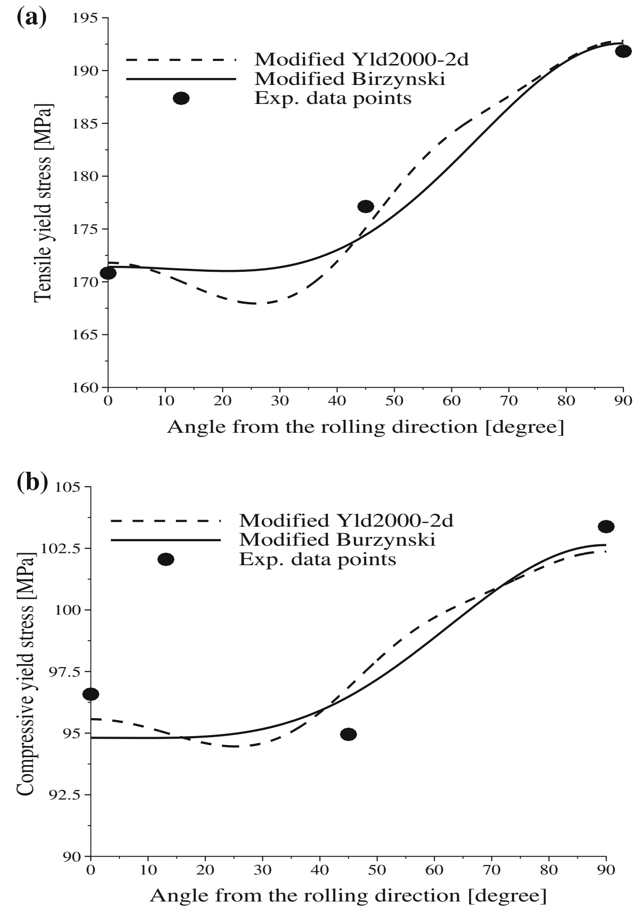

Figure 11. Comparison of the yield stress directionality for AZ31: (**a**) uniaxial tensile yield stress and (**b**) uniaxial compressive yield stress.

In table 3c, the obtained errors of modified Burzynski and Modified Yld2000-2d in predicting tensile and compressive yield stresses, compared with experimental results, are shown and it is seen that the proposed criterion is also successful for a HCP material. Furthermore, the initial yield surface is not sufficient for the correctness of a selected function and it is important to find the subsequent yield surface in an appropriate yield function [3, 5]. Therefore, five more HCP anisotropic materials such as Mg–0.5% Th alloy, Mg–4% Li alloy and pure textured magnesium, textured magnesium and Ti–4Al–1/4O_2 are taken into account as case studies with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

The experimental data points are presented in table 4. Because only five experimental data points have been reported, the other five experimental data points are assumed to be isotropic. That is, four *R*-values of $R_0^T, R_{45}^T, R_{90}^T, R_b^T$ are set to unity and the uniaxial tensile yield stress in the diagonal direction denoted as σ_{45}^T is assumed to be identical to that in RD denoted as σ_0^T [18].

Then these data are used to calibrate 10 material parameters in the modified Burzynski criterion yield function as presented in tables 4, 5 and 6 for Mg–0.5% Th alloy, Mg–4% Li alloy, pure textured magnesium, textured

Table 4. Experimental data points of (a) Mg–0.5% Th alloy, (b) Mg–4% Li alloy, (c) pure textured magnesium, (d) textured magnesium and (e) Ti–4Al–1/4O_2 [18].

Material (%)	σ_0^T	σ_b^T	σ_{90}^T	σ_0^C	σ_{90}^C
(a)					
1	187.6360	150.2715	169.2040	97.0909	98.6159
5	203.7930	197.3405	207.609	125	125.1720
10	217.091	215.7045	223.448	210.545	194.487
(b)					
1	94.5455	94.6630	80.1724	67.2727	66.3793
5	138.182	157.947	121.552	93.6364	93.1034
10	153.08	191.229	151.724	151.268	146.552
(c)					
1	9.89308	9.57815	19.1184	4.15727	3.79666
5	19	18.2918	24	8	10.0339
10	23	21.9490	28.9796	21.4286	25.5782
(d)					
1	10.1515	9.6473	19.2115	4.24242	3.7276
5	19.0189	18.2285	23.9427	7.84906	9.7491
10	23.0938	21.8698	28.8172	21.2836	25.233
(e)					
1	653.43	961.038	693.396	594.059	530.66
5	705.446	1121.295	725.352	757.426	711.268
10	771.513	1300.52	760.563	867.953	873.239

Table 5. Material parameters for modified Burzynski of (a) Mg–0.5% Th alloy, (b) Mg–4% Li alloy, (c) pure textured magnesium, (d) textured magnesium and (e) Ti–4Al–1/4O_2 titanium alloy.

$\bar{\varepsilon}^p$(%)	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
(a)										
1	0.0277	0.0117	0.0066	−0.0007	0.0065	−0.0109	−0.0093	0.8285	−2.0086	1.3632
5	0.0284	0.0121	0.0069	−0.0007	0.0075	−0.0092	−0.0094	0.5869	−2.3613	1.0138
10	0.0241	0.0139	0.0062	−0.0010	0.0075	−0.0021	−0.0097	0.3938	−4.7004	0.2055
(b)										
1	0.0243	0.0120	0.0106	−0.0002	0.0110	−0.0077	−0.0046	1.1200	−2.0948	1.6778
5	0.0244	0.0115	0.0103	−0.0003	0.0108	−0.0068	−0.0050	0.6221	−2.5890	1.5153
10	0.0170	0.0096	0.0092	−0.0005	0.0114	−0.0011	−0.0033	0.3717	−5.3699	0.2139
(c)										
1	−0.2824	−0.2033	0.1195	0.0057	0.1730	−0.0321	−0.0485	0.7154	−11.6698	13.0330
5	−0.2167	−0.1451	0.0784	0.0070	0.0845	−0.0455	−0.0365	0.6863	−23.8968	4.7684
10	−0.3009	−0.2055	0.0612	0.0034	0.0707	−0.0453	−0.0653	0.3871	−17.5900	0.2110
(d)										
1	−0.2845	−0.2043	0.1157	0.0059	0.1686	−0.0310	−0.0489	0.7163	−11.8483	13.2634
5	−0.3302	−0.2041	0.0906	0.0057	0.0985	−0.0555	−0.0451	0.5108	−26.9404	4.0435
10	−0.4207	−0.2830	0.0896	0.0099	0.1002	−0.0258	−0.0345	0.1912	−56.8526	0.4289
(e)										
1	0.0274	0.0458	0.0617	−0.0004	−0.1003	0.0016	0.0047	0.0003	0.2636	−0.2801
5	0.0398	0.0370	0.0410	−0.0014	−0.0618	0.0016	−0.0005	0.0006	0.1694	0.1812
10	−0.0237	0.0093	0.0017	−0.0099	0.0288	0.0447	0.0526	0.0022	0.0003	0.0097

magnesium and Ti–4Al–1/4O_2. The 10 coefficients $\alpha_i (i = 1 - 10)$ of these anisotropic HCP materials in $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ are collected in table 5 for the modified Burzynski criterion.

The yield functions in $\sigma_{xx} - \sigma_{yy}$ plane, the diagrams of modified Burzynski criterion in $\bar{\sigma}_e - \bar{\sigma}_m$ plane and also the tensile/compressive yield stress directionality of these materials with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ for these materials can be shown according the procedure explained before.

In figures 12, 15, 18, 21 and 24 the yield functions, in figures 13, 16, 19, 22 and 25 the modified Burzynski criterion in $\bar{\sigma}_e - \bar{\sigma}_m$ plane and finally in figures 14, 17, 20, 23

Table 6. Discrepancy of different $\bar{\varepsilon}^p$ compared to experimental results for (a) Mg–0.5% Th alloy, (b) Mg–4% Li alloy, (c) pure textured magnesium, (d) textured magnesium and (e) Ti–4Al–1/$4O_2$ titanium alloy (in percentage) and the modified Burzynski.

$\bar{\varepsilon}^p$(%)	E_σ^T	E_σ^C	E_σ^b
(a)			
1	0.3485	0.1875	0.3166
5	0.5700	0.3236	1.9955
10	0.2494	0.2270	0.6512
(b)			
1	0.1388	0.1763	0.1708
5	0.2054	0.1494	0.5122
10	0.1206	0.1412	0.0972
(c)			
1	0.0150	0.0114	0.0259
5	0.0606	0.0235	0.0265
10	0.2888	0.3789	0.1530
(d)			
1	0.0311	0.0164	0.0691
5	0.0725	0.0417	0.1355
10	0.2302	0.2638	1.0502
(e)			
1	1.3014	1.4674	1.6152
5	0.9772	1.1215	1.9055
10	0.4879	0.7077	2.7998

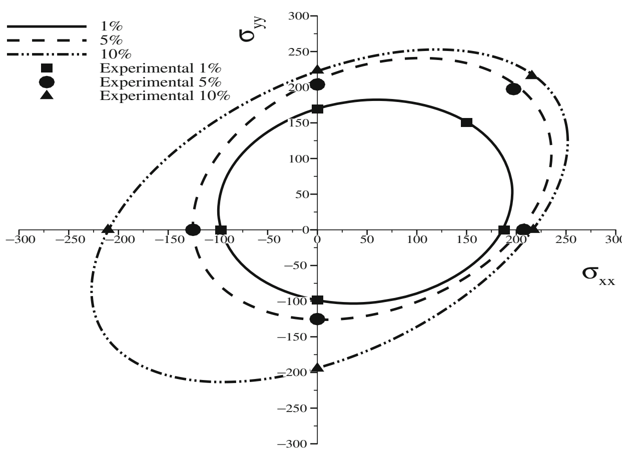

Figure 12. Comparison of the yield surfaces for Mg–0.5% Th alloy with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

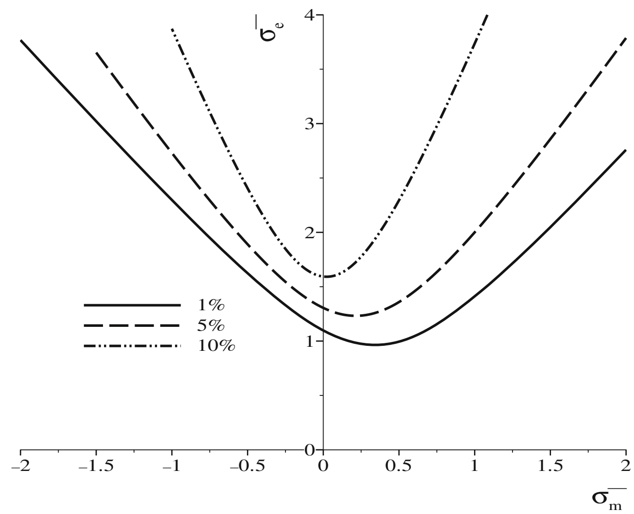

Figure 13. The Burzynski yield condition for Mg–0.5% Th alloy in $\bar{\sigma}_e - \bar{\sigma}_m$ plane for $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

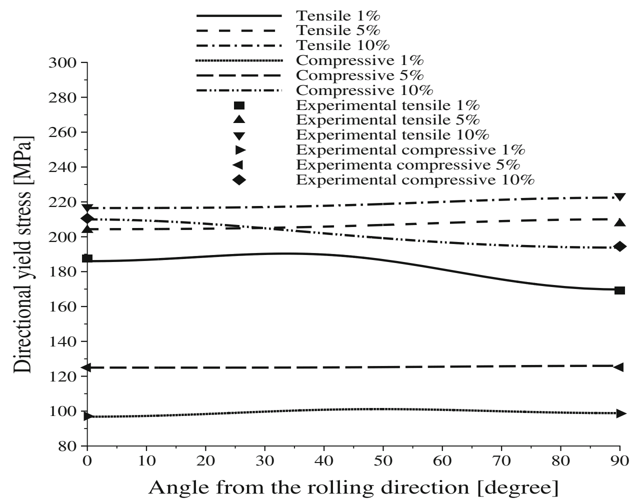

Figure 14. Comparison of the tensile/compressive yield stress directionality for Mg–0.5% Th alloy with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

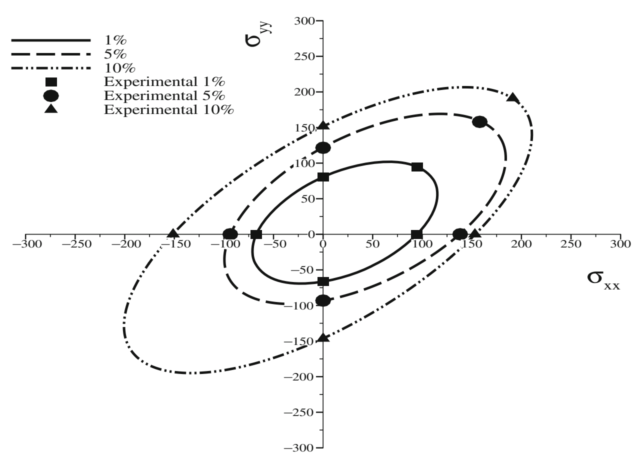

Figure 15. Comparison of the yield surfaces for Mg–4% Li alloy with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

and 26 the tensile/compressive yield stress directionality of these materials for $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ of Mg–0.5% Th alloy, Mg–4% Li alloy, pure textured magnesium, textured magnesium and Ti–4Al–1/$4O_2$ are shown.

It is observed that all anisotropic materials follow an identical pattern in subsequent yielding with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ except for pure textured magnesium and textured magnesium. These two materials have the same pattern for $\bar{\varepsilon}^p = 5\%, 10\%$ and vary from $\bar{\varepsilon}^p = 1\%$ for pure textured magnesium, figures 18 and 19, and for textured

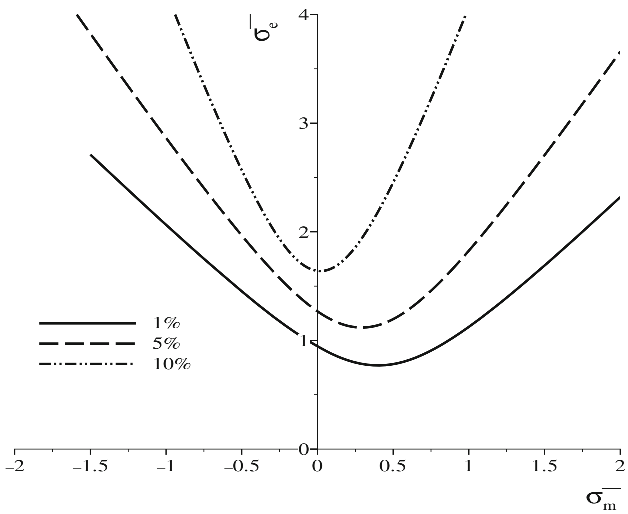

Figure 16. The Burzynski yield condition for Mg–4% Li alloy in $\bar{\sigma}_e - \bar{\sigma}_m$ plane with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ and the modified Burzynski hyperboloid.

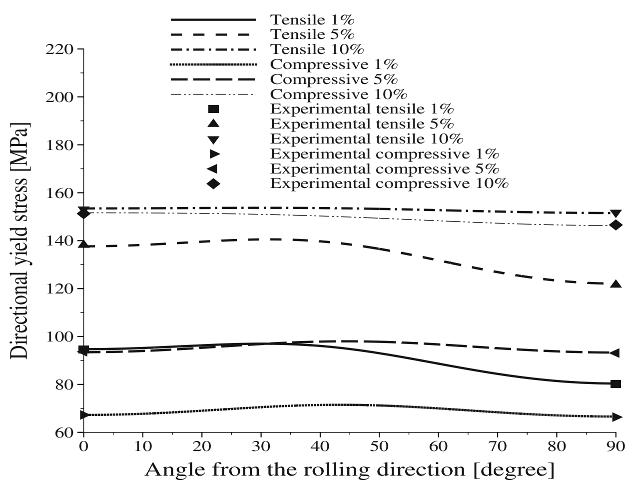

Figure 17. Comparison of the tensile/compressive yield stress directionality for Mg–4% Li alloy with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

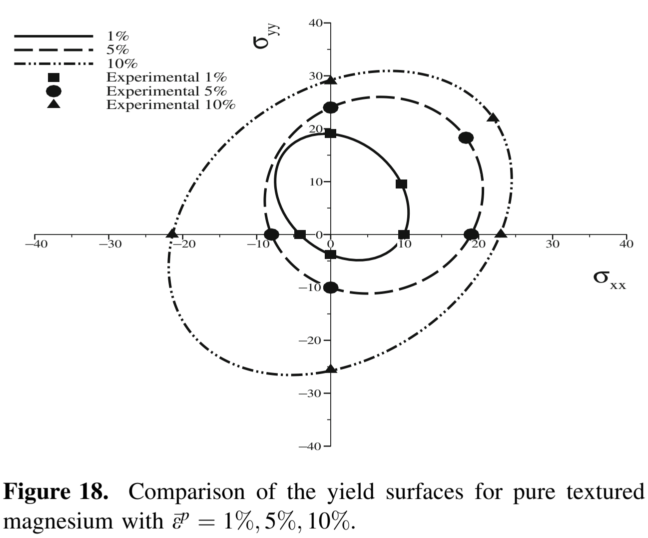

Figure 18. Comparison of the yield surfaces for pure textured magnesium with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

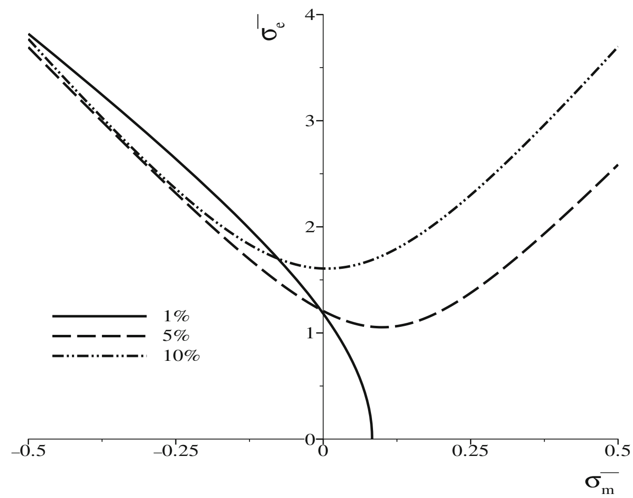

Figure 19. The Burzynski yield condition for pure textured magnesium in $\bar{\sigma}_e - \bar{\sigma}_m$ plane for $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$. For $\bar{\varepsilon}^p = 1\%$ the modified Burzynski–Torre paraboloid and for $\bar{\varepsilon}^p = 5\%, 10\%$ the modified Burzynski hyperboloid.

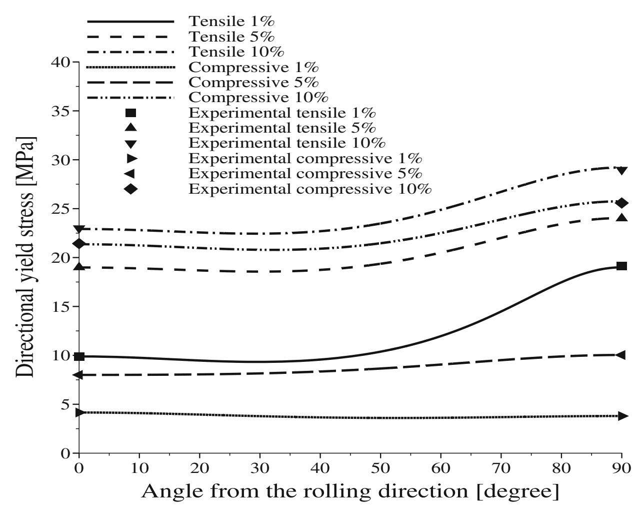

Figure 20. Comparison of the tensile/compressive yield stress directionality for pure textured magnesium with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

magnesium, figures 21 and 22. This issue may be attributed to the enormous change in the mechanical structure of these anisotropic materials from $\bar{\varepsilon}^p = 1\%$ to $\bar{\varepsilon}^p = 5\%$.

Finally, to validate the modified Burzunski to predict the subsequent yield stresses, the three relative errors are proposed as Eq. (36). In this relation, E_σ^T, E_σ^C and E_σ^b are the relative errors of directional tensile and compressive yield stresses and biaxial tensile stress, respectively. It should be mentioned that due to lack of experimental data points for these materials it is assumed that $\left(\sigma_{45}^T\right)_{exp.}$ is identical to $\left(\sigma_0^T\right)_{exp.}$ [18]. The relative errors are presented in table 6 for Mg–0.5% Th alloy, Mg–4% Li alloy, pure textured magnesium, textured magnesium and Ti–4Al–1/4O_2. It is seen

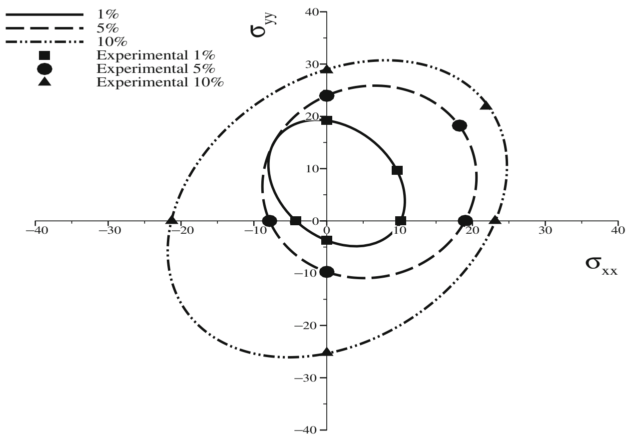

Figure 21. Comparison of the yield surfaces for textured magnesium with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

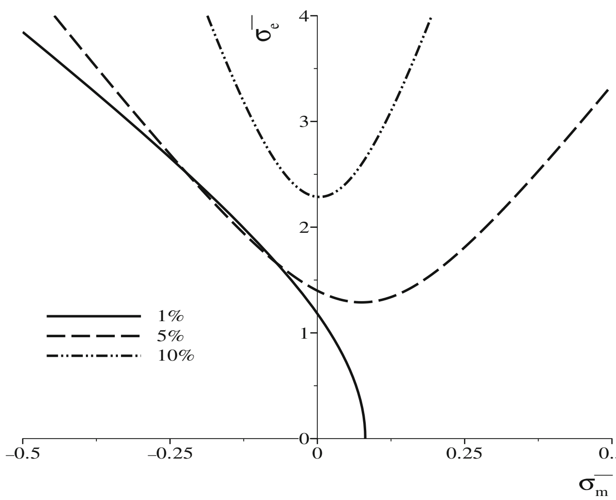

Figure 22. The Burzynski yield condition for textured magnesium in $\bar{\sigma}_e - \bar{\sigma}_m$ plane with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$. For $\bar{\varepsilon}^p = 1\%$ the modified Burzynski–Torre paraboloid and for $\bar{\varepsilon}^p = 5\%, 10\%$ the Modified Burzynski hyperbolid.

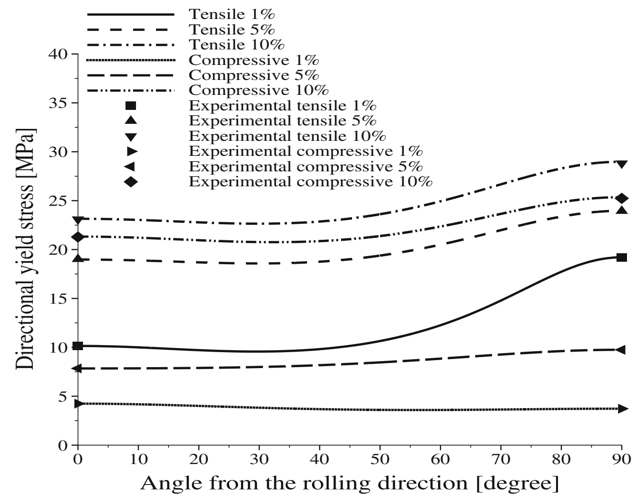

Figure 23. Comparison of the tensile/compressive yield stress directionality for textured magnesium with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

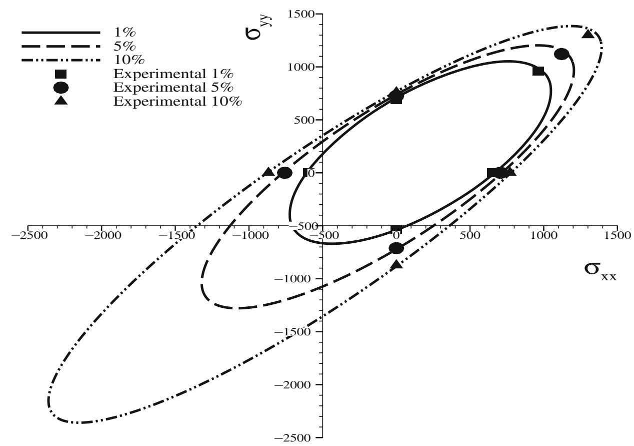

Figure 24. Comparison of the yield surfaces for Ti–4Al–1/4O_2 titanium alloy with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

that the values of relative errors are very small and the modified Burzynski is successful for predicting subsequent yield stresses for these materials.

7. Conclusions

An asymmetric yield function, the Burzynski criterion for isotropic materials, is extended for an asymmetric anisotropic yield function, the modified Burzynski criterion. In

$$\begin{cases} E_\sigma^T = \dfrac{1}{3}\left(\left[\dfrac{(\sigma_0^T)_{exp.} - (\sigma_0^T)_{pred.}}{(\sigma_0^T)_{exp.}}\right]^2 + \left[\dfrac{(\sigma_0^T)_{exp.} - (\sigma_{45}^T)_{pred.}}{(\sigma_0^T)_{exp.}}\right]^2 + \left[\dfrac{(\sigma_{90}^T)_{exp.} - (\sigma_{90}^T)_{pred.}}{(\sigma_{90}^T)_{exp.}}\right]^2\right)^{\frac{1}{2}} \times 100 \\ E_\sigma^C = \dfrac{1}{2}\left(\left[\dfrac{(\sigma_0^C)_{exp.} - (\sigma_0^C)_{pred.}}{(\sigma_0^C)_{exp.}}\right]^2 + \left[\dfrac{(\sigma_{90}^C)_{exp.} - (\sigma_{90}^C)_{pred.}}{(\sigma_{90}^C)_{exp.}}\right]^2\right)^{\frac{1}{2}} \times 100 \\ E_\sigma^b = \left|\dfrac{(\sigma_b^T)_{exp.} - (\sigma_b^T)_{pred.}}{(\sigma_b^T)_{exp.}}\right| \times 100 \end{cases} \tag{36}$$

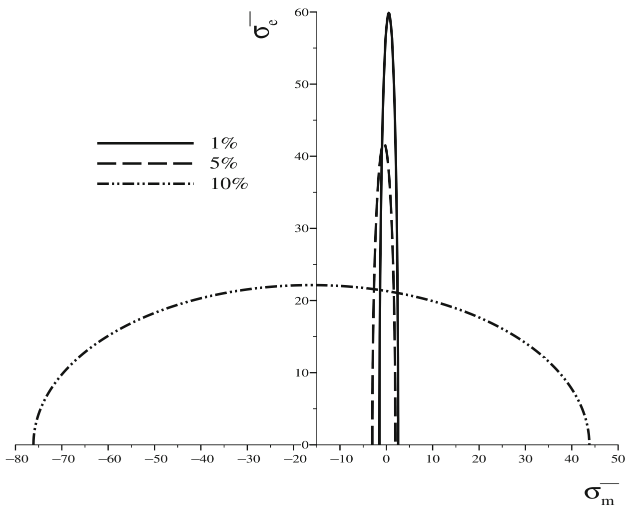

Figure 25. The Burzynski yield condition for Ti–4Al–1/4O_2 titanium alloy in $\bar{\sigma}_e - \bar{\sigma}_m$ plane with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ and the modified Burzynski ellipsoid.

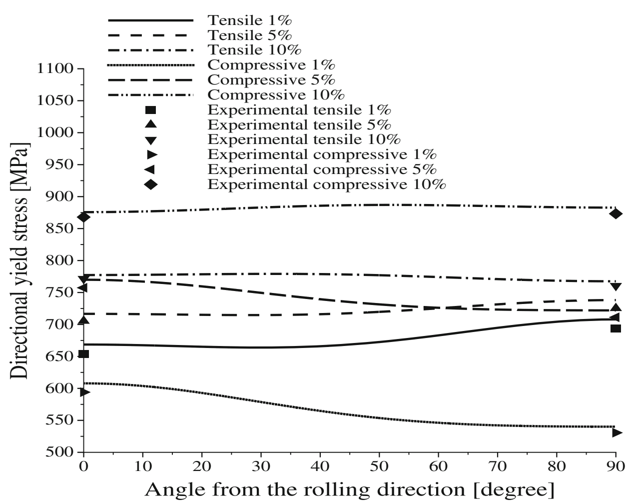

Figure 26. Comparison of the tensile/compressive yield stress directionality for Ti–4Al–1/4O_2 titanium alloy with $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$.

this extension a linear transformation for transforming isotropic deviatoric stress with five coefficients to anisotropic modified deviatoric stress and two coefficients for transforming isotropic hydrostatic stress to anisotropic hydrostatic stress are considered. These modified anisotropic deviatoric and hydrostatic stresses are weighted in the modified Burzynski criterion with three coefficients. Finally with 10 experimental data points these parameters are evaluated for Al 2008-T4 (BCC), Al 2090-T3 (FCC) and AZ31 (HCP) and it is shown that this criterion is well fitted especially for materials with the modified Burzynski–Torre paraboloid.

In addition, to show the accuracy of the modified Burzynski in predicting the subsequent yield stresses, five other HCP anisotropic materials such as Mg–0.5% Th alloy, Mg–4% Li alloy and pure textured magnesium, textured magnesium and Ti–4Al–1/4O_2 are examined for $\bar{\varepsilon}^p = 1\%, 5\%, 10\%$ and it is shown that the modified Burzynski is a powerful criterion to predict the subsequent yield surfaces as well.

Nomenclature

L_{ij} linear transformation matrix
s_{ij} deviatoric stress tensor
$\bar{s}_{ij}$ modified deviatoric stress tensor
R plastic strain increment ratio
R_b^T tensile equibiaxial R-value
R_θ^T tensile R-value in θ direction

Greek symbols

α_i anisotropy parameters
$\bar{\varepsilon}^p$ effective plastic strain
$d\varepsilon_{ij}^p$ increment of plastic strain tensor
θ angle from the rolling direction
Φ Burzynski criterion
$\bar{\Phi}$ modified Burzynski criterion
σ_e effective stress
$\bar{\sigma}_e$ modified effective stress
σ_{ij} stress tensor
σ_m hydrostatic stress
$\bar{\sigma}_m$ modified hydrostatic stress
σ_b^T tensile equibiaxial yield stress
σ_θ^C compressive yield stress in θ direction
σ_θ^T tensile yield stress in θ direction

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