Topology and its Applications 219 (2017) 17–28

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topological coarse shape homotopy groups

Fateme Ghanei [a], Hanieh Mirebrahimi [a,*], Behrooz Mashayekhy [a], Tayyebe Nasri [b]

[a] *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran*
[b] *Department of Pure Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran*

A R T I C L E I N F O

Article history:
Received 1 March 2016
Received in revised form 11 December 2016
Accepted 23 December 2016
Available online 3 January 2017

MSC:
55Q07
55P55
54C56
54H11
18A30

Keywords:
Topological coarse shape homotopy group
Coarse shape group
Shape group
Topological group
Inverse limit

A B S T R A C T

Cuchillo-Ibanez et al. introduced a topology on the sets of shape morphisms between arbitrary topological spaces in 1999. In this paper, applying a similar idea, we introduce a topology on the set of coarse shape morphisms $Sh^*(X,Y)$, for arbitrary topological spaces X and Y. In particular, we can consider a topology on the coarse shape homotopy group of a topological space (X,x), $Sh^*((S^k,*),(X,x)) = \check{\pi}_k^*(X,x)$, which makes it a Hausdorff topological group. Moreover, we study some properties of these topological coarse shape homotopy groups such as second countability, movability and in particular, we prove that $\check{\pi}_k^{*^{top}}$ preserves finite product of compact Hausdorff spaces. Also, we show that for a pointed topological space (X,x), $\check{\pi}_k^{top}(X,x)$ can be embedded in $\check{\pi}_k^{*^{top}}(X,x)$.

1. Introduction and motivation

Suppose that (X,x) is a pointed topological space. We know that $\pi_k(X,x)$ has a quotient topology induced by the natural map $q : \Omega^k(X,x) \to \pi_k(X,x)$, where $\Omega^k(X,x)$ is the kth loop space of (X,x) with the compact-open topology. With this topology, $\pi_k(X,x)$ is a quasitopological group, denoted by $\pi_k^{qtop}(X,x)$ and for some spaces it becomes a topological group (see [5–7,15]).

Calcut and McCarthy [8] proved that for a path connected and locally path connected space X, $\pi_1^{qtop}(X)$ is a discrete topological group if and only if X is semilocally 1-connected (see also [6]). Pakdaman et al. [24] showed that for a locally $(n-1)$-connected space X, $\pi_n^{qtop}(X,x)$ is discrete if and only if X is

* Corresponding author.
E-mail addresses: fatemeh.ghanei91@gmail.com (F. Ghanei), h_mirebrahimi@um.ac.ir (H. Mirebrahimi), bmashf@um.ac.ir (B. Mashayekhy), tnasri72@yahoo.com (T. Nasri).

http://dx.doi.org/10.1016/j.topol.2016.12.015
0166-8641/

semilocally n-connected at x (see also [15]). Fabel [12,13] and Brazas [6] presented some spaces for which their quasitopological homotopy groups are not topological groups. Moreover, despite Fabel's result [12] that says the quasitopological fundamental group of the Hawaiian earring is not a topological group, Ghane et al. [16] proved that the topological nth homotopy group of an n-Hawaiian like space is a prodiscrete metrizable topological group, for all $n \geq 2$.

Cuchillo-Ibanez et al. [10] introduced a topology on the set of shape morphisms between arbitrary topological spaces X, Y, $Sh(X,Y)$. Moszyńska [21] showed that for a compact Hausdorff space (X,x), the kth shape group $\check{\pi}_k(X,x)$, $k \in \mathbb{N}$, is isomorphic to the set $Sh((S^k,*),(X,x))$ and Bilan [2] mentioned that the result can be extended for all topological spaces. The authors [22], considering the latter topology on the set of shape morphisms between pointed spaces, obtained a topology on the shape homotopy groups of arbitrary spaces, denoted by $\check{\pi}_k^{top}(X,x)$ and showed that with this topology, the kth shape group $\check{\pi}_k^{top}(X,x)$ is a Hausdorff topological group, for all $k \in \mathbb{N}$. Moreover, they obtained some topological properties of these groups under some conditions such as movability, $\mathbb{N}$-compactness and compactness. In particular, they proved that $\check{\pi}_k^{top}$ commutes with finite product of compact Hausdorff spaces. Also, they presented two spaces X and Y with the same shape homotopy groups such that their topological shape homotopy groups are not isomorphic.

The aim of this paper is to introduce a topology on the coarse shape homotopy groups $\check{\pi}_k^*(X,x)$ and to provide some topological properties of these groups. First, similarly to the techniques in [10], we introduce a topology on the set of coarse shape morphisms $Sh^*(X,Y)$, for arbitrary topological spaces X and Y. Several properties of this topology such as continuity of the map $\Omega : Sh^*(X,Y) \times Sh^*(Y,Z) \longrightarrow Sh^*(X,Z)$ given by the composition $\Omega(F^*,G^*) = G^* \circ F^*$ and the equality $Sh^*(X,Y) = \varprojlim Sh^*(X,Y_\mu)$, for an HPol-expansion $\mathbf{q} : Y \to (Y_\mu, q_{\mu\mu'}, M)$ of Y, are proved which are useful to hereinafter results. Moreover, we show that this topology can also be induced from an ultrametric similarly to the process in [9].

By the above topology, we can consider a topology on the coarse shape homotopy group $\check{\pi}_k^{*^{top}}(X,x) = Sh^*((S^k,*),(X,x))$ which makes it a Hausdorff topological group, for all $k \in \mathbb{N}$ and any pointed topological space (X,x). It is known that if X and Y are compact Hausdorff spaces, then $X \times Y$ is a product in the coarse shape category [23, Theorem 2.2]. In this case, we show that the kth topological coarse shape group commutes with finite product, for all $k \in \mathbb{N}$. Also, we prove that movability of $\check{\pi}_k^{*^{top}}(X,x)$ can be concluded from the movability of (X,x), for topological space (X,x) with some conditions. As previously mentioned, $\check{\pi}_k(X,x)$ with the topology defined by Cuchillo-Ibanez et al. [10] on the set of shape morphisms, is a topological group. We show that this topology also coincides with the topology induced by $\check{\pi}_k^{*^{top}}(X,x)$ on the subspace $\check{\pi}_k(X,x)$.

2. Preliminaries

Recall from [1] some of the main notions concerning the coarse shape category and pro*-category. Let $\mathcal{T}$ be a category and let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in the category $\mathcal{T}$. An *S^*-morphism* of inverse systems, $(f, f_\mu^n) : \mathbf{X} \to \mathbf{Y}$, consists of an index function $f : M \to \Lambda$ and of a set of $\mathcal{T}$-morphisms $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, such that for every related pair $\mu \leq \mu'$ in M, there exist a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$, and an $n \in \mathbb{N}$ so that for every $n' \geq n$,

$$q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda} = f_\mu^{n'} p_{f(\mu)\lambda}.$$

If $M = \Lambda$ and $f = 1_\Lambda$, then $(1_\lambda, f_\lambda^n)$ is said to be a *level S^*-morphism*. The *composition* of S*-morphisms $(f, f_\mu^n) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu^n) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is an S*-morphism $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) : \mathbf{X} \to \mathbf{Z}$, where $h = fg$ and $h_\nu^n = g_\nu^n f_{g(\nu)}^n$, for all $n \in \mathbb{N}$. The *identity S^*-morphism* on $\mathbf{X}$ is an S*-morphism $(1_\Lambda, 1_{X_\lambda}^n) : \mathbf{X} \to \mathbf{X}$, where 1_Λ is the identity function and $1_{X_\lambda}^n = 1_{X_\lambda}$ in $\mathcal{T}$, for all $n \in \mathbb{N}$ and every $\lambda \in \Lambda$.

An S*-morphism $(f, f_\mu^n) : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to an S*-morphism $(f', f'^n_\mu) : \mathbf{X} \to \mathbf{Y}$, denoted by $(f, f_\mu^n) \sim (f', f'^n_\mu)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$ and $n \in \mathbb{N}$ such that $\lambda \geq f(\mu), f'(\mu)$ and for every $n' \geq n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = f'^{n'}_\mu p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation among S*-morphisms of inverse systems in $\mathcal{T}$. The *category* pro*-$\mathcal{T}$ has as objects all inverse systems $\mathbf{X}$ in $\mathcal{T}$ and as morphisms all equivalence classes $\mathbf{f}^* = [(f, f_\mu^n)]$ of S*-morphisms (f, f_μ^n). The composition in pro*-$\mathcal{T}$ is well defined by putting

$$\mathbf{g}^*\mathbf{f}^* = \mathbf{h}^* = [(h, h_\nu^n)],$$

where $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) = (fg, g_\nu^n f_{g(\nu)}^n)$. For every inverse system $\mathbf{X}$ in $\mathcal{T}$, the identity morphism in pro*-$\mathcal{T}$ is $\mathbf{1}_\mathbf{X}^* = [(1_\Lambda, 1_{X_\Lambda}^n)]$.

In particular, if (X) and (Y) are two rudimentary inverse systems in HTop, then every set of mappings $f^n : X \to Y$, $n \in \mathbb{N}$, induces a map $\mathbf{f}^* : (X) \to (Y)$ in pro*-HTop.

A functor $\underline{\mathcal{J}} = \underline{\mathcal{J}}_\mathcal{T} : pro\text{-}\mathcal{T} \to pro^*\text{-}\mathcal{T}$ is defined as follows: For every inverse system $\mathbf{X}$ in $\mathcal{T}$, $\underline{\mathcal{J}}(\mathbf{X}) = \mathbf{X}$ and if $\mathbf{f} \in pro\text{-}\mathcal{T}(\mathbf{X}, \mathbf{Y})$ is represented by (f, f_μ), then $\underline{\mathcal{J}}(\mathbf{f}) = \mathbf{f}^* = [(f, f_\mu^n)] \in pro^* - \mathcal{T}(\mathbf{X}, \mathbf{Y})$ is represented by the S*-morphism (f, f_μ^n), where $f_\mu^n = f_\mu$ for all $\mu \in M$ and $n \in \mathbb{N}$. Since the functor $\underline{\mathcal{J}}$ is faithful, we may consider the category pro-$\mathcal{T}$ as a subcategory of pro*-$\mathcal{T}$.

Let $\mathcal{P}$ be a subcategory of $\mathcal{T}$. A $\mathcal{P}$*-expansion* of an object X in $\mathcal{T}$ is a morphism $\mathbf{p} : X \to \mathbf{X}$ in pro-$\mathcal{T}$, where $\mathbf{X}$ belongs to pro-$\mathcal{P}$ characterised by the following two properties:

(E1) For every object P of $\mathcal{P}$ and every map $h : X \to P$ in $\mathcal{T}$, there is a $\lambda \in \Lambda$ and a map $f : X_\lambda \to P$ in $\mathcal{P}$ such that $fp_\lambda = h$;
(E2) If $f_0, f_1 : X_\lambda \to P$ in $\mathcal{P}$ satisfy $f_0 p_\lambda = f_1 p_\lambda$, then there exists a $\lambda' \geq \lambda$ such that $f_0 p_{\lambda\lambda'} = f_1 p_{\lambda\lambda'}$.

The subcategory $\mathcal{P}$ is said to be *pro-reflective* (*dense*) subcategory of $\mathcal{T}$ provided that every object X in $\mathcal{T}$ admits a $\mathcal{P}$-expansion $\mathbf{p} : X \to \mathbf{X}$.

Let $\mathcal{P}$ be a pro-reflective subcategory of $\mathcal{T}$. Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{p}' : X \to \mathbf{X}'$ be two $\mathcal{P}$-expansions of the same object X in $\mathcal{T}$, and let $\mathbf{q} : Y \to \mathbf{Y}$ and $\mathbf{q}' : Y \to \mathbf{Y}'$ be two $\mathcal{P}$-expansions of the same object Y in $\mathcal{T}$. Then there exist two natural (unique) isomorphisms $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ and $\mathbf{j} : \mathbf{Y} \to \mathbf{Y}'$ in pro-$\mathcal{P}$ with respect to $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{q}$, $\mathbf{q}'$, respectively. Consequently $\underline{\mathcal{J}}(\mathbf{i}) : \mathbf{X} \to \mathbf{X}'$ and $\underline{\mathcal{J}}(\mathbf{j}) : \mathbf{Y} \to \mathbf{Y}'$ are isomorphisms in pro*-$\mathcal{P}$. A morphism $\mathbf{f}^* : \mathbf{X} \to \mathbf{Y}$ is said to be *pro*-$\mathcal{P}$ equivalent* to a morphism $\mathbf{f}'^* : \mathbf{X}' \to \mathbf{Y}'$, denoted by $\mathbf{f}^* \sim \mathbf{f}'^*$, provided that the following diagram in pro*-$\mathcal{P}$ commutes:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\underline{\mathcal{J}}(\mathbf{i})} & \mathbf{X}' \\ \Big\downarrow \mathbf{f}^* & & \mathbf{f}'^* \Big\downarrow \\ \mathbf{Y} & \xrightarrow{\underline{\mathcal{J}}(\mathbf{j})} & \mathbf{Y}'. \end{array} \tag{1}$$

This is an equivalence relation on the appropriate subclass of Mor(pro*-$\mathcal{P}$). Now, the *coarse shape category* $\mathrm{Sh}^*_{(\mathcal{T},\mathcal{P})}$ for the pair $(\mathcal{T}, \mathcal{P})$ is defined as follows: The objects of $\mathrm{Sh}^*_{(\mathcal{T},\mathcal{P})}$ are all objects of $\mathcal{T}$. A morphism $F^* : X \to Y$ is the pro*-$\mathcal{P}$ equivalence class $< \mathbf{f}^* >$ of a mapping $\mathbf{f}^* : \mathbf{X} \to \mathbf{Y}$ in pro*-$\mathcal{P}$. The *composition* of $F^* =< \mathbf{f}^* >: X \to Y$ and $G^* =< \mathbf{g}^* >: Y \to Z$ is defined by the representatives, i.e., $G^*F^* =< \mathbf{g}^*\mathbf{f}^* >: X \to Z$. The *identity coarse shape morphism* on an object X, $1_X^* : X \to X$, is the pro*-$\mathcal{P}$ equivalence class $< \mathbf{1_X}^* >$ of the identity morphism $\mathbf{1_X}^*$ in pro*-$\mathcal{P}$.

The faithful functor $\mathcal{J} = \mathcal{J}_{(\mathcal{T},\mathcal{P})} : Sh_{(\mathcal{T},\mathcal{P})} \to Sh^*_{(\mathcal{T},\mathcal{P})}$ is defined by keeping objects fixed and whose morphisms are induced by the inclusion functor $\underline{\mathcal{J}} = \underline{\mathcal{J}}_\mathcal{T} : pro\text{-}\mathcal{P} \to pro^*\text{-}\mathcal{P}$.

Remark 2.1. Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ be $\mathcal{P}$-expansions of X and Y respectively. For every morphism $f : X \to Y$ in $\mathcal{T}$, there is a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{P}$ such that the following diagram commutes in pro-$\mathcal{P}$:

$$\begin{tikzcd} \mathbf{X} \arrow[d, "\mathbf{f}"] & X \arrow[l, "\mathbf{p}"'] \arrow[d, "f"'] \\ \mathbf{Y} & Y. \arrow[l, "\mathbf{q}"] \end{tikzcd} \tag{2}$$

If we take other $\mathcal{P}$-expansions $\mathbf{p}' : X \to \mathbf{X}'$ and $\mathbf{q}' : Y \to \mathbf{Y}'$, we obtain another morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in pro-$\mathcal{P}$ such that $\mathbf{f}'\mathbf{p}' = \mathbf{q}'f$ and so we have $\mathbf{f} \sim \mathbf{f}'$ and hence $\underline{\mathcal{J}}(\mathbf{f}) \sim \underline{\mathcal{J}}(\mathbf{f}')$ in pro*-$\mathcal{P}$. Therefore, every morphism $f \in \mathcal{T}(X, Y)$ yields an pro*-$\mathcal{P}$ equivalence class $< \underline{\mathcal{J}}(\mathbf{f}) >$, i.e., a coarse shape morphism $F^* : X \to Y$, denoted by $\mathcal{S}^*(f)$. If we put $\mathcal{S}^*(X) = X$ for every object X of $\mathcal{T}$, then we obtain a functor $\mathcal{S}^* : \mathcal{T} \to Sh^*$, which is called the *coarse shape functor*.

Since the homotopy category of polyhedra HPol is pro-reflective (dense) in the homotopy category HTop [19, Theorem 1.4.2], the coarse shape category $\mathrm{Sh}^*_{(HTop,HPol)} = \mathrm{Sh}^*$ is well defined.

3. A topology on the set of coarse shape morphisms

Similarly to the method of [10], we can define a topology on the set of coarse shape morphisms. Let X and Y be topological spaces. Assume $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an inverse system in pro-HPol and $\mathbf{p} : X \to \mathbf{X}$ is an HPol-expansion of X. For every $\lambda \in \Lambda$ and $F^* \in Sh^*(Y, X)$ put $V_\lambda^{F^*} = \{G^* \in Sh^*(Y, X)|\ \ \mathcal{S}^*(p_\lambda) \circ F^* = \mathcal{S}^*(p_\lambda) \circ G^*\}$. First, we prove the following results.

Proposition 3.1. *The family* $\{V_\lambda^{F^*}|\ \ F^* \in Sh^*(Y, X)$ *and* $\lambda \in \Lambda\}$ *is a basis for a topology* $T_{\mathbf{p}}$ *on* $Sh^*(Y, X)$*. Moreover, if* $\mathbf{p}' : X \to \mathbf{X}' = (X_\nu, p_{\nu\nu'}, N)$ *is another HPol-expansion of* X*, then the identity map* $(Sh^*(Y, X), T_{\mathbf{p}}) \longrightarrow (Sh^*(Y, X), T_{\mathbf{p}'})$ *is a homeomorphism which shows that this topology depends only on* X *and* Y*.*

Proof. We know that $F^* \in V_\lambda^{F^*}$ for every $\lambda \in \Lambda$ and every $F^* \in Sh^*(Y, X)$. Suppose $F^*, G^* \in Sh^*(Y, X)$ and $\lambda_1, \lambda_2 \in \Lambda$ and $H^* \in V_{\lambda_1}^{F^*} \cap V_{\lambda_2}^{G^*}$. Since $H^* \in V_{\lambda_1}^{F^*}$, then $\mathcal{S}^*(p_{\lambda_1}) \circ F^* = \mathcal{S}^*(p_{\lambda_1}) \circ H^*$. We show that $V_{\lambda_1}^{F^*} = V_{\lambda_1}^{H^*}$. Suppose $K^* \in V_{\lambda_1}^{F^*}$, so $\mathcal{S}^*(p_{\lambda_1}) \circ K^* = \mathcal{S}^*(p_{\lambda_1}) \circ F^* = \mathcal{S}^*(p_{\lambda_1}) \circ H^*$. Therefore $K^* \in V_{\lambda_1}^{H^*}$ and hence $V_{\lambda_1}^{F^*} \subseteq V_{\lambda_1}^{H^*}$. Conversely, if $K^* \in V_{\lambda_1}^{H^*}$, then we have $\mathcal{S}^*(p_{\lambda_1}) \circ K^* = \mathcal{S}^*(p_{\lambda_1}) \circ H^* = \mathcal{S}^*(p_{\lambda_1}) \circ F^*$. So $K^* \in V_{\lambda_1}^{F^*}$ and hence $V_{\lambda_1}^{H^*} \subseteq V_{\lambda_1}^{F^*}$. Similarly, since $H^* \in V_{\lambda_1}^{G^*}$, we have $V_{\lambda_1}^{G^*} = V_{\lambda_1}^{H^*}$ and so $H^* \in V_{\lambda_1}^{H^*} \cap V_{\lambda_2}^{H^*}$. We know that there exists a $\lambda \in \Lambda$ such that $\lambda \geq \lambda_1, \lambda_2$. We show that $H^* \in V_\lambda^{H^*} \subseteq V_{\lambda_1}^{H^*} \cap V_{\lambda_2}^{H^*}$ which completes the proof of the first assertion.

Given $K^* \in V_\lambda^{H^*}$. We have $p_{\lambda_1\lambda}p_\lambda = p_{\lambda_1}$ and $p_{\lambda_2\lambda}p_\lambda = p_{\lambda_2}$. Since $K^* \in V_\lambda^{H^*}$, so $\mathcal{S}^*(p_\lambda) \circ K^* = \mathcal{S}^*(p_\lambda) \circ H^*$ and therefore $\mathcal{S}^*(p_{\lambda_1}) \circ K^* = \mathcal{S}^*(p_{\lambda_1}) \circ H^*$. Hence $K^* \in V_{\lambda_1}^{H^*}$. Similarly $K^* \in V_{\lambda_2}^{H^*}$ and so $K^* \in V_{\lambda_1}^{H^*} \cap V_{\lambda_2}^{H^*}$.

Now, suppose that $\mathbf{p}' : X \to \mathbf{X}'$ is another HPol-expansion of X. Then there exists a unique isomorphism $\mathbf{i} : \mathbf{X} \longrightarrow \mathbf{X}'$ given by (i_ν, ϕ) such that $\mathbf{i} \circ \mathbf{p} = \mathbf{p}'$. To complete the proof, it is sufficient to show that $V_\nu^{F^*} = V_{\phi(\nu)}^{F^*}$, for every $\nu \in N$ and $F^* \in Sh^*(Y, X)$. For each $G^* \in V_{\phi(\nu)}^{F^*}$, we have $\mathcal{S}^*(p_{\phi(\nu)}) \circ G^* = \mathcal{S}^*(p_{\phi(\nu)}) \circ F^*$ and so $\mathcal{S}^*(p'_\nu) \circ G^* = \mathcal{S}^*(p'_\nu) \circ F^*$. Hence $G^* \in V_\nu^{F^*}$ and therefore $V_{\phi(\nu)}^{F^*} \subseteq V_\nu^{F^*}$. Similarly, one can show that $V_\nu^{F^*} \subseteq V_{\phi(\nu)}^{F^*}$. $\square$

Corollary 3.2. *Let* $X \in Obj(HPol)$*. Then* $Sh^*(Y, X)$ *is discrete, for every topological space* Y*.*

Example 3.3. Let $P = \{*\}$ be a singleton and $Q = \{*\}\dot{\cup}\{*\}$ (disjoint union). Then $card(Sh(P,Q)) = 2$ while $card(Sh^*(P,Q)) = 2^{\aleph_0}$ (see [1, Example 7.4]). It shows that $Sh(P,Q)$ is a countable discrete space while $Sh^*(P,Q)$ is an uncountable discrete space.

Theorem 3.4. *The map $\Omega : Sh^*(X,Y) \times Sh^*(Y,Z) \longrightarrow Sh^*(X,Z)$ given by the composition $\Omega(F^*,G^*) = G^* \circ F^*$ is continuous, for arbitrary topological spaces X, Y and Z.*

Proof. Consider HPol-expansions $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{q} : Y \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ and $\mathbf{r} : Z \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ of X, Y and Z, respectively. Let $F_0^* \in Sh^*(X,Y)$ and $G_0^* \in Sh^*(Y,Z)$ given by (f_μ^n, f) and (g_ν^n, g), respectively. Let $\nu \in N$ and $G_0^* \circ F_0^* \in V_\nu^{G_0^* \circ F_0^*}$. We show that $\Omega(V_{g(\nu)}^{F_0^*} \times V_\nu^{G_0^*}) \subseteq V_\nu^{G_0^* \circ F_0^*}$. To do this, we must show that for any $F^* \in V_{g(\nu)}^{F_0^*}$ and $G^* \in V_\nu^{G_0^*}$, $\mathcal{S}^*(r_\nu) \circ G^* \circ F^* = \mathcal{S}^*(r_\nu) \circ G_0^* \circ F_0^*$. Since $F^* \in V_{g(\nu)}^{F_0^*}$, we have $\mathcal{S}^*(q_{g(\nu)}) \circ F_0^* = \mathcal{S}^*(q_{g(\nu)}) \circ F^*$ and since $G^* \in V_\nu^{G_0^*}$, we have $\mathcal{S}^*(r_\nu) \circ G_0^* = \mathcal{S}^*(r_\nu) \circ G^*$. Note that $\mathcal{S}^*(r_\nu) \circ G_0^*$ is a coarse shape morphism which is given by $[(g_\nu^n, g\alpha_\nu)]$, where $\alpha_\nu : \{\nu\} \longrightarrow N$ is the inclusion map. Define $\alpha : Y_{g(\nu)} \longrightarrow Z_\nu$ as a coarse shape morphism given by $[(g_\nu^n, \beta_\nu)]$, where $\beta_\nu : \{\nu\} \longrightarrow \{g(\nu)\}$. We have $\mathcal{S}^*(r_\nu) \circ G_0^* = \alpha \circ \mathcal{S}^*(q_{g(\nu)})$ and so $\mathcal{S}^*(r_\nu) \circ G_0^* \circ F_0^* = \alpha \circ \mathcal{S}^*(q_{g(\nu)}) \circ F_0^* = \alpha \circ \mathcal{S}^*(q_{g(\nu)}) \circ F^* = \mathcal{S}^*(r_\nu) \circ G_0^* \circ F^* = \mathcal{S}^*(r_\nu) \circ G^* \circ F^*$. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.5. *Let X and Y be topological spaces and let $F^* : X \longrightarrow Y$ be a coarse shape morphism. Let Z be a topological space and consider $\hat{F}^* : Sh^*(Y,Z) \longrightarrow Sh^*(X,Z)$ and $\tilde{F}^* : Sh^*(Z,X) \longrightarrow Sh^*(Z,Y)$ to be defined by $\hat{F}^*(H^*) = H^* \circ F^*$ and $\tilde{F}^*(G^*) = F^* \circ G^*$.*

(i) $\hat{F}^*$ and $\tilde{F}^*$ are continuous, $(\widetilde{G^* \circ F^*}) = \tilde{G}^* \circ \tilde{F}^*$, $(\widehat{G^* \circ F^*}) = \hat{F}^* \circ \hat{G}^*$ and $\hat{Id}^*$ and $\tilde{Id}^*$ are the corresponding identity maps.
(ii) Assume $Sh^*(X) \geq Sh^*(Y)$. Then $Sh^*(Y,Z)$ is homeomorphic to a retract of $Sh^*(X,Z)$ and $Sh^*(Z,Y)$ is homeomorphic to a retract of $Sh^*(Z,X)$, for every topological space Z.
(iii) Assume $Sh^*(X) = Sh^*(Y)$. Then $Sh^*(Y,Z)$ is homeomorphic to $Sh^*(X,Z)$ and $Sh^*(Z,Y)$ is homeomorphic to $Sh^*(Z,X)$, for every topological space Z.

Now, we want to prove the following theorem which is useful to study the topological properties of the space of coarse shape morphisms.

Theorem 3.6. *Let X and Y be topological spaces and let $\mathbf{p} : X \longrightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : Y \longrightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be HPol-expansions of X and Y, respectively. Take $\mathbf{Sh}^*(X,Y) = (Sh^*(X,Y_\mu), (q_{\mu\mu'})_*, M)$ and consider the morphism $\mathbf{q}_* : Sh^*(X,Y) \longrightarrow \mathbf{Sh}^*(X,Y)$ induced by $\mathbf{q}$. Then $\mathbf{q}_*$ is an inverse limit of $\mathbf{Sh}^*(X,Y)$ in Top.*

Proof. Let Z be a topological space and let $\mathbf{g} : Z \longrightarrow (Sh^*(X,Y_\mu), (q_{\mu\mu'})_*, M)$ be a morphism in pro-Top. We must show that there is a unique continuous map $\alpha : Z \longrightarrow Sh^*(X,Y)$ such that $\mathbf{q}_* \circ \alpha = \mathbf{g}$ in pro-Top. We know that $g_\mu(z) \in Sh^*(X,Y_\mu)$, for every $z \in Z$ and $\mu \in M$. Suppose $g_\mu(z) =< [(g_{\mu,z}^n, const = \lambda_{\mu,z})] >$ and define $h^z : M \longrightarrow \Lambda$ by $h^z(\mu) = \lambda_{\mu,z}$. We define $\alpha(z) =< [(g_{\mu,z}^n, h^z)] >$. Since $(q_{\mu\mu'})_* \circ g_{\mu'} = g_\mu$, so for every $z \in Z$, $((q_{\mu\mu'})_* \circ g_{\mu'})(z) = g_\mu(z)$. Thus, there is a $\lambda \geq \lambda_{\mu,z}, \lambda_{\mu',z}$ and $n \in \mathbb{N}$ such that for every $n' \geq n$, $q_{\mu\mu'} \circ g_{\mu',z}^{n'} \circ p_{\lambda_{\mu',z}\lambda} = g_{\mu,z}^{n'} \circ p_{\lambda_{\mu,z}\lambda}$. It follows that $\alpha(z)$ is a coarse shape morphism. It is clear that $\mathbf{q}_* \circ \alpha = \mathbf{g}$. To complete the proof, we show that α is continuous. Let $z \in Z$, $\mu \in M$ and $F^* = \alpha(z) \in V_\mu^{F^*}$. We have $\alpha^{-1}(V_\mu^{F^*}) = \{z' \in Z : \alpha(z') \in V_\mu^{F^*}\} = \{z' \in Z : (q_\mu)_* \circ \alpha(z') = (q_\mu)_* \circ \alpha(z)\} = \{z' \in Z : g_\mu(z) = g_\mu(z')\} = g_\mu^{-1}(g_\mu(z))$. Since $Sh^*(X,Y_\mu)$ is discrete, $\{g_\mu(z)\}$ is an open subset of $Sh^*(X,Y_\mu)$ and since g_μ is continuous, we have $g_\mu^{-1}(g_\mu(z))$ is open subset of Z. It follows that α is continuous. □

Corollary 3.7. *Let X and Y be two topological spaces. Then $Sh^*(X,Y)$ is a Tychonoff space.*

Suppose that $(M, \leq)$ is a directed set. From [9], we denote by $L(M)$ the set of all lower classes in M ordered by inclusion, in which $\Delta \subseteq M$ is called a lower class if for every $\delta \in \Delta$ and $\mu \in M$ with $\mu \leq \delta$, then $\mu \in \Delta$. Moreover, for any two lower classes $\Delta, \Delta' \in L(M)$, we say that $\Delta \leq \Delta'$ if and only if $\Delta \supset \Delta'$. Then $(L(M), \leq)$ is a partially ordered set with the least element M which is denoted by 0. Furthermore, $L(M)^* = L(M) - 0$ is downward directed (see [9, proposition 2.1]).

Let X be a set and $(\Gamma, \leq)$ be a partial ordered set with a least element 0. Recall from [17] that a Γ-ultrametric on X is a map $d : X \times X \to \Gamma$ such that for all $x, y \in X$ and $\gamma \in \Gamma$, the following hold:

1) $d(x,y) = 0 \Longleftrightarrow x = y$.
2) $d(x,y) = d(y,x)$.
3) If $d(x,y) \leq \gamma$ and $d(y,z) \leq \gamma$, then $d(x,z) \leq \gamma$.

Now, using the same idea as in [9], we can prove the following theorem:

Theorem 3.8. *Let X and Y be topological spaces. Assume $\mathbf{q} : Y \longrightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is an HPol-expansion of Y. For every $F^*, G^* \in Sh^*(X,Y)$ take*

$$d(F^*, G^*) = \{\mu \in M : \mathcal{S}^*(q_\mu) \circ F^* = \mathcal{S}^*(q_\mu) \circ G^*\}.$$

Then we have an $L(M)$-ultrametric $d : Sh^*(X,Y) \times Sh^*(X,Y) \to (L(M), \leq)$.

Proof. First, we show that $d(F^*, G^*)$ is a lower class. Suppose $\mu \in d(F^*, G^*)$ and $\mu' \in M$ such that $\mu' \leq \mu$. Then $q_{\mu'} = q_{\mu'\mu} q_\mu$ and we have

$$\mathcal{S}^*(q_{\mu'}) \circ F^* = \mathcal{S}^*(q_{\mu'\mu}) \circ \mathcal{S}^*(q_\mu) \circ F^* = \mathcal{S}^*(q_{\mu'\mu}) \circ \mathcal{S}^*(q_\mu) \circ G^* = \mathcal{S}^*(q_{\mu'}) \circ G^*.$$

It follows that $\mu' \in d(F^*, G^*)$. Now, let $F^*, G^* \in Sh^*(X,Y)$ such that $d(F^*, G^*) = 0$. It is equivalent to $\mathcal{S}^*(q_\mu) \circ F^* = \mathcal{S}^*(q_\mu) \circ G^*$, for every $\mu \in M$ or equivalently $F^* = G^*$. Other conditions can also be proved easily. □

Let $(M, \leq)$ be a directed set and $(L(M), \leq)$ be the corresponding ordered set of lower classes in M. For every $\mu \in M$, consider $\{\mu' \in M : \mu \geq \mu'\}$ as the lower class generated by μ, which is denote by $[\mu]$ and define $\phi : (M, \leq) \to (L(M), \leq)$ that maps μ to $[\mu]$. If $\mu \geq \mu'$, then $[\mu] \leq [\mu']$ and $(\phi(M), \leq)$ is a partial ordered set and also is downward directed in $L(M)$ (see [9]).

Now, we have:

Proposition 3.9. *Let X and Y be topological spaces. Suppose $\mathbf{q} : Y \longrightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is an HPol-expansion of Y. For every $\mu \in M$ and $F^* \in Sh^*(X,Y)$ take*

$$B_{[\mu]}(F^*) = \{G^* \in Sh^*(X,Y) : d(F^*, G^*) \leq [\mu]\}.$$

Then the family $\{B_{[\mu]}(F^*) : F^* \in Sh^*(X,Y), \mu \in M\}$ is a basis for a topology in $Sh^*(X,Y)$. Moreover, this topology is independent of the fixed HPol-expansion of Y and it coincides with the topology defined previously.

Proof. It is obvious that $F^* \in B_{[\mu]}(F^*)$, for all $\mu \in M$. Suppose $F^*, G^* \in Sh^*(X,Y)$ and $\mu_1, \mu_2 \in M$ and $H^* \in B_{[\mu_1]}(F^*) \cap B_{[\mu_2]}(G^*)$. Therefore $d(H^*, F^*) \leq [\mu_1]$ and $d(H^*, G^*) \leq [\mu_2]$. Let $K^* \in B_{[\mu_1]}(H^*)$, then we have $d(K^*, H^*) \leq [\mu_1]$ and $d(H^*, F^*) \leq [\mu_1]$ and so by the definition, $d(K^*, F^*) \leq [\mu_1]$. It shows that $K^* \in B_{[\mu_1]}(F^*)$ and $B_{[\mu_1]}(H^*) \subseteq B_{[\mu_1]}(F^*)$. Conversely, we can show that $B_{[\mu_1]}(F^*) \subseteq B_{[\mu_1]}(H^*)$ and hence we have $B_{[\mu_1]}(H^*) = B_{[\mu_1]}(F^*)$. Similarly, we can conclude that $B_{[\mu_2]}(H^*) = B_{[\mu_2]}(G^*)$. Hence, to prove the first assertion, it is enough to consider $\mu \in M$ such that $\mu \geq \mu_1, \mu_2$, then $[\mu] \leq [\mu_1], [\mu_2]$ and this easily implies that $H^* \in B_{[\mu]}(H^*) \subseteq B_{[\mu_1]}(H^*) \cap B_{[\mu_2]}(H^*) = B_{[\mu_1]}(F^*) \cap B_{[\mu_2]}(G^*)$.

Now, suppose that $\mathbf{q}' : Y \to \mathbf{Y}' = (Y_\nu, q_{\nu\nu'}, N)$ is another HPol-expansion of Y. Then there exists a unique isomorphism $\mathbf{j} : \mathbf{Y} \longrightarrow \mathbf{Y}'$ given by (j_ν, ϕ) such that $\mathbf{j} \circ \mathbf{q} = \mathbf{q}'$ (we can assume that ϕ is an increasing map). Let $\nu \in N$ and $F^* \in Sh^*(X, Y)$, then $\phi(\nu) \in M$. Given $G^* \in B_{[\nu]}(F^*)$, so by the above argument $B_{[\nu]}(F^*) = B_{[\nu]}(G^*)$. For each $H^* \in B_{[\phi(\nu)]}(G^*)$, we have $d(G^*, H^*) \leq [\phi(\nu)]$, i.e., if $\mu \leq \phi(\nu)$, then $\mathcal{S}^*(q_\mu) \circ G^* = \mathcal{S}^*(q_\mu) \circ H^*$. Given $\nu' \in N$ such that $\nu' \leq \nu$, then $\phi(\nu') \leq \phi(\nu)$ and so $\mathcal{S}^*(q_{\phi(\nu')}) \circ G^* = \mathcal{S}^*(q_{\phi(\nu')}) \circ H^*$. It implies that $\mathcal{S}^*(q'_{\nu'}) \circ G^* = \mathcal{S}^*(q'_{\nu'}) \circ H^*$ and thus $H^* \in B_{[\nu]}(G^*)$. Therefore $G^* \in B_{[\phi(\nu)]}(G^*) \subseteq B_{[\nu]}(G^*) = B_{[\nu]}(F^*)$ and it follows that the topology corresponding to HPol-expansion $\mathbf{q}$ is stronger than the topology corresponding to HPol-expansion $\mathbf{q}'$. Similarly, we can prove that the converse is true.

Finally, we want to show that the topology induced by d coincides with the topology T_q studied in Proposition 3.1. It is easy to see that $V_\mu^{F^*} = B_{[\mu]}(F^*)$, for every $\mu \in M$ and $F^* \in Sh^*(X, Y)$ which completes the proof. □

4. The topological coarse shape homotopy groups

Let X be a topological space and $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an HPol-expansion of X. We know that the kth coarse shape group $\check{\pi}_k^*(X, x)$, $k \in \mathbb{N}$, is the set of all coarse shape morphisms $F^* : (S^k, *) \to (X, x)$ with the following binary operation

$$F^* + G^* =< \mathbf{f}^* > + < \mathbf{g}^* >=< \mathbf{f}^* + \mathbf{g}^* >=< [(f_\lambda^n)] + [(g_\lambda^n)] >=< [(f_\lambda^n + g_\lambda^n)] >,$$

where coarse shape morphisms F^* and G^* are represented by morphisms $\mathbf{f}^* = [(f, f_\lambda^n)]$ and $\mathbf{g}^* = [(g, g_\lambda^n)] : (S^k, *) \to (\mathbf{X}, \mathbf{x})$ in pro*-HPol$_*$, respectively (see [2]).

Now, we show that $\check{\pi}_k^*(X, x) = Sh^*((S^k, *), (X, x))$ with the above topology is a topological group which is denoted by $\check{\pi}_k^{*^{top}}(X, x)$, for all $k \in \mathbb{N}$.

Theorem 4.1. *Let (X, x) be a pointed topological space. Then $\check{\pi}_k^{*^{top}}(X, x)$ is a topological group, for all $k \in \mathbb{N}$.*

Proof. First, we show that $\phi : \check{\pi}_k^{*^{top}}(X, x) \to \check{\pi}_k^{*^{top}}(X, x)$ given by $\phi(F^*) = F^{*^{-1}}$ is continuous, where F^* and $F^{*^{-1}} : (S^k, *) \to (X, x)$ are represented by $\mathbf{f}^* = (f, f_\lambda^n)$ and $\mathbf{f}^{*^{-1}} = (f, f_\lambda^{n^{-1}}) : (S^k, *) \to (\mathbf{X}, \mathbf{x})$, respectively and $f_\lambda^{n^{-1}} : (S^k, *) \to (X_\lambda, x_\lambda)$ is the inverse loop of f_λ^n. Let $V_\lambda^{F^{*^{-1}}}$ be an open neighbourhood of $F^{*^{-1}}$ in $\check{\pi}_k^{*^{top}}(X, x)$. We know that for any $G^* =< [(g, g_\lambda^n)] >\in V_\lambda^{F^*}$, $\mathcal{S}^*(p_\lambda) \circ G^* = \mathcal{S}^*(p_\lambda) \circ F^*$. So there is an $n' \in \mathbb{N}$ such that for any $n \geq n'$, $g_\lambda^n \simeq f_\lambda^n$ rel $\{*\}$ by [1, Claim 1 and Claim 2]. Then for any $n \geq n'$, $g_\lambda^{n^{-1}} \simeq f_\lambda^{n^{-1}}$ rel $\{*\}$ and so $\mathcal{S}^*(p_\lambda) \circ G^{*^{-1}} = \mathcal{S}^*(p_\lambda) \circ F^{*^{-1}}$. Thus $\phi(G^*) \in V_\lambda^{F^{*^{-1}}}$. Therefore, the map ϕ is continuous.

Second, we show that the map $m : \check{\pi}_k^{*^{top}}(X, x) \times \check{\pi}_k^{*^{top}}(X, x) \to \check{\pi}_k^{*^{top}}(X, x)$ given by $m(F^*, G^*) = F^* + G^*$ is continuous, where $F^* + G^*$ is the coarse shape morphism represented by $\mathbf{f}^* + \mathbf{g}^* = (f, f_\lambda^n + g_\lambda^n) : (S^k, *) \to (\mathbf{X}, \mathbf{x})$ and $f_\lambda^n + g_\lambda^n$ is the concatenation of paths. Let $V_\lambda^{F^*+G^*}$ be an open neighbourhood of $F^* + G^*$ in $\check{\pi}_k^{*^{top}}(X, x)$. For any $(K^*, H^*) \in V_\lambda^{F^*} \times V_\lambda^{G^*}$, we have $\mathcal{S}^*(p_\lambda) \circ (K^* + H^*) = (\mathcal{S}^*(p_\lambda) \circ K^*) + (\mathcal{S}^*(p_\lambda) \circ H^*) = (\mathcal{S}^*(p_\lambda) \circ F^*) + (\mathcal{S}^*(p_\lambda) \circ G^*) = \mathcal{S}^*(p_\lambda) \circ (F^* + G^*)$. Hence $m(K^*, H^*) \in V_\lambda^{F^*+G^*}$ and so m is continuous. □

Using Corollary 3.5, we can conclude the following results:

Corollary 4.2. *If $F^* : (X,x) \to (Y,y)$ is a coarse shape morphism, then $\tilde{F^*} : \check{\pi}_k^{*^{top}}(X,x) \to \check{\pi}_k^{*^{top}}(Y,y)$ is continuous.*

Corollary 4.3. *If (X,x) and (Y,y) are two pointed topological spaces and $Sh^*(X,x) = Sh^*(Y,y)$, then $\check{\pi}_k^{*^{top}}(X,x) \cong \check{\pi}_k^{*^{top}}(Y,y)$ as topological groups.*

Corollary 4.4. *For any $k \in \mathbb{N}$, $\check{\pi}_k^{*^{top}}(-)$ is a functor from the pointed coarse shape category of spaces to the category of Hausdorff topological groups.*

Bilan in [4], showed that every coarse shape group can be obtained as the inverse limit of an inverse system of groups and also proved that, for an inverse systems of compact polyhedra, the coarse shape group functor commutes with the inverse limit. Now, we generalise this result for topological coarse shape groups in the following:

Corollary 4.5. *Let X be a topological space and $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an HPol-expansion of X. By Theorem 3.6, we know that $\check{\pi}_k^{*^{top}}(X,x) \cong \underleftarrow{\lim}\, \check{\pi}_k^{*^{top}}(X_\lambda, x_\lambda)$ as topological groups, for all $k \in \mathbb{N}$. Since every $\check{\pi}_k^{*^{top}}(X_\lambda, x_\lambda)$ is discrete and Hausdorff, $\check{\pi}_k^{*^{top}}(X,x)$ is prodiscrete and Hausdorff, for every topological space (X,x).*

Corollary 4.6. *Let $(X,x) = \underleftarrow{\lim}(X_i, x_i)$, where X_i's are compact polyhedra. Then for all $k \in \mathbb{N}$,*

$$\check{\pi}_k^{*^{top}}(X,x) \cong \underleftarrow{\lim}\, \check{\pi}_k^{*^{top}}(X_i, x_i).$$

Proof. It can be proved similarly to the Corollary 3.8 in [22]. □

Corollary 4.7. *Let $\mathbf{p} : (X,x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ be an HPol$_*$-expansion of a pointed topological space (X,x). Then the following statements hold for all $k \in \mathbb{N}$:*

(i) If the cardinal number of Λ is $\aleph_0$ and $\check{\pi}_k^{*^{top}}(X_\lambda, x_\lambda)$ is second countable for every $\lambda \in \Lambda$, then $\check{\pi}_k^{*^{top}}(X,x)$ is second countable.
(ii) If $\check{\pi}_k^{*^{top}}(X_\lambda, x_\lambda)$ is totally disconnected for every $\lambda \in \Lambda$, then so is $\check{\pi}_k^{*^{top}}(X,x)$.

Proof. The results follow from the fact that the product and the subspace topologies preserve the properties of being second countable and totally disconnected. □

Remark 4.8. The authors proved a similar result to the above corollary for shape homotopy groups [22, Corollary 3.9]. Note that in that case, we can omit the assumption of second countability of $\pi_k^{qtop}(X_\lambda, x_\lambda)$, for all $\lambda \in \Lambda$. Indeed, if X is a polyhedron, so X is second countable and hence $\Omega^k(X,x)$ is second countable, for all $x \in X$ and $k \in \mathbb{N}$ (see [11]). Since $\pi_k^{qtop}(X,x)$ is discrete, then the map $q : \Omega^k(X,x) \to \pi_k^{qtop}(X,x)$ is a bi-quotient map and therefore $\pi_k^{qtop}(X,x)$ is also second countable, for all $k \in \mathbb{N}$ (see [20]).

Let X be a topological space and let $x_0, x_1 \in X$. A coarse shape path in X from x_0 to x_1 is a bi-pointed coarse shape morphism $\Omega^* : (I, 0, 1) \to (X, x_0, x_1)$. X is said to be coarse shape path connected, if for every pair $x, x' \in X$, there is a coarse shape path from x to x'. If X is a coarse shape path connected space, then $\check{\pi}_k^*(X,x) \cong \check{\pi}_k^*(X,x')$, for any two points $x, x' \in X$ and every $k \in \mathbb{N}$ [3, Corollary 1].

Now, we show that these two groups are isomorphic as topological groups, if X is a coarse shape path connected, paracompact and locally compact space.

Theorem 4.9. *Let X be a coarse shape path connected, paracompact and locally compact space. Then $\check{\pi}_k^{*^{top}}(X,x) \cong \check{\pi}_k^{*^{top}}(X,x')$, for every pair $x,x' \in X$ and all $k \in \mathbb{N}$.*

Proof. If X is a topological space admitting a metrizable polyhedral resolution and for a pair $x,x' \in X$ there exists a coarse shape path in X from x to x', then (X,x) and (X,x') are isomorphic pointed spaces in $Sh^*_\star$ (see [3, Theorem 3]). Since coarse shape path connected, paracompact and locally compact spaces satisfy in the above condition [25], $Sh^*(X,x) \cong Sh^*(X,x')$. Hence by Corollary 4.3 we have $\check{\pi}_k^{*^{top}}(X,x) \cong \check{\pi}_k^{*^{top}}(X,x')$, for every pair $x,x' \in X$ and all $k \in \mathbb{N}$. □

5. Main results

It is well-known that if the Cartesian product of two spaces X and Y admits an HPol-expansion, which is the Cartesian product of HPol-expansions of these space, then $X \times Y$ is a product in the shape category (see [18]). In this case, the authors showed that the kth topological shape group commutes with finite products, for all $k \in \mathbb{N}$ [22, Theorem 4.1].

Also, if X and Y admit HPol-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$, respectively, such that $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is an HPol-expansion, then $X \times Y$ is a product in the coarse shape category [23, Theorem 2.2]. Mardešić [18] proved that if $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ are HPol-expansions of compact Hausdorff spaces X and Y, respectively, then $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is an HPol-expansion and so in this case, $X \times Y$ is a product in the coarse shape category.

Now, we show that under the above condition, the kth topological coarse shape group commutes with finite products, for all $k \in \mathbb{N}$.

Theorem 5.1. *If X and Y are coarse shape path connected spaces with HPol-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ such that $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is an HPol-expansion, then $\check{\pi}_k^{*^{top}}(X \times Y) \cong \check{\pi}_k^{*^{top}}(X) \times \check{\pi}_k^{*^{top}}(Y)$, for all $k \in \mathbb{N}$.*

Proof. Let $\mathcal{S}^*(\pi_X) : X \times Y \to X$ and $\mathcal{S}^*(\pi_Y) : X \times Y \to Y$ be the induced coarse shape morphisms of canonical projections and assume that $\phi_X : \check{\pi}_k^{*^{top}}(X \times Y) \to \check{\pi}_k^{*^{top}}(X)$ and $\phi_Y : \check{\pi}_k^{*^{top}}(X \times Y) \to \check{\pi}_k^{*^{top}}(Y)$ are the induced continuous homomorphisms by $\mathcal{S}^*(\pi_X)$ and $\mathcal{S}^*(\pi_Y)$, respectively. Then the induced homomorphism $\phi : \check{\pi}_k^{*^{top}}(X \times Y) \to \check{\pi}_k^{*^{top}}(X) \times \check{\pi}_k^{*^{top}}(Y)$ is continuous. Since $X \times Y$ is a product in Sh*, we can define a homomorphism $\psi : \check{\pi}_k^{*^{top}}(X) \times \check{\pi}_k^{*^{top}}(Y) \to \check{\pi}_k^{*^{top}}(X \times Y)$ by $\psi(F^*, G^*) = \lfloor F^*, G^* \rfloor$, where $\lfloor F^*, G^* \rfloor : S^k \to X \times Y$ is a unique coarse shape morphism with $\mathcal{S}^*(\pi_X)(\lfloor F^*, G^* \rfloor) = F^*$ and $\mathcal{S}^*(\pi_Y)(\lfloor F^*, G^* \rfloor) = G^*$. In fact, if $F^* = \langle \mathbf{f}^* = (f, f_\lambda^n) \rangle$ and $G^* = \langle \mathbf{g}^* = (g, g_\mu^n) \rangle$, then $\lfloor F^*, G^* \rfloor = \langle \lfloor \mathbf{f}^*, \mathbf{g}^* \rfloor \rangle$, where $\lfloor \mathbf{f}^*, \mathbf{g}^* \rfloor$ is given by $\lfloor f, g \rfloor_{\lambda\mu}^n = f_\lambda^n \times g_\mu^n : S^k \to X_\lambda \times Y_\mu$. By the proof of [23, Theorem 2.4], the homomorphism ψ is well define and moreover, $\phi \circ \psi = id$ and $\psi \circ \phi = id$.

To complete the proof, it is enough to show that ψ is continuous. Let $\lfloor F^*, G^* \rfloor \in V_{\lambda\mu}^{\lfloor F^*, G^* \rfloor}$ be a basis open in the topology on $\check{\pi}_k^{*^{top}}(X \times Y)$. Considering open sets $F^* \in V_\lambda^{F^*}$ and $G^* \in V_\mu^{G^*}$, we show that $\psi(V_\lambda^{F^*} \times V_\mu^{G^*}) \subseteq V_{\lambda\mu}^{\lfloor F^*, G^* \rfloor}$. Let $H^* \in V_\lambda^{F^*}$ and $K^* \in V_\mu^{G^*}$, then $\mathcal{S}^*(p_\lambda) \circ H^* = \mathcal{S}^*(p_\lambda) \circ F^*$ and $\mathcal{S}^*(q_\mu) \circ K^* = \mathcal{S}^*(q_\mu) \circ G^*$. By a straight computation, we can conclude that $p_\lambda \times q_\mu(\lfloor H^*, K^* \rfloor) = p_\lambda \times q_\mu(\lfloor F^*, G^* \rfloor)$ which implies that $\psi(H^*, K^*) = \lfloor H^*, K^* \rfloor \in V_{\lambda\mu}^{\lfloor F^*, G^* \rfloor}$. □

Theorem 5.2. *Let (X,x) be a pointed topological space. Then for all $k \in \mathbb{N}$,*

(i) If $(X,x) \in HPol_*$, then $\check{\pi}_k^{*^{top}}(X,x)$ is discrete.
(ii) If $\mathbf{p} : (X,x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ is an $HPol_*$-expansion of (X,x) and $\check{\pi}_k^{*^{top}}(X,x)$ is discrete, then $\check{\pi}_k^{*^{top}}(X,x) \leq \check{\pi}_k^{*^{top}}(X_\lambda, x_\lambda)$, for some $\lambda \in \Lambda$.

Proof. (i) This follows from Corollary 3.2.

(ii) Since $\check{\pi}_k^{*^{top}}(X,x)$ is a discrete group, $\{E_x^*\}$ is an open set of identity point of $\check{\pi}_k^{*^{top}}(X,x)$. Thus $\{E_x^*\} = \cup_{\lambda\in\Lambda_0} V_\lambda^{F^*}$, where $\Lambda_0 \subseteq \Lambda$. Consider the induced homomorphism $p_{\lambda *} : \check{\pi}_k^{*^{top}}(X,x) \to \check{\pi}_k^{*^{top}}(X_\lambda, x_\lambda)$ given by $p_{\lambda *}(F^*) = \mathcal{S}^*(p_\lambda) \circ F^*$. Let $G^* \in kerp_{\lambda *}$, i.e., $\mathcal{S}^*(p_\lambda) \circ G^* = E_{x_\lambda}^* = \mathcal{S}^*(p_\lambda) \circ E_x^*$. Thus $G^* \in V_\lambda^{E_x^*} \subseteq \cup_{\lambda\in\Lambda_0} V_\lambda^{F^*} = \{E_x^*\}$ and so $G^* = E_x^*$. Therefore $p_{\lambda *}$ is injective, for all $\lambda \in \Lambda_0$ and $k \in \mathbb{N}$. □

Recall that an inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-HTop is said to be movable if every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ such that each $\lambda'' \geq \lambda$ admits a morphism $r : X_{\lambda'} \to X_{\lambda''}$ of HTop with $p_{\lambda\lambda''} \circ r \simeq p_{\lambda\lambda'}$. We say that a topological space X is movable provided that it admits an HPol-expansion $\mathbf{p} : X \to \mathbf{X}$ such that $\mathbf{X}$ is a movable inverse system of pro-HPol [19]. We know that under some conditions, movability can be transferred from a pointed topological space (X,x) to $\check{\pi}_k^{top}(X,x)$ (see [22]) and now we show that it can be transferred to $\check{\pi}_k^{*^{top}}(X,x)$ too.

Lemma 5.3. *If* $(\mathbf{X},\mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ *is a movable (uniformly movable) inverse system, then the inverse system*

$$\mathbf{Sh}^*((S^k,*),(X,x)) = (Sh^*((S^k,*),(X_\lambda,x_\lambda)),(p_{\lambda\lambda'})_*,\Lambda)$$

in pro-Top is also movable (uniformly movable), for all $k \in \mathbb{N}$.

Proof. Let $\lambda \in \Lambda$. Since $(\mathbf{X},\mathbf{x})$ is a movable inverse system, there is a $\lambda' \geq \lambda$ such that for every $\lambda'' \geq \lambda$ there is a map $r : (X_{\lambda'}, x_{\lambda'}) \to (X_{\lambda''}, x_{\lambda''})$ such that $p_{\lambda\lambda''} \circ r \simeq p_{\lambda\lambda'}$ rel $\{x_{\lambda'}\}$. We consider $r_* : Sh^*((S^k,*),(X_{\lambda'},x_{\lambda'})) \to Sh^*((S^k,*),(X_{\lambda''},x_{\lambda''}))$. Hence $(p_{\lambda\lambda''})_* \circ r_* = (p_{\lambda\lambda'})_*$ and so $\mathbf{Sh}^*((S^k,*),(X,x))$ is movable. □

Remark 5.4. Let (X,x) be a movable space. Then there exists an HPol$_*$-expansion $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x})$ such that $(\mathbf{X},\mathbf{x})$ is a movable inverse system. Suppose $\mathbf{p}_* : Sh^*((S^k,*),(X,x)) \to \mathbf{Sh}^*((S^k,*),(X,x))$ is an HPol$_*$-expansion, then using Lemma 5.3, we can conclude that $\check{\pi}_k^{*^{top}}(X,x)$ is a movable topological group, for all $k \in \mathbb{N}$. By Theorem 3.6, if $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x})$ is an HPol$_*$-expansion of X, then $\mathbf{p}_* : Sh^*((S^k,*),(X,x)) \to \mathbf{Sh}^*((S^k,*),(X,x))$ is an inverse limit of $\mathbf{Sh}^*((S^k,*),(X,x)) = (Sh^*((S^k,*),(X_\lambda,x_\lambda)),(p_{\lambda\lambda'})_*,\Lambda)$. Now, if $Sh^*((S^k,*),(X_\lambda,x_\lambda))$ is a compact polyhedron for all $\lambda \in \Lambda$, then by [14, Remark 1] $\mathbf{p}_*$ is an HPol$_*$-expansion of $Sh^*((S^k,*),(X,x))$ and therefore in this case, movability of (X,x) implies movability of $\check{\pi}_k^{*^{top}}(X,x)$.

Remark 5.5. Suppose (X,x) is a topological space and $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x}) = ((X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda)$ is an HPol$_*$-expansion of (X,x). Consider $J : \check{\pi}_k^{top}(X,x) \longrightarrow \check{\pi}_k^{*^{top}}(X,x)$ given by $J(F =< (f,f_\lambda) >) = F^*$, where $F^* =< (f, f_\lambda^n = f_\lambda) >$. Then J is an embedding. To prove this, we show that for each $\lambda \in \Lambda$ and for all $F \in \check{\pi}_k^{top}(X,x)$, $J(V_\lambda^F) = V_\lambda^{J(F)} \cap J(\check{\pi}_k^{top}(X,x))$. Suppose $G =< (g,g_\lambda) > \in V_\lambda^F$, so $\mathcal{S}^*(p_\lambda) \circ G = \mathcal{S}^*(p_\lambda) \circ F$ or equivalently $g_\lambda \simeq f_\lambda$. We know that $J(G) =< g_\lambda^n = g_\lambda >$ and $J(F) =< f_\lambda^n = f_\lambda >$. So for all $n \in \mathbb{N}$, $g_\lambda^n \simeq f_\lambda^n$ and it follows that $\mathcal{S}^*(p_\lambda) \circ J(G) = \mathcal{S}^*(p_\lambda) \circ J(F)$. Hence $J(G) \in V_\lambda^{J(F)} \cap J(\check{\pi}_k^{top}(X,x))$. Conversely, suppose that $G^* =< g_\lambda'^n > \in V_\lambda^{J(F)} \cap J(\check{\pi}_k^{top}(X,x))$. Since $G^* \in J(\check{\pi}_k^{top}(X,x))$, there exists a $G \in \check{\pi}_k^{top}(X,x)$ such that $J(G) = G^*$. If $G =< g_\lambda >$, then $J(G) =< g_\lambda^n = g_\lambda >$. Since $J(G) = G^*$, we can conclude that there is an $n' \in \mathbb{N}$ such that for every $n \geq n'$, $g_\lambda'^n \simeq g_\lambda$. On the other hand, we have $\mathcal{S}^*(p_\lambda) \circ G^* = \mathcal{S}^*(p_\lambda) \circ J(F)$, i.e., there is an $n'' \in \mathbb{N}$ such that for every $n \geq n''$, $g_\lambda'^n \simeq f_\lambda$. It follows that $g_\lambda \simeq f_\lambda$ and hence $\mathcal{S}^*(p_\lambda) \circ G = \mathcal{S}^*(p_\lambda) \circ F$. Therefore, $G \in V_\lambda^F$ and $G^* = J(G) \in J(V_\lambda^F)$. Hence $J(V_\lambda^F) = V_\lambda^{J(F)} \cap J(\check{\pi}_k^{top}(X,x))$ which completes the proof.

Let (X,x) be a topological space. We know that the induced homomorphism $\phi : \pi_k^{qtop}(X,x) \to \check{\pi}_k^{top}(X,x)$ is continuous, for all $k \in \mathbb{N}$. Consider the composition $J \circ \phi : \pi_k^{qtop}(X,x) \to \check{\pi}_k^{*^{top}}(X,x)$ in which J is the embedding defined in Remark 5.5. If (X,x) is shape injective, then the homomorphism ϕ is an embedding and hence we have an embedding from $\pi_k^{qtop}(X,x)$ to $\check{\pi}_k^{*^{top}}(X,x)$.

Let $X \subseteq Y$ and $r : Y \to X$ be a retraction. Consider the inclusion map $j : X \to Y$. It is known that $j_* : \check{\pi}_k^{top}(X,x) \to \check{\pi}_k^{top}(Y,x)$ is a topological embedding [22, Theorem 4.2] and similarly to the proof of it, we can conclude that the induced map $j_* : \check{\pi}_k^{*^{top}}(X,x) \to \check{\pi}_k^{*^{top}}(Y,x)$ is also a topological embedding.

In the following, we present examples whose topological coarse shape homotopy groups are not discrete.

Example 5.6. Let $(HE, p = (0,0)) = \varprojlim (X_i, p_i)$ be the Hawaiian Earring where $X_j = \vee_{i=1}^{j} S_i^1$. The first shape homotopy group $\check{\pi}_1^{top}(HE,p)$ is not discrete (see [22, Example 4.5]). So the above Remark follows that $\check{\pi}_1^{*^{top}}(HE,p)$ is not discrete.

Example 5.7. Let $k \in \mathbb{N}$ and let $\mathbf{X} = (X_n, p_{nn+1}, \mathbb{N})$, where $X_n = \prod_{j=1}^{n} S_j^k$ is the product of n copies of k-sphere S^k, for all $n \in \mathbb{N}$ and the bonding morphisms of $\mathbf{X}$ are the projection maps. Put $X = \varprojlim X_n$. Refer to [22], $\check{\pi}_k^{top}(X) \cong \varprojlim \pi_k^{qtop}(X_n) \cong \prod \mathbb{Z}$ is not discrete. Since $\check{\pi}_k^{top}(X)$ is a subspace of $\check{\pi}_k^{*^{top}}(X)$ and it is not discrete, then $\check{\pi}_k^{*^{top}}(X)$ is not discrete.

Acknowledgements

This research was supported by a grant from Ferdowsi University of Mashhad-Graduate Studies (No. 41763).

References

[1] N.K. Bilan, N. Uglesic, The coarse shape, Glas. Mat. 42 (2007) 145–187.
[2] N.K. Bilan, The coarse shape groups, Topol. Appl. 157 (2010) 894–901.
[3] N.K. Bilan, Towards the algebraic characterization of (coarse) shape path connectedness, Topol. Appl. 160 (2013) 538–545.
[4] N.K. Bilan, Continuity of coarse shape groups, Homol. Homotopy Appl. 18 (2016) 209–215.
[5] D. Biss, The topological fundamental group and generalized covering spaces, Topol. Appl. 124 (2002) 355–371.
[6] J. Brazas, The topological fundamental group and free topological groups, Topol. Appl. 158 (2011) 779–802.
[7] J. Brazas, The fundamental group as topological group, Topol. Appl. 160 (2013) 170–188.
[8] J.S. Calcut, J.D. McCarthy, Discreteness and homogeneity of the topological fundamental group, Topol. Proc. 34 (2009) 339–349.
[9] E. Cuchillo-Ibanez, M.A. Morón, F.R. Ruiz del Portal, Ultrametric spaces, valued and semivalued groups arising from the theory of shape, Mathematical contributions in honor of Juan Tarrés, Univ. Complut. Madr., Fac. Mat. (2012) 81–92 (in Spanish).
[10] E. Cuchillo-Ibanez, M.A. Morón, F.R. Ruiz del Portal, J.M.R. Sanjurjo, A topology for the sets of shape morphisms, Topol. Appl. 94 (1999) 51–60.
[11] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston, Mass, 1966.
[12] P. Fabel, Multiplication is discontinuous in the Hawaiian Earring group (with the quotient topology), Bull. Pol. Acad. Sci., Math. 59 (2011) 77–83.
[13] P. Fabel, Compactly generated quasi-topological homotopy groups with discontinuous mutiplication, Topol. Proc. 40 (2012) 303–309.
[14] H. Fischer, A. Zastrow, The fundamental groups of subsets of closed surfaces inject into their first shape groups, Algebraic Geom. Topol. 5 (2005) 1655–1676.
[15] H. Ghane, Z. Hamed, B. Mashayekhy, H. Mirebrahimi, Topological homotopy groups, Bull. Belg. Math. Soc. Simon Stevin 15 (2008) 455–464.
[16] H. Ghane, Z. Hamed, B. Mashayekhy, H. Mirebrahimi, On topological homotopy groups of n-Hawaiian like spaces, Topol. Proc. 36 (2010) 255–266.
[17] U. Heckmanns, Ultrametric Spaces, English version of the Doctoral dissertation, preprint, Munchen, 1996.
[18] S. Mardešić, Strong expansions of products and products in strong shape, Topol. Appl. 140 (2004) 81–110.
[19] S. Mardešić, J. Segal, Shape Theory, North-Holland, Amsterdam, 1982.
[20] E. Michael, Bi-quotient maps and Cartesian products of quotient maps, Ann. Inst. Fourier (Grenoble) 18 (1968) 287–302.
[21] M. Moszyńska, Various approaches to the fundamental groups, Fundam. Math. 78 (1973) 107–118.
[22] T. Nasri, F. Ghanei, B. Mashayekhy, H. Mirebrahimi, On topological shape homotopy groups, Topol. Appl. 198 (2016) 22–33.

[23] T. Nasri, B. Mashayekhy, H. Mirebrahimi, On products in the coarse shape category, Glas. Mat. 50 (70) (2015) 223–231.
[24] A. Pakdaman, H. Torabi, B. Mashayekhy, On H-groups and their applications to topological fundamental groups, preprint, arXiv:1009.5176v1.
[25] B. Červar, N. Uglešić, Paracompact locally compact spaces as inverse limits of polyhedra, Publ. Math. (Debr.) 52 (1998) 69–78.