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On subsemicovering

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Abstract

In this paper, by reviewing the concept of subcovering and semicovering maps, we extend the notion of subcovering map to subsemicovering map. We present necessary and sufficient condition for a local homeomorphism to be a subsemicovering map.

Keywords: local homeomorphism, covering map, semicovering map, subcovering map, subsemicovering map.

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1 Introduction

Steinberg [7, Section 4.2] defined a map $p : \tilde{X} \longrightarrow X$ of locally path connected spaces a *subcovering map* if there exists a covering map $p' : \tilde{Y} \longrightarrow X$ and a topological embedding $i : \tilde{X} \longrightarrow \tilde{Y}$ such that $p' \circ i = p$. He presented a necessary and sufficient condition for a local homeomorphism $p : \tilde{X} \longrightarrow X$ to be a subcovering. More precisely, he proved that a continuous map $p : \tilde{X} \longrightarrow X$ of locally path connected and semilocally simply connected spaces is a subcovering if and only if $p : \tilde{X} \longrightarrow X$ is a local homeomorphism and any path f in $\tilde{X}$ with $p \circ f$ null homotopic (in X) is closed, i.e, $f(0) = f(1)$ (see [7, Theorem 4.6]).

Brazas [1, Definition 3.1] extended the concept of covering map to semicovering map. A *semicovering map* is a local homeomorphism with continuous lifting of paths and homotopies. Klevdal in [5, Definition 7] simplified the notion of semicovering map as a local homeomorphism with unique path lifting and path lifting properties.

In this paper, we extend the notion of subcovering map to subsemicovering map. We call a local homeomorphism $p : \tilde{X} \longrightarrow X$ a *subsemicovering map* if it can be extended to a semicovering map $q : \tilde{Y} \longrightarrow X$, i.e, there exists a topological embedding $\varphi : \tilde{X} \to \tilde{Y}$ such that $q \circ \varphi = p$.

In this paper, all maps $f : X \longrightarrow Y$ between topological spaces X and Y are continuous. We recall that a continuous map $p : \tilde{X} \longrightarrow X$ is called a *local homeomorphism* if for every point $\tilde{x} \in \tilde{X}$, there exists an open neighborhood $\tilde{W}$ of $\tilde{x}$ such that $p(\tilde{W})$ is open in X and the restriction map $p|_{\tilde{W}} : \tilde{W} \longrightarrow p(\tilde{W})$ is a

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homeomorphism. In this paper, we denote a local homeomorphism $p : \tilde{X} \longrightarrow X$ by $(\tilde{X}, p)$ and assume that $\tilde{X}$ is path connected and locally path connected.

Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map and $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of f. The map p has *path lifting property* (PLP for short) if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f. Also, the map p has the *unique path lifting property* (UPLP for short) if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f. (See [6].)

Brazas [1, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \longrightarrow X$ has *continuous lifting of paths* if $\rho_p : (\rho Y)_y \longrightarrow (\rho X)_{p(y)}$ defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism, for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0, 1] \longrightarrow Y | \alpha(0) = y\}$. Also, a map $p : Y \longrightarrow X$ *has continuous lifting of homotopies* if $\Phi_p : (\Phi Y)_y \longrightarrow (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism, for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. He also simplified the definition of semicovering maps by showing that having continuous lifting of paths implies having continuous lifting of homotopies. (See [2, Remark 2.5].)

The following theorem can be found in [5, Lemma 2.1] and [3, Theorem 3.1].

Theorem 1.1. *(Local Homeomorphism Homotopy Theorem for Paths).*
Let $(\tilde{X}, p)$ be a local homeomorphism of X with UPLP and PLP. Consider the diagram of continuous maps

$$\begin{tikzcd} I \arrow[r, "\tilde{f}"] \arrow[d, "j"'] & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ I \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & (X, x_0), \end{tikzcd}$$

where $j(t) = (t, 0)$ for all $t \in I$. Then there exists a unique continuous map $\tilde{F} : I \times I \longrightarrow \tilde{X}$ making the diagram commute.

The following corollary is a consequence of the above theorem.

Corollary 1.2. *Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism with UPLP and PLP. Let $x_0, x_1 \in X$ and $f, g : I \longrightarrow X$ be paths such that $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$ and $\tilde{x}_0 \in p^{-1}(x_0)$. If $F : f \simeq g$ rel $\dot{I}$ and $\tilde{f}$ and $\tilde{g}$ are the lifting of f and g, respectively, with $\tilde{f}(0) = \tilde{x}_0 = \tilde{g}(0)$, then $\tilde{F} : \tilde{f} \simeq \tilde{g}$ rel $\dot{I}$.*

The following theorem can be found in [1, Corollary 2.6 and Proposition 6.2].

Theorem 1.3. *(Lifting Criterion Theorem for Semicovering Maps).*
If Y is connected and locally path connected, $f : (Y, y_0) \longrightarrow (X, x_0)$ is continuous and $p : \tilde{X} \longrightarrow X$ is a

semicovering map where $\tilde{X}$ is path connected, then there exists a unique $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

The following theorem can be concluded from [5, Definition 7, Corollary 2.1].

Theorem 1.4. *A map $p : \tilde{X} \longrightarrow X$ is a semicovering map if and only if it is a local homeomorphism with UPLP and PLP.*

Note that there exists a local homeomorphism without UPLP and PLP and so it is not a semicovering map.

Example 1.5. ([4, Example 2.4]). Let $\tilde{X} = ([0,1] \times \{0\}) \cup (\{1/2\} \times [0, 1/2))$ with coherent topology with respect to $\{[0, 1/2] \times \{0\}, (1/2, 1] \times \{0\}, \{1/2\} \times (0, 1/2)\}$ and let $X = [0, 1]$. Define $p : \tilde{X} \longrightarrow X$ by $p(s,t) = \begin{cases} s & t = 0 \\ s + 1/2 & s = 1/2 \end{cases}$. It is routine to check that p is a local homeomorphism which does not have UPLP and PLP.

2 Main Results

Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism. We are interested in finding some conditions on p or $\tilde{X}$ under which the map p can be extended to a semicovering map $q : \tilde{Y} \longrightarrow X$. We recall that Steinberg [7, Section 4.2] defined a map $p : \tilde{X} \longrightarrow X$ of locally path connected and semilocally simply connected spaces a *subcovering map (and $\tilde{X}$ a subcover)* if there exists a covering map $p' : \tilde{Y} \longrightarrow X$ and a topological embedding $i : \tilde{X} \longrightarrow \tilde{Y}$ such that $p' \circ i = p$. We are going to extend this definition as follows:

Definition 2.1. Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism. We say that p *can be extended to a local homeomorphism* $q : \tilde{Y} \longrightarrow X$, if there exists an embedding map $\varphi : \tilde{X} \hookrightarrow \tilde{Y}$ such that $q \circ \varphi = p$. In particular, if q is a covering map, then p is called a *subcovering map* (see [7, Section 4.2]) and if q is a semicovering map, then we call the map p a *subsemicovering map.*

Note that since every covering map is a semicovering map, and every subcovering map is a subsemicovering map. Also, if $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ can be extended to $q : (\tilde{Y}, \tilde{y}_0) \longrightarrow (X, x_0)$ via $\varphi : (\tilde{X}, \tilde{x}_0) \longrightarrow (\tilde{Y}, \tilde{y}_0)$, then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a subgroup of $q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$.

The following example shows that a local homeomorphism may be extended to various covering maps.

Example 2.2. Let $X = S^1 \vee S^1 = \{e^{2\pi i t} + 1 | t \in \mathbb{R}\} \cup \{e^{2\pi i t} - 1 | t \in \mathbb{R}\}$ be the figure eight space, $\tilde{X} = \mathbb{R} \times \{0\} \cup \bigcup_{n \in \mathbb{R}} \{(-1, 1) \times \{n\}\}$ and $p : \tilde{X} \longrightarrow X$ defined by

$$p(t,s) = \begin{cases} e^{2\pi i t} + 1 & s = 0 \\ e^{2\pi i s} - 1 & s \neq 0. \end{cases}$$

Then p is a subcovering map since p can be extended to the universal cover of figure eight space introduced in [6, Section 1.3] which we denote it by $h : \tilde{Z} \longrightarrow X$. Note that one can extend p to the covering $q : \tilde{Y} \longrightarrow X$ where $\tilde{Y} = (\mathbb{R} \times \{0\}) \cup \bigcup_{n\in\mathbb{N}}(S^1 \times \{n\})$ via an embedding map $\varphi : \tilde{X} \longrightarrow \tilde{Y}$ defined by

$$\varphi(t, s) = \begin{cases} (t, 0) & s = 0 \\ (e^{2\pi it}, t) & s \neq 0. \end{cases}$$

Hence p can be extended to two coverings which are not equivalent since $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = h_*(\pi_1(\tilde{Z}, \tilde{z}_0)) = \{1\}$ but $\{1\} = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \lneqq q_*(\pi_1(\tilde{Y}, \tilde{y}_0)) \leq \pi_1(X, x_0)$.

Steinberg [7, Theorem 4.6] proved that the condition *"if f is a path in $\tilde{X}$ with $p \circ f$ null homotopic (in X), then $f(0) = f(1)$"* is a necessary condition for a local homeomorphism $p : \tilde{X} \longrightarrow X$ to be subcovering. In the following theorem, we show that this condition is also a necessary condition for a local homeomorphism to be a subsemicovering.

Theorem 2.3. *If $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a subsemicovering map, then*

1. *$p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a local homeomorphism;*
2. *if f is a path in $\tilde{X}$ with $p \circ f$ null homotopic (in X), then $f(0) = f(1)$.* ($\bigstar$)

In the following, we are going to find a sufficient condition for extending a local homeomorphism to a semicovering map. For this purpose first, note that Steinberg in [7, Theorem 4.6] presented a necessary and sufficient condition for a local homeomorphism $p : \tilde{X} \longrightarrow X$ to be a subcovering. More precisely, he proved that a continuous map $p : \tilde{X} \longrightarrow X$ of locally path connected and semilocally simply connected spaces is a subcovering if and only if $p : \tilde{X} \longrightarrow X$ is a local homeomorphism and any path f in $\tilde{X}$ with $p \circ f$ null homotopic (in X) is closed, i.e, $f(0) = f(1)$. We will show that the latter condition on a local homeomorphism $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a sufficient condition for p to be a subsemicovering provided that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is an open subgroup of the quasitopological fundamental group $\pi_1^{qtop}(X, x_0)$.

Theorem 2.4. *Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a map such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is an open subgroup of $\pi_1^{qtop}(X, x_0)$. Then p is a subsemicovering map if and only if*

1. *$p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a local homeomorphism;*
2. *if f is a path in $\tilde{X}$ with $p \circ f$ null homotopic (in X), then $f(0) = f(1)$.*

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