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Some Results on Stochastic Properties of Spacings of Generalized Order Statictics

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Abstract

Sample spacings play important roles in reliability theory, life testing, data analysis, goodness-of-fit tests, and other related areas. Generalized order statistics unify the study of order statistics, record values, k-records, Pfeifer's records and several other cases of ordered random variables, so it is natural and interesting to obtain stochastic properties of spacings of generalized order statistics by analogy with ordinary order statistics. In this paper we prove when $X_{(1,n,\widetilde{m}_n,k)}$ is increasing mean residual life (IMRL), the spacings of generalized order statistics ordered by variance, and that the covariance between successive spacings is nonnegative. Also, we compare variance of successive generalized order statistics. Furthermore, the mean residual life order between the normalized spacings of generalized order statistics from two sample sequences are built as well.

Keywords: Generalized order statistics, Mean residual life order, Excess wealth order, DMRL, IMRL.

1 Introduction

The concept of generalized order statistics was introduced in Kamps ($1995a, b$) as a unification of several models of random variables arranged in ascending order of magnitude with different interpretations and statistical applications. Ordinary order statistics and the corresponding sample spacings play important roles in many areas of statistics such as goodness-of-fit test, auction theory, life testing and reliability. Interested readers may refer to Balakrishnan and Rao (1998 a, b), Paul and Gutierrez (2004), and Li (2005) for discussions on this issue. Stochastic comparisons between ordinary order statistics and sample spacings are interesting topics investigated in the literature by various authors. We refer the readers to Kochar (2012) and Balakrishnan and Zhao (2013) for reviews of the recent developments on this topic. Because of generality of generalized order statistics, several results on stochastic comparisons of ordinary order statistics and the corresponding sample spacings have been extended to generalized order statistics and their spacings. For example, Franco *et al.* (2002), established univariate stochastic comparisons of generalized order statistics and their normalizing spacings under restrictive assumptions on the parameters of generalized order statistics. Belzunce *et al.* (2005) proved one comparison of general p-spacings of generalized order statistics in the usual stochastic order. Hu and Zhuang (2005) investigated the conditions on the parameters which enable one to establish several stochastic comparisons of general p-spacings for a subclass of generalized order statistics in the univariate likelihood ratio and the hazard rate orders. Fang *et al.*(2006) investigated conditions on the underlying distribution function F and the parameters k and $\widetilde{m}_n$ to obtain the multivariate likelihood ratio and the multivariate usual stochastic orderings of simple spacing vectors of generalized order statistics with the m_i's unequal. Xie and Hu (2009) investigated less restrictive conditions on the model parameters and the underlying distribution function upon which the generalized order statistics are based, which enable one to establish the likelihood ratio and the hazard ratio orderings for p-spacings of generalized order statistics. Xie and Zhuang (2011) established the mean residual life and the excess wealth orders for simple spacings of generalized order statistics in one sample problems.

In the foolowing, we recall the definition of generalized order statistics, increasing (decreasing) mean residual life class, the mean residual life order and two lemmas which be used in the sequel.

Uniform generalized order statistics are defined via some joint density function on a cone of the $\Re^n$. Generalized order statistics based on an arbitrary distribution function F are defined by means of the inverse function of F.

Definition 1.1. Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \cdots, m_{n-1} \in \Re$, be parameters such that $\gamma_{r,n} = k + \sum_{j=r}^{n-1}(m_j+1) \geq 1$, for $r = 1, \cdots, n-1$ and $\gamma_{n,n} = k$ and let $\widetilde{m}_n = (m_1, \cdots, m_{n-1})$ if $n \geq 2$ ($\widetilde{m}_n$ arbitrary if $n = 1$). If the random variables $U_{(r,n,\widetilde{m}_n,k)}, r = 1, \cdots, n$ possess a joint density of the form

$$f_{U_{(1,n,\widetilde{m}_n,k)},\cdots,U_{(n,n,\widetilde{m}_n,k)}}(u_1, \cdots, u_n) = k\left(\prod_{j=1}^{n-1}\gamma_{j,n}\right)\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

The random variables

$$X_{(r,n,\widetilde{m}_n,k)} = F^{-1}(U_{(r,n,\widetilde{m}_n,k)}), \quad r = 1, \cdots, n,$$

are called the generalized order statistics based on F, where F^{-1} is the inverse of F defined by $F^{-1}(u) = \sup\{x : F(x) \leq u\}$ for $u \in [0, 1]$

Generalized order statistics provide an unified approach for examining distributional and moment properties of many models of ordered random variables, which can actually be regarded as special cases. For example, ordinary order statistics of a random sample from a distribution F are a particular case of generalized order statistics when $k = 1$ and $m_r = 0$ for all $r = 1, \cdots, n-1$. When $k = 1$ and $m_1 = \cdots = m_{n-1} = -1$, then we get the first n record values from a sequence of random variables with distribution F. Choosing the parameters appropriately, the model of generalized order statistics also yields k-record values, sequential order statistics, Pfeifer's records, progressive Type II censored order statistics, etc., as special cases. One may refer to Kamps (1995 b) for order statistics with non-integral sample size, k-record values, sequential order statistics and Pfeifer's records, to Balakrishnan *et al.* (2001) for progressive Type II censored order statistics, and to Belzunce *et al.* (2005) and references therein for order statistics under multivariate imperfect repair.

It is well-known that generalized order statistics from a continuous distribution form a Markov chain with transition probabilities

$$P\left[X_{(r,n,\widetilde{m}_n,k)} > t | X_{(r-1,n,\widetilde{m}_n,k)} = s\right] = \left[\frac{\overline{F}(t)}{\overline{F}(s)}\right]^{\gamma_{r,n}} \tag{1.1}$$

for $t \geq s$ and $r = 2, \cdots, n$.

Let $\{X_{(i,n,\widetilde{m}_n,k)},\ i = 1, \cdots, n\}$ and $\{Y_{(j,m,\widetilde{m}_m,k)},\ j = 1, \cdots, m\}$, be generalized order statistics based on absolutely continuous distribution functions F and G, respectively. The random variables $U_{i,n} = X_{(i,n,\widetilde{m}_n;k)} - X_{(i-1,n,\widetilde{m}_n,k)}$ and $\widetilde{U}_{i,n} = \gamma_{i,n}U_{i,n},\ i = 1, \cdots, n$, with $X_{(0,n,\widetilde{m}_n,k)} \equiv 0$, are the simple spacings and the normalized spacings of the generalized order statistics $\{X_{(i,n,\widetilde{m}_n,k)},\ i = 1, \cdots, n\}$. Similarly, define $V_{j,m} = Y_{(j,m,\widetilde{m}_m,k)} - Y_{(j-1,m,\widetilde{m}_m,k)}$ and $\widetilde{V}_{j,m} = \gamma_{j,m}V_{j,m}$ for $j = 1, \cdots, m$, where $Y_{(0,m,\widetilde{m}_m,k)} \equiv 0$.

Since the accurate distribution of the life of an unit or a system is often unavailable in practical situations, nonparametric aging classes of life distributions, such as IMRL and DMRL have been founded to be quite useful in maintain policy and system analysis.

Let X and Y be two nonnegative random variables with absolutely continuous distribution functions F and G and mean residual life functions $\mu_X(t)$ and $\mu_Y(t)$, respectively.

Definition 1.2. A random variable X is said to have an increasing (decreasing) mean residual life (IMRL (DMRL)), if the mean residual life of X, i.e.

$$\mu_X(t) = E(X - t | X > t) = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)},$$

is increasing (decreasing) in t.

Definition 1.3. A random variable X is smaller than a random variable Y in the mean residual life order (denoted by $X \leq_{mrl} Y$), if $\mu_X(t) \leq \mu_Y(t)$ for all t.

(i) mean residual life order (denoted by $X \leq_{mrl} Y$), if $\mu_X(t) \leq \mu_Y(t)$ for all t.

(ii) excess wealth order (denoted by $X \leq_{ew} Y$), if

$$\int_{F^{-1}(p)}^\infty \overline{F}(x)dx \leq \int_{G^{-1}(p)}^\infty \overline{G}(x)dx$$

where F^{-1} is the inverse of F defined by $F^{-1}(u) = \sup\{x : F(x) \leq u\}$ for $u \in [0, 1)$.

The following lemma used in deriving the main results.

Lemma 1.4. *(Xie and Zhuag, 2011) Let $X_{(1,n,\widetilde{m}_n,k)}, \cdots, X_{(n,n,\widetilde{m}_n,k)}$ be generalized order statistics based on F, where $m_i \geq -1$ for each i, and denote $\widetilde{m}^*_{r,n} = (m_{r+1}, \cdots, m_{n-1})$ for $r = 1, \cdots, n-2$.*

(i) If $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL (DMRL), then $X_{(1,n-r,\widetilde{m}^*_{r,n},k)}$ is IMRL (DMRL) for each r.

(ii) If $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL, then $U_{r,n}$ is IMRL for each r.

The next proposition due to Shaked and Shanthikumar (2007) shows that the mean residual life order can be characterized by way of the excess wealth order.

Proposition 1.5. *Let X and Y be two random variables with distribution functions F and G, respectively. If $X \leq_{mrl} Y$ and either X or Y or both are IMRL, then $X \leq_{ew} Y$.*

2 Main Results

The purpose of this section is to investigate conditions on the distributions and the parameters to establish some comparisons between the simple (normalized) spacings of generalized order statistics based on the same distribution, and then based on two different distributions. We first give a lemma will be useful throughout this paper.

Lemma 2.1. *Let $\{X_{(r,n,\widetilde{m}_n,k)}, r = 1, \cdots, n\}$ be generalized order statistics based on an absolutely continuous distribution function F. Further, let $\mu_{1,n}(t)$ denote the mean residual life function of $X_{(1,n,\widetilde{m}_n,k)}$. If F is DMRL (IMRL), then $\mu_{1,n}(t) \geq (\leq) \frac{\mu(t)}{\gamma_{1,n}},\ \forall t$, where $\mu(t)$ denotes the mean residual life function of X.*

Theorem 2.2. *Let $X_{(1,n,\widetilde{m}_n,k)}, \cdots, X_{(n,n,\widetilde{m}_n,k)}$ be generalized order statistics based on absolutely continuous distribution function F, where $m_i \geq -1$ for each i. If $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL and $E(X^2) < \infty$, then*

(i) $var(U_{r+1,n}) \geq var(U_{r,n})$, if $m_1 \geq \cdots \geq m_n$.

(ii) $cov(U_{r,n}, U_{r+1,n}) \geq 0$.

Next we prove the following result concerning variances of successive generalized order statistics, when $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL.

Theorem 2.3. *Let $X_{(1,n,\widetilde{m}_n,k)}, \cdots, X_{(n,n,\widetilde{m}_n,k)}$ be generalized order statistics based on absolutely continuous distribution function F, where $m_i \geq -1$ for each i. Further, suppose that $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL and $E(X^2) < \infty$, then*

$$var(X_{(i+1,n,\widetilde{m}_n,k)}) \geq var(X_{(i,n,\widetilde{m}_n,k)})$$

Theorem 2.4. *Let $X_{(1,n,\widetilde{m}_n,k)}, \cdots, X_{(n,n,\widetilde{m}_n,k)}$ be generalized order statistics based on absolutely continuous distribution function F. If F is IMRL and $E(X^2) < \infty$, then*

$$var(X_{(1,n,\widetilde{m}_n,k)}) \leq \frac{1}{\gamma_{1,n}^2} var(X)$$

The following theorem establishes the mean residual life order among normalized spacings of two samples of generalized order statistics.

Theorem 2.5. *Let $X \leq_{mrl} Y$ and X and Y are IMRL and DMRL respectively; Then*

$$\widetilde{U}_{i,n} \leq_{mrl} \widetilde{V}_{j,m}\ ,\ for\ i = 1, \cdots, n,\ j = 1, \cdots, m.$$

In view of Theorem 2.5 , it is natural to get the following theorem by using Proposition 1.5.

Theorem 2.6. *under the same conditions as in Theorem 2.5, if, in addition, $X_{(1,n,\widetilde{m}_n,k)}$ or $Y_{(1,m,\widetilde{m}_m,k)}$ or both are IMRL, then*

$$\widetilde{U}_{i,n} \leq_{ew} \widetilde{V}_{j,m}\ ,\quad for\ i = 1, \cdots, n,\ j = 1, \cdots, m.$$

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