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Nonlinear transversely vibrating beams by the improved energy balance method and the global residue harmonic balance method

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ABSTRACT

Mathematical modeling of many engineering systems such as beam structures often leads to nonlinear ordinary or partial differential equations. Nonlinear vibration analysis of the beam structures is very important in mechanical and industrial applications. This paper presents the high order frequency-amplitude relationship for nonlinear transversely vibrating beams with odd and even nonlinearities using the improved energy balance method and the global residue harmonic balance method. The accuracy of the energy balance method is improved based on combining features of collocation method and Galerkin–Petrov method, and an improved harmonic balance method is proposed which is called the global residue harmonic balance method. Unlike other harmonic balance methods, all the former global residual errors are introduced in the present approximation to improve the accuracy. Finally, preciseness of the present analytic procedures is evaluated in contrast with numerical calculations methods, giving excellent results.

1. Introduction

Beams are very important element in structures because of its widespread usage in steel construction such as airplane wings, helicopter rotor blades, flexible satellites, space craft antennae, robot arms, high rise buildings, long span bridges, drill strings and vibratory drilling. As the amplitude of oscillation increases, these structures are subjected to non-linear vibrations which often lead to material fatigue and structural damage. These effects become more significant around the natural frequencies of the system. If natural frequency of a system becomes distinct we are empowered to prevent divulging of resonance and its undesirable consequences.

The vibration problems of uniform Euler–Bernoulli beams can be solved by analytical or approximate approaches [1–5]. It is very important to provide an accurate analysis towards the understanding of the nonlinear vibration characteristics of these structures.

Nowadays, interests of researchers for using analytical techniques to solve nonlinear problems are increased. The reason lies in the fact that every physical process is really a nonlinear process, and should be described using nonlinear equations, and many analytical methods have been implemented to solve nonlinear oscillation systems. For example, perturbation methods [6,7] are mentioned to be the most versatile tools in nonlinear analysis of engineering problems. These methods are constantly being developed and applied to ever more complex problems.

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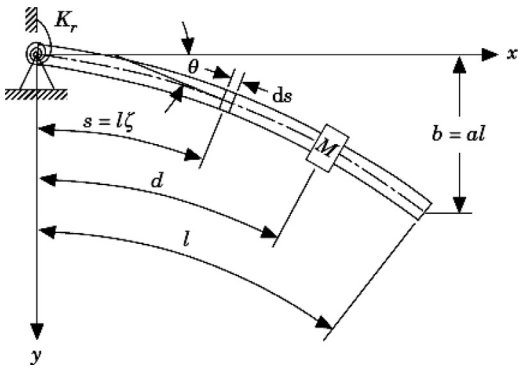

Fig. 1. Configuration of a beam carrying a lumped mass.

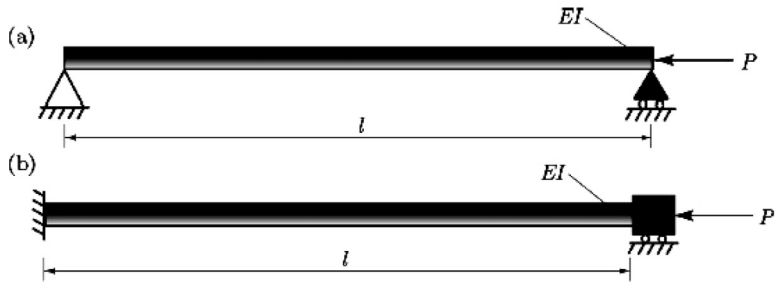

Fig. 2. Schematic representation of transversely vibrating quintic nonlinear beam.

However, traditional perturbation methods have many shortcomings and are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have been proposed in the literature, such as harmonic based methods [8–15], coupled homotopy–variational formulation [16–18], variational approach [19–21], amplitude– frequency formulation [22–24], optimal homotopy asymptotic method [25–27], Energy Balance Method [28–31], Iteration perturbation method [32], Homotopy perturbation method (HPM) [33–38], Multistage Adomian Decomposition Method [39], Variational iteration method [40–42].

This paper intends to promote the application of modern analytical approaches to the governing equation of transversely vibrating quintic nonlinear beams. Two new studied methods are the improved energy balance method and the global residue harmonic balance method. The powerful analytical approaches are used to obtain the nonlinear frequency– amplitude relationship for dynamic behavior of vibrating beams with quintic nonlinearity. Comparison of the result which is obtained by these methods with the obtained result by the numerical solution reveals that these approaches are very effective and convenient.

First order approximation of these methods by simple calculation yields good accuracy, but interest to reduce the relative error induced the researchers to implement higher order of approximations.

2. Mathematical formulation

In order to exhibit the accuracy of the improved energy balance method and the global residue harmonic balance method to approximate the solution of nonlinear oscillators, different nonlinear ordinary differential equations of vibrating beams are considered in this paper.

2.1. Uniform beam carrying a lumped mass

Fig. 1 shows the schematic representation of the free vibration of a beam carrying a lumped mass along its span. The uniform beam considered in this section has length l and mass m per unit length, hinged at the base to a rotational spring with stiffness K_r and carries a lumped mass Mat an arbitrary intermediate point$d$along the beam span. By neglecting the rotary inertia and shear deformation, the following non-dimensional parameters are introduced:

$$\zeta = \frac{s}{l}, \eta = \frac{d}{l}, \mu = \frac{M}{ml}. \tag{1}$$

The kinetic and potential energies of the beam can be expressed as [43]

$$T = \frac{ml}{2}\int_0^1 [1 + \mu\delta(\zeta - \eta)]\left(\dot{x}^2 + \dot{y}^2\right)d\zeta \tag{2}$$

Table 1
Values of dimensionless parameters ε_i in Eq. (7) for three modes [44].

Mode	ε_1	ε_2	ε_3	ε_4
1	0.326845	0.129579	0.232598	0.087584
2	1.642033	0.913055	0.313561	0.204297
3	4.051486	1.665232	0.281418	0.149677

and

$$U = \frac{EI\lambda^4}{2}\int_0^1 [y''^2 + (\lambda y' y'')^2 + \lambda^4 y'^4 y''^2]d\zeta, \tag{3}$$

where$\delta(\zeta-\eta)$ is Dirac's function, $\lambda = 1/l$ and prime represents differentiation with respect to the dimensionless arc length ζ. Using the standard method of separation of variables, the solution can be assumed in the form $y(\zeta, t) = u(t)\varphi(\zeta)$, with $\varphi(\zeta)$ is a normalized eigenfunction of the corresponding linear problem, $u(t)$ is the time dependent variable. Using Rayleigh-Ritz procedure with single linear mode, the following Lagrangian function is obtained

$$L = \frac{ml}{2}\left[\alpha_1\dot{u}^2 + \alpha_3\lambda^2u^2\dot{u}^2 + \alpha_4\lambda^2u^2\dot{u}^2 + \alpha_5\lambda^4u^4\dot{u}^2 + \alpha_6\lambda^4u^4\dot{u}^2 - \frac{EI}{m}\lambda^4\left(\alpha_2u^2 + \alpha_7\lambda^2u^4 + \alpha_8\lambda^4u^6\right)\right] \tag{4}$$

in which

$$\alpha_1 = \int_0^1 \phi^2 d\zeta + \mu\phi^2(\eta), \alpha_2 = \int_0^1 \phi''^2 d\zeta, \alpha_3 = \int_0^1 \left(\int_0^\zeta \phi'^2 d\zeta\right)^2 d\zeta$$
$$\alpha_4 = \mu\left[\left(\int_0^\zeta \phi'^2 d\zeta\right)^2\right]_{\zeta=\eta}, \alpha_5 = \int_0^1 \left[\left(\int_0^\zeta \phi'^2 d\zeta\right)\left(\int_0^\zeta \phi'^4 d\zeta\right)\right]d\zeta$$
$$\alpha_6 = \mu\left[\left(\int_0^\zeta \phi'^2 d\zeta\right)\left(\int_0^\zeta \phi'^4 d\zeta\right)\right]_{\zeta=\eta}, \alpha_7 = \int_0^1 \phi'^2\phi''^2 d\zeta, \alpha_8 = \int_0^1 \phi'^4\phi''^2 d\zeta. \tag{5}$$

In order to eliminate the variable x in the governing equations, the following constraint equation for the in extensional beams is used [44]:

$$(1 + \lambda x'^2)^2 + (\lambda y'^2) = 1. \tag{6}$$

Applying Euler–Lagrange equation, the following temporal governing equation for a restrained uniform beam carrying an intermediate lumped mass is obtained as:

$$\ddot{q} + \lambda q + \varepsilon_1 q^2\ddot{q} + \varepsilon_1 q\dot{q}^2 + \varepsilon_2 q^4\ddot{q} + 2\varepsilon_2 q^3\dot{q}^2 + \varepsilon_3 q^3 + \varepsilon_4 q^5 = 0. \tag{7}$$

So let us consider such initial conditions

$$q(0) = A, \dot{q}(0) = 0 \tag{8}$$

where

$$\varepsilon_1 = \frac{\alpha_3\alpha_4}{\bar{p}^4\alpha_1}, \varepsilon_2 = \frac{\alpha_5\alpha_6}{\bar{p}^4\alpha_1}, \varepsilon_3 = \frac{2\alpha_7}{\alpha_2}, \varepsilon_4 = \frac{2\alpha_8}{\alpha_2}. \tag{9}$$

In Eq. (7) dots denote derivatives with respect to the new dimensionless time τ, and q is the dimensionless beam displacement. ε_1,ε_2, ε_3 and ε_4 are positive parameters associated with each of the three calculation modes are shown in Table 1 [44].

$$\tau = \sqrt{\left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{EI\lambda^4}{m}\right)}t \tag{10}$$

and

$$q = \frac{\bar{p}u}{l}, \bar{p} = \sqrt[4]{\frac{m\bar{\omega}^2}{EI}}l, \tag{11}$$

where $\bar{\omega}$ is the fundamental frequency of the beam [44].

In Ref. [44] the parameters ε_i in Eq. (7) for various values of base stiffness S mass ratio μ and position η of the intermediate mass M has been presented. Some examples of the results for some selected values are shown in Table 1.

2.2. Transversely vibrating quintic nonlinear beam

Consider the Euler-Bernoulli beam of length l, a moment of inertia I, mass per unit length m and a modulus of elasticity E, which is axially compressed by a loading P as shown in Fig. 1. If w is the transverse deflection then the differential equation governing the equilibrium in the deformed situation is derived as [1]

$$\frac{d^2}{dx^2}\left\{\frac{EIw''(x,t)}{\sqrt{1+w'^2(x,t)^3}}\right\}+Pw''(x,t)\left[1+\frac{3}{2}w'^2(x,t)\right]+m\ddot{w}(x,t)=0, \tag{12}$$

where $w''(x,t)/\sqrt{1+w'^2(x,t)^3}$ is the "exact" expression for the curvature, taking into account this approximation

$$\frac{w''(x,t)}{1+w'^2(x,t)^{\frac{3}{2}}}\cong w''(x,t)\left[1-\frac{3}{2}w'^2(x,t)+\frac{15}{8}w'^4(x,t)\right], \tag{13}$$

which is subjected to the following boundary conditions:

a) for simply supported (S–S) beam

$$\begin{aligned} &w(0,t)=w''(0,t)=0\\ &w(l,t)=w''(l,t)=0, \end{aligned} \tag{14}$$

b) for clamped-clamped (C–C) beam

$$\begin{aligned} &w(0,t)=w(l,t)=0\\ &w'(0,t)=w'(l,t)=0. \end{aligned}$$

First, the governing non-linear partial differential equation using Galerkin method was reduced to a single non-linear ordinary differential equation. It was then assumed that only fundamental mode was excited. The later equation was solved analytically in time domain using improved energy balance method and the global residue harmonic balance method, assuming

$$w(x,t)=\varphi(x)\bar{q}(t), \tag{15}$$

where $\varphi(x)$ is the first eigenmode of the beam vibration, it can be expressed as [43]

a) for simply supported (S–S) beam

$$\varphi(x)=\sin\left(\frac{\pi x}{l}\right), \tag{16a}$$

b) for clamped-clamped (C–C) beam

$$\left(\frac{x}{l}\right)^2\left(1-\frac{x}{l}\right)^2, \tag{16b}$$

applying the Galerkin method yields

$$\begin{aligned} \int_0^1 \Big(&EIw^{(4)}\left[1-\frac{3}{2}w'^2(x,t)+\frac{15}{8}w'^4(x,t)\right]-9EIw'''(x,t)w''(x,t)w'(x,t)\\ &+\frac{45}{2}EIw''^3(x,t)w'^2(x,t)+Pw''(x,t)\left[1+\frac{3}{2}w'^2(x,t)\right]+m\ddot{w}(x,t)\Big)\varphi(x)dx=0. \end{aligned} \tag{17}$$

By introducing the following variables:

$$\tau=\sqrt{\frac{EI}{ml^4}}t,\ q=\frac{\bar{q}}{l}. \tag{18}$$

The non-dimensional nonlinear equation of motion can be written as

$$\frac{d^2}{dt^2}q(\tau)+aq(\tau)+bq(\tau)^3+cq(\tau)^5=0, \tag{19}$$

where

a) for simply supported (S–S) beam

$$\begin{aligned} a&=\pi^4-\frac{Pl^2\pi^2}{EI}\\ b&=-\frac{3}{8}\pi^6-\frac{3}{8}\frac{Pl^2\pi^4}{EI}\\ c&=\frac{15}{64}\pi^8, \end{aligned} \tag{20}$$

b) for clamped-clamped (C–C) beam

$$a = 500.534 - \frac{12.142Pl^2}{EI}$$
$$b = -6.654 - \frac{0.1694Pl^2}{EI}$$
$$c = -0.3673.$$

Oscillatory systems contain two important physical parameters, that is, the frequency ω and the amplitude of oscillation A. So let us consider such initial conditions

$$q(0) = A, \dot{q}(0) = 0. \tag{21}$$

In Eq. (19), if $\varepsilon_1 = \varepsilon_2 = 0$,then Eq. (19) and Eq. (7) are similar, and they are cubic-quintic Duffing oscillator.

3. Application of the energy balance method (EBM)

Energy balance method was first proposed by professor He [30]; in this method, a variational principle for the nonlinear oscillation is established, and then, a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. It is easy to establish a variational principle for Eq. (7), which reads

$$J = \int_0^{\frac{2\pi}{\omega}} \left(-\frac{1}{2}\left(\frac{d}{dt}q(\tau)\right)^2 \left(1 + \varepsilon_1 q(\tau)^2 + \varepsilon_2 q(\tau)^4\right) + \frac{1}{2}\lambda q(\tau)^2 + \frac{1}{4}\varepsilon_3 q(\tau)^4 + \frac{1}{6}\varepsilon_4 q(\tau)^6 \right) d\tau. \tag{22}$$

From which its Hamiltonian can be obtained immediately [45]

$$H = \frac{1}{2}\left(\frac{d}{dt}q(\tau)\right)^2 \left(1 + \varepsilon_1 q(\tau)^2 + \varepsilon_2 q(\tau)^4\right) + \frac{1}{2}\lambda q(\tau)^2 + \frac{1}{4}\varepsilon_3 q(\tau)^4 + \frac{1}{6}\varepsilon_4 q(\tau)^6, \tag{23}$$

or

$$R = \frac{1}{2}\left(\frac{d}{dt}q(\tau)\right)^2 \left(1 + \varepsilon_1 q(\tau)^2 + \varepsilon_2 q(\tau)^4\right) + \frac{1}{2}\lambda q(\tau)^2 + \frac{1}{4}\varepsilon_3 q(\tau)^4 + \frac{1}{6}\varepsilon_4 q(\tau)^6 - \left(\frac{1}{2}\lambda A^2 + \frac{1}{4}\varepsilon_3 A^4 + \frac{1}{6}\varepsilon_4 A^6\right) = 0. \tag{24}$$

3.1. First order approximate solution

We consider the solution of Eq. (7) as follows:

$$q(\tau) = A\cos(\omega\tau). \tag{25}$$

Substituting Eq. (25) into Eq. (24) where $\omega\tau = \pi/4$, the following residual equation is obtained:

$$R = \frac{1}{4}A^2\omega^2\left(1 + \frac{1}{2}\varepsilon_1 A^2 + \frac{1}{4}\varepsilon_2 A^4\right) - \frac{1}{4}\lambda A^2 - \frac{3}{16}\varepsilon_3 A^4 - \frac{7}{48}\varepsilon_4 A^6 = 0. \tag{26}$$

The first order approximate solution is obtained, which reads

$$\omega^{(1)}{}_{ebm} = \frac{\sqrt{3}}{3}\sqrt{\frac{7A^4\varepsilon_4 + 9A^2\varepsilon_3 + 12\lambda}{A^4\varepsilon_2 + 2A^2\varepsilon_1 + 4}}. \tag{27}$$

Similarly, for Eq. (19), we obtain

$$\omega^{(1)}{}_{ebm} = \sqrt{\frac{7}{12}A^4 c + \frac{3}{4}A^2 b + a}. \tag{28}$$

3.2. Second order approximate solution

In the present study, the accuracy of energy balance method for analysis of conservative nonlinear oscillator is improved based on combining features of collocation method and Galerkin–Petrov method [46].

In order to improve the accuracy of energy balance method, we consider the solution of Eq. (7) as follows:

$$q(\tau) = B\cos(\omega\tau) + (A - B)\cos(3\omega\tau). \tag{29}$$

It should be noted that method does not have any limitation for trial solution, and other functions could be considered as trial solution in Eq. (7).

By inserting Eq. (29) into Eq. (7), residual is obtained. Obtained residual contain two unknown parameters, one of them is ω and other is B. In order to determine unknown parameters, we need two equations; the first equation obtained based on collocation method as follows:

$$\lim_{\omega\tau\to\frac{\pi}{4}} R(\tau)=0 \tag{30}$$

and the second equation obtained based on Galerkin–Petrov method

$$\int_0^{\frac{T}{4}} R(\tau)\cos(\omega\tau)d\tau=0, \tag{31}$$

where $T=\frac{2\pi}{\omega}$ is period of the nonlinear oscillator. Finally, by simultaneously solution of Eqs. (30) and (31), unknown parameters are determined for different value of A.

Then, we can write the residual as follows:

$$\begin{aligned}R(\tau)=&-\frac{1}{2}\left(-\frac{1}{2}B\sqrt{2}\omega-\frac{3}{2}(A-B)\sqrt{2}\omega\right)^2\left(1+\varepsilon_1\left(\frac{1}{2}B\sqrt{2}-\frac{1}{2}(A-B)\sqrt{2}\right)^2+\varepsilon_2\left(\frac{1}{2}B\sqrt{2}-\frac{1}{2}(A-B)\sqrt{2}\right)^4\right)\\&+\frac{1}{2}\lambda\left(\frac{1}{2}B\sqrt{2}-\frac{1}{2}(A-B)\sqrt{2}\right)^2+\frac{1}{4}\varepsilon_3\left(\frac{1}{2}B\sqrt{2}-\frac{1}{2}(A-B)\sqrt{2}\right)^4+\frac{1}{6}\varepsilon_4\left(\frac{1}{2}B\sqrt{2}-\frac{1}{2}(A-B)\sqrt{2}\right)^6=0.\end{aligned} \tag{32}$$

The second order approximate solution is obtained, which reads

$$\begin{aligned}\omega^{(2)}{}_{ebm}&=\frac{\sqrt{3}}{3}\sqrt{\frac{\Gamma_1}{\Gamma_2}}\frac{(A-2B)}{(3A-2B)},\\\Gamma_1&=A^4\varepsilon_4-8A^3b\varepsilon_4+24A^2b^2\varepsilon_4-32Ab^3\varepsilon_4+16b^4\varepsilon_4+3A^2\varepsilon_3-12Ab\varepsilon_3+12b^2\varepsilon_3+12\lambda\\\Gamma_2&=A^4\varepsilon_2-8A^3b\varepsilon_2+24A^2b^2\varepsilon_2-32Ab^3\varepsilon_2+16b^4\varepsilon_2+2A^2\varepsilon_1-8Ab\varepsilon_1+8b^2\varepsilon_1+4.\end{aligned} \tag{33}$$

Similarly, for Eq. (19), the second order approximate solution is obtained, which reads

$$\begin{aligned}\omega^{(2)}{}_{ebm}&=\frac{\sqrt{3\left(A^2+4AB-4B^2\right)(\Gamma)}}{18A-12B},\\\Gamma&=7A^4c-16A^3Bc+32A^2B^2c-32AB^3c+16B^4c+9A^2b-12ABb+12B^2b+12a.\end{aligned} \tag{34}$$

4. Application of the global residue harmonic balance method (GRHBM)

In this section, we put forward a novel approximate method, namely the global residue harmonic balance method, to determine the periodic solutions of the Eq. (7). To obtain higher-order analytical approximations, all the residual errors are considered in the process of every order approximation. Introducing a new independent variable [47]

$$\bar{\tau}=\omega\tau. \tag{35}$$

Eq. (7) becomes

$$\begin{aligned}&\omega^2\left(\frac{d^2}{d\tau^2}q(\bar{\tau})\right)+\lambda q(\bar{\tau})+\omega^2\varepsilon_1q(\bar{\tau})^2\left(\frac{d^2}{d\tau^2}q(\bar{\tau})\right)+\omega^2\varepsilon_1\left(\frac{d}{d\tau}q(\bar{\tau})\right)^2q(\bar{\tau})\\&+\omega^2\varepsilon_2q(\bar{\tau})^4\left(\frac{d^2}{d\tau^2}q(\bar{\tau})\right)+2\omega^2\varepsilon_2q(\bar{\tau})^3\left(\frac{d}{d\tau}q(\bar{\tau})\right)^2+\varepsilon_3q(\bar{\tau})^3+\varepsilon_4q(\bar{\tau})^5=0,\end{aligned} \tag{36}$$

and, Eq. (19) becomes

$$\omega^2\left(\frac{d^2}{d\tau^2}q(\bar{\tau})\right)+aq(\bar{\tau})+bq(\bar{\tau})^3+cq(\bar{\tau})^5=0. \tag{37}$$

4.1. First-order approximation

The initial approximation with initial conditions Eq. (7) is

$$q_1(\bar{\tau})=A\cos(\bar{\tau}). \tag{38}$$

Substituting Eq. (38) into Eq. (36) yields

$$\begin{aligned}&\left(-\frac{3}{16}\omega^2\varepsilon_2A^5+\frac{1}{16}\varepsilon_4A^5\right)\cos(5\bar{\tau})+\left(-\frac{1}{2}\omega^2\varepsilon_1A^3-\frac{7}{16}\omega^2\varepsilon_2A^5+\frac{1}{4}\varepsilon_3A^3+\frac{5}{16}\varepsilon_4A^5\right)\cos(3\bar{\tau})\\&+\left(-\omega^2A+\lambda A-\frac{1}{2}\omega^2\varepsilon_1A^3-\frac{3}{8}\omega^2\varepsilon_2A^5+\frac{3}{4}\varepsilon_3A^3+\frac{5}{8}\varepsilon_4A^5\right)\cos(\bar{\tau})=0.\end{aligned} \tag{39}$$

Equating the coefficient of $\cos(\bar{\tau})$ to zero, we obtain the first-order analytical approximate frequency

$$\omega^{(1)}{}_{grhbm} = \sqrt{\frac{5A^4\varepsilon_4 + 6A^2\varepsilon_3 + 8\lambda}{3A^4\varepsilon_2 + 4A^2\varepsilon_1 + 8}}. \tag{40}$$

Similarly, for Eq. (19), we obtain

$$\omega^{(1)}{}_{grhbm} = \sqrt{\frac{5}{8}A^4c + \frac{3}{4}A^2b + a}. \tag{41}$$

Taking $q_1(\bar{\tau})$ and $\omega^{(1)}{}_{grhbm}$ into the left-hand side of Eq. (39), we have the residual error

$$\begin{aligned} R_1(\bar{\tau}) = &\left(-\frac{3}{16}\frac{\left(5A^4\varepsilon_4 + 6A^2\varepsilon_3 + 8\lambda\right)\varepsilon_2A^5}{3A^4\varepsilon_2 + 4A^2\varepsilon_1 + 8} + \frac{1}{16}\varepsilon_4A^5\right)\cos(5\bar{\tau}) \\ &+\left(-\frac{1}{2}\frac{\left(5A^4\varepsilon_4 + 6A^2\varepsilon_3 + 8\lambda\right)\varepsilon_1A^3}{3A^4\varepsilon_2 + 4A^2\varepsilon_1 + 8} - \frac{7}{16}\frac{\left(5A^4\varepsilon_4 + 6A^2\varepsilon_3 + 8\lambda\right)\varepsilon_2A^5}{3A^4\varepsilon_2 + 4A^2\varepsilon_1 + 8} + \frac{1}{4}\varepsilon_3A^3 + \frac{5}{16}\varepsilon_4A^5\right)\cos(3\bar{\tau}). \end{aligned} \tag{42}$$

4.2. The second order approximate solution

To obtain the rest of the nonlinear correction, we consider [47]

$$q(\bar{\tau}) = q_1(\bar{\tau}) + pq_2(\bar{\tau}) \tag{43}$$

and

$$\omega^2 = {\omega_1}^2 + p\omega_2. \tag{44}$$

Where p is a bookkeeping parameter. We assume

$$q_2(\bar{\tau}) = a_{13}(\cos(\bar{\tau}) - \cos(3\bar{\tau})). \tag{45}$$

Where $a_{13}$and the parameter ω_2 are two unknown constants that will be determined later. Substituting Eq. (45) into Eq. (36) and taking the coefficients of p; we obtain a function $F_1(\bar{\tau}, a_{13}, \omega_2)$

$$\begin{aligned} F_1(\bar{\tau}, a_{13}, \omega_2) = &\frac{1}{48A^4\varepsilon_2 + 64A^2\varepsilon_1 + 128}(\Gamma_1\cos(\bar{\tau}) + \Gamma_2\cos(3\bar{\tau}) + \Gamma_3\cos(5\bar{\tau}) + \Gamma_4\cos(7\bar{\tau})) \\ \Gamma_1 = &-18A^9\varepsilon_2^2\omega_1 + 100A^8a_{13}\varepsilon_2\varepsilon_4 - 48A^7\varepsilon_1\varepsilon_2\omega_1 + 100A^6a_{13}\varepsilon_1\varepsilon_4 + 102A^6a_{13}\varepsilon_2\varepsilon_3 \\ &-32A^5\varepsilon_1^2\omega_1 - 96A^5\varepsilon_2\omega_1 + 88A^4a_{13}\lambda\varepsilon_2 + 96A^4a_{13}\varepsilon_1\varepsilon_3 + 120A^4a_{13}\varepsilon_4 - 128A^3\varepsilon_1\omega_1 \\ &+64A^2a_{13}\lambda\varepsilon_1 + 96A^2a_{13}\varepsilon_3 - 128A\omega_1 \\ \Gamma_2 = &-21A^9\varepsilon_2^2\omega_1 + 140A^8a_{13}\varepsilon_2\varepsilon_4 - 52A^7\varepsilon_1\varepsilon_2\omega_1 + 260A^6a_{13}\varepsilon_1\varepsilon_4 + 150A^6a_{13}\varepsilon_2\varepsilon_3 \\ &-32A^5\varepsilon_1^2\omega_1 - 56A^5\varepsilon_2\omega_1 + 200A^4a_{13}\lambda\varepsilon_2 + 288A^4a_{13}\varepsilon_1\varepsilon_3 + 680A^4a_{13}\varepsilon_4 - 64A^3\varepsilon_1\omega_1 \\ &+384A^2a_{13}\lambda\varepsilon_1 + 768A^2a_{13}\varepsilon_3 + 1024a_{13}\lambda \\ \Gamma_3 = &-9A^9\varepsilon_2^2\omega_1 + 260A^8a_{13}\varepsilon_2\varepsilon_4 - 12A^7\varepsilon_1\varepsilon_2\omega_1 + 300A^6a_{13}\varepsilon_1\varepsilon_4 + 330A^6a_{13}\varepsilon_2\varepsilon_3 - 24A^5\varepsilon_2\omega_1 \\ &+488A^4a_{13}\lambda\varepsilon_2 + 384A^4a_{13}\varepsilon_1\varepsilon_3 - 120A^4a_{13}\varepsilon_4 + 576A^2a_{13}\lambda\varepsilon_1 - 96A^2a_{13}\varepsilon_3 \\ \Gamma_4 = &\ 140\varepsilon_2A^8a_{13}\varepsilon_4 - 20A^6a_{13}\varepsilon_1\varepsilon_4 + 186\varepsilon_2A^6a_{13}\varepsilon_3 + 248\varepsilon_2A^4a_{13}\lambda - 40A^4a_{13}\varepsilon_4. \end{aligned} \tag{46}$$

Considering the following equation:

$$F_1(\bar{\tau}, a_{13}, \omega_2) + R_0(\bar{\tau}) = 0. \tag{47}$$

Equating the coefficients of $\cos(\bar{\tau})$; $\cos(3\bar{\tau})$ to zero in Eq. (47), we obtain the constants a_{13} and ω_2. The second-order analytical approximate frequency

$$\begin{aligned} \omega^{(2)}{}_{grhbm} &= \sqrt{\frac{5A^4\varepsilon_4 + 6A^2\varepsilon_3 + 8\lambda}{3A^4\varepsilon_2 + 4A^2\varepsilon_1 + 8} + \omega_1} \\ \omega_1 &= \frac{\sigma_1}{\sigma_2}, \end{aligned} \tag{48}$$

$$\begin{aligned} \sigma_1 = &\ 2A^4\left(10A^6\varepsilon_2\varepsilon_4 + 10A^4\varepsilon_1\varepsilon_4 + 15A^4\varepsilon_2\varepsilon_3 + 28A^2\lambda\varepsilon_2 + 16A^2\varepsilon_1\varepsilon_3 - 20A^2\varepsilon_4 + 32\lambda\varepsilon_1 - 16\varepsilon_3\right) \\ &\times\left(3A^4\varepsilon_2 + 4A^2\varepsilon_1 + 8\right)\times\left(50A^6\varepsilon_2\varepsilon_4 + 50A^4\varepsilon_1\varepsilon_4 + 51A^4\varepsilon_2\varepsilon_3 + 44A^2\lambda\varepsilon_2\right. \\ &\left.+48A^2\varepsilon_1\varepsilon_3 + 60A^2\varepsilon_4 + 32\lambda\varepsilon_1 + 48\varepsilon_3\right), \end{aligned} \tag{49}$$

Table 2
Values of coefficients in Eq. (19) for three examples.

Example	$\frac{pl^2}{EI}$	a	b	c
1 (C–C)	10	379.1140	−8.3480	−0.3673
2 (S–S)	0.5	92.4743	−364.1738	2223.874
3 (S–S)	5	48.0611	−543.1630	2223.874

$$\begin{aligned}\sigma_2 = &\left(70A^{12}\varepsilon_2^2\varepsilon_4 + 590A^{10}\varepsilon_1\varepsilon_2\varepsilon_4 + 93A^{10}\varepsilon_2^2\varepsilon_3 + 292A^8\lambda\varepsilon_2^2\right.\\ &+ 640A^8\varepsilon_1^2\varepsilon_4 + 720A^8\varepsilon_1\varepsilon_2\varepsilon_3 + 2740A^8\varepsilon_2\varepsilon_4 + 1376A^6\lambda\varepsilon_1\varepsilon_2\\ &+ 768A^6\varepsilon_1^2\varepsilon_3 + 4320A^6\varepsilon_1\varepsilon_4 + 3168A^6\varepsilon_2\varepsilon_3 + 1280A^4\lambda\varepsilon_1^2\\ &+ 4672A^4\lambda\varepsilon_2 + 4992A^4\varepsilon_1\varepsilon_3 + 5440A^4\varepsilon_4\\ &+ 7168A^2\lambda\varepsilon_1 + 6144A^2\varepsilon_3 + 8192\lambda\Big)\\ &\times\left(9A^8\varepsilon_2^2 + 24A^6\varepsilon_1\varepsilon_2 + 16A^4\varepsilon_1^2 + 48A^4\varepsilon_2 + 64A^2\varepsilon_1 + 64\right).\end{aligned} \tag{50}$$

Equating the coefficients of $\cos(\bar{\tau})$; $\cos(3\bar{\tau})$to zero for Eq. (19), one can obtain

$$\begin{cases}\frac{15}{16}A^4a_{13}c + \frac{3}{4}A^2a_{13}b - A\omega_2 = 0.\\ \frac{85}{16}A^4a_{13}c + 6A^2a_{13}b + 8a_{13}a + \frac{1}{4}bA^3 + \frac{5}{16}cA^5 = 0.\end{cases} \tag{51}$$

From Eq. (51), we obtain the constants $a_{13}$and ω_2 for Eq. (19) as below

$$\omega^{(2)}{}_{grhbm} = \sqrt{\frac{5}{8}A^4c + \frac{3}{4}A^2b + a - \frac{3}{16}\frac{A^4\left(5A^2c + 4b\right)^2}{85A^4c + 96A^2b + 128a}} \tag{52}$$

and

$$a_{13} = -\frac{A^3\left(5A^2c + 4b\right)}{85A^4c + 96A^2b + 128a}. \tag{53}$$

The further higher-order approximation can be obtained by the same technique.

5. Exact numerical solution of the quintic nonlinear beam vibrations

The exact numerical solution of Eq. (19) could be represented as [48]

$$\omega_N(A) = \frac{\pi k_1}{2\int_0^{\frac{\pi}{2}}\left(1 + k_2\sin^2\tau + k_3\sin^4\tau\right)^{-\frac{1}{2}}d\tau}, \tag{54}$$

where

$$\begin{aligned}k_1 &= \sqrt{a + b\frac{A^2}{2} + c\frac{A^4}{3}}\\ k_2 &= \frac{3bA^2 + 2cA^4}{6a + 3bA^2 + 2cA^4}\\ k_3 &= \frac{2cA^4}{6a + 3bA^2 + 2cA^4}.\end{aligned} \tag{55}$$

6. Discussion of the solution methods

To show the efficiency of the presented methods, in comparison with numerical solution, three examples are given in Table 2.

In Tables 3 and 4 the approximate analytical solutions are compared with an accurate numerical solution, using fourth-order Runge–Kutta method.

In Tables 5–7 the results, obtained by global residue harmonic balance method (GRHBM), and energy balance method (EBM), and exact numerical solution are compared.

Sedighi et al. [1] used parameter expansion method to solve the nonlinear differential equation that governs the oscillations of the quintic nonlinear beam in Eq. (19). They analyzed this problem to illustrate the usefulness and effectiveness of the technique and found the approximate solution for the frequency of the oscillator.

Table 3
Comparison between global residue harmonic balance method and energy balance method, and exact numerical solution for mode 1 in Table 1. ($\lambda = 0.2$).

A	$\omega^{(1)}{}_{grhbm}$	$\omega^{(2)}{}_{grhbm}$	$\omega^{(1)}{}_{ebm}$	B	$\omega^{(2)}{}_{ebm}$	ω_N
0.01	0.4472	0.4472	0.4472	0.01002	0.4472	0.4472
0.05	0.4476	0.4477	0.4476	0.05002	0.4479	0.4479
0.1	0.4488	0.4490	0.4488	0.10009	0.4491	0.4490
0.5	0.4865	0.4921	0.4865	0.50995	0.4923	0.4921
1	0.5951	0.6203	0.5965	1.06610	0.6215	0.6211
5	1.0457	1.6354	1.2002	6.35851	1.6394	1.6383

Table 4
Comparison between global residue harmonic balance method and energy balance method, and exact numerical solution for mode 1 in Table 1. ($\lambda = 0.5$).

A	$\omega^{(1)}{}_{grhbm}$	$\omega^{(2)}{}_{grhbm}$	$\omega^{(1)}{}_{ebm}$	B	$\omega^{(2)}{}_{ebm}$	ω_N
0.01	0.7071	0.7071	0.7071	0.01001	0.7071	0.7071
0.05	0.7073	0.7074	0.7073	0.05001	0.7076	0.7075
0.1	0.7078	0.7081	0.7078	0.10005	0.7084	0.7082
0.5	0.7239	0.7327	0.7241	0.50541	0.7336	0.7333
1	0.7757	0.8111	0.7789	1.04274	0.8116	0.8115
5	1.0498	1.6399	1.2052	6.37150	1.6719	1.6712

Table 5
Comparison between global residue harmonic balance method and energy balance method, and exact numerical solution for example 1 in Table 2.

A	$\omega^{(1)}{}_{grhbm}$	$\omega^{(2)}{}_{grhbm}$	$\omega^{(1)}{}_{ebm}$	B	$\omega^{(2)}{}_{ebm}$	ω_N
0.1	19.4692	19.4692	19.4692	0.1001	19.4692	19.4692
0.3	19.4563	19.4563	19.4563	0.3001	19.4563	19.4563
0.5	19.4302	19.4302	19.4303	0.5002	19.4302	19.4302
1	19.3035	19.3033	19.3039	1.0100	19.2935	19.2936
3	17.4405	17.4125	17.4760	3.0440	17.4109	17.4309

Table 6
Comparison between global residue harmonic balance method and energy balance method, and exact numerical solution for example 2 in Table 2.

A	$\omega^{(1)}{}_{grhbm}$	$\omega^{(2)}{}_{grhbm}$	$\omega^{(1)}{}_{ebm}$	B	$\omega^{(2)}{}_{ebm}$	ω_N
0.1	6.6428	6.6417	6.6421	0.1001	6.6417	6.6417
0.3	4.7598	4.6860	4.6803	0.3001	4.6794	4.6792
0.5	5.7522	5.6755	5.2246	0.5503	5.5322	5.5985
1	32.1031	30.4251	30.6259	1.1485	30.4009	30.4103
3	330.0990	314.8993	318.5275	3.4300	314.8963	314.8970
5	926.5876	884.4603	884.4603	5.7250	878.0995	881.1248

Table 7
Comparison between global residue harmonic balance method and energy balance method, and exact numerical solution for example 3 in Table 2.

A	$\omega^{(1)}{}_{grhbm}$	$\omega^{(2)}{}_{grhbm}$	$\omega^{(1)}{}_{ebm}$	B	$\omega^{(2)}{}_{ebm}$	ω_N
0.1	9.4806	9.4805	9.4801	0.1001	9.4799	9.4797
0.3	8.8967	8.8949	8.8544	0.3001	8.8542	8.8540
0.5	10.5386	10.4731	10.2601	0.5285	10.3888	10.3891
1	34.7745	33.2227	33.4156	1.1425	33.2310	33.2373
3	331.9908	316.8226	320.4877	3.4280	317.2103	317.2111
5	928.4207	886.3174	896.6893	4.9500	886.2122	884.2139

The results obtained for parameter expansion method, and first order global residue harmonic balance method are similar.

The comparison between global residue harmonic balance method and energy balance method presented in Figs. 3–6 for different parameters.

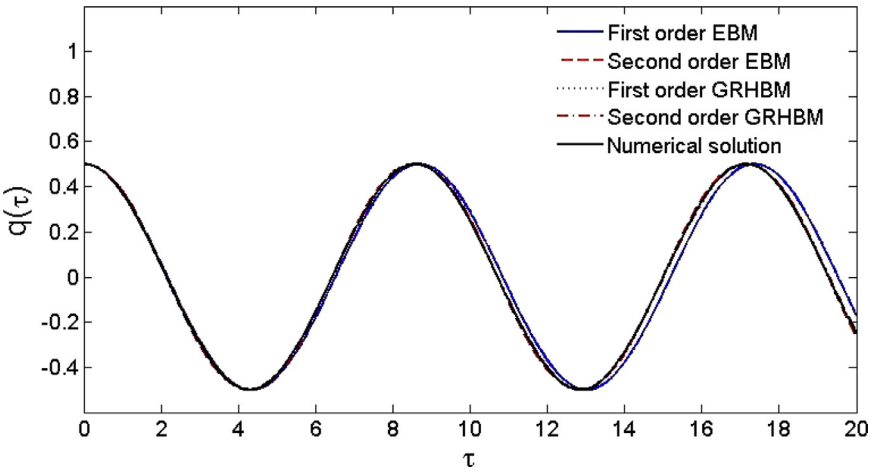

Fig. 3. Comparison between global residue harmonic balance method and energy balance method for mode 1 in Table 1. ($\lambda = 0.5$), ($A = 0.5$).

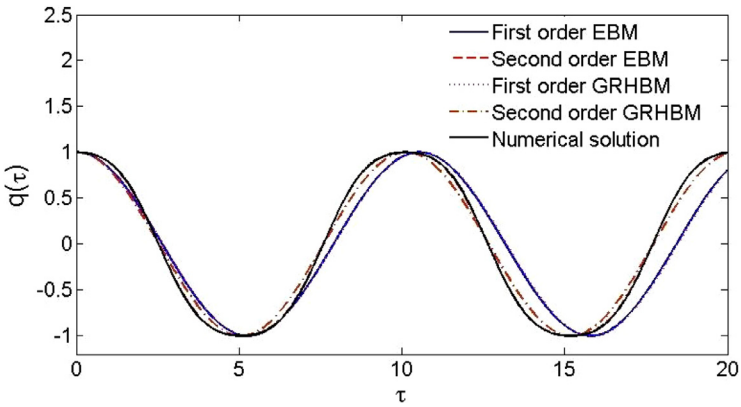

Fig. 4. Comparison between global residue harmonic balance method and energy balance method for mode 1 in Table 1. ($\lambda = 0.2$), ($A = 1$).

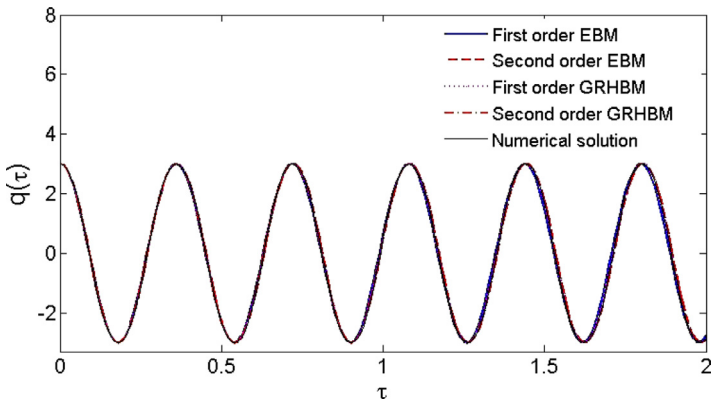

Fig. 5. Comparison between global residue harmonic balance method and energy balance method for example 1 in Table 2. ($A = 3$).

7. Conclusion

In this paper, calculating natural frequencies of nonlinear beam vibration and its relation with vibration amplitude is considered. Since system's natural frequency depends on vibration amplitude in nonlinear systems subtle calculating of natural frequency is prominent.

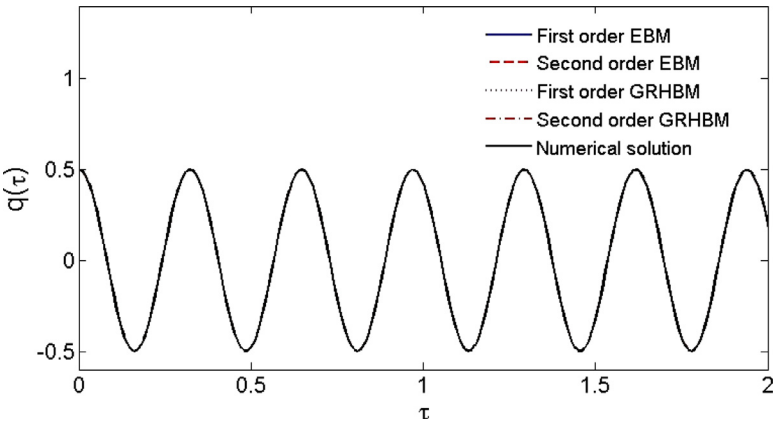

Fig. 6. Comparison between global residue harmonic balance method and energy balance method for example 1 in Table 2. ($A=0.5$).

Two different methods are employed to propose first order and second order approximate solutions for different governing equation of nonlinear vibrating beams. If the first order approximate solution is required, all the proposed methods can be applied by university students with manual calculation without the requirement of advanced calculus.

Comparing the approximate frequency obtained by second order energy balance method with the exact one, we see that this method is more accurate, but it should be noted that in this method obtained residual contain two unknown parameters, one of them is ω and other is B. In order to determine unknown parameters, we need two equations; the first equation obtained based on collocation method and the second equation obtained based on Galerkin–Petrov method. Finding the parameter B in this method is more complex than finding the parameter a_{13}in global residual harmonic balance method. First order and second order of energy balance method are more accurate, but the second order and higher order of global residual harmonic balance method can be applied with the requirement of easier calculus than second order of energy balance method, and global residual harmonic balance method is more suitable for the computer programming.

We think these methods have great potential and can be applied to other types of nonlinear engineering problems.

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