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A SPECTRAL PARAMETRIC ITERATION METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this work, we investigate a spectral parametric iteration method for solving nonlinear partial differential equations. The efficiency of this approach is illustrated by numerical experiments on the Burgers' equation.

1. Introduction

Numerical methods for partial differential equations (PDEs) can be classified into the local and global categories. The finite-difference and finite-element methods are based on local arguments, whereas the spectral method is global in character. In practice, finite-element methods are particularly well-suited to problems in complex geometries, whereas spectral methods can provide superior accuracy, at the expense of domain flexibility. We emphasize that there are many numerical approaches, such as hp finite-elements and spectral-elements, which combine advantages of both the global and local methods.

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Here, we introduce the basic principles of the spectral parametric iteration method (PIM) [1] for solving the PDEs in one spatial dimension. In particular, the adaptive version of the PIM is used to solve a simple model problem- an initial-boundary value problem consisting of Burgers' equation

$$\begin{cases} u_t = \varepsilon\, u_{xx} - \left(\dfrac{u^2}{2}\right)_x, & x \in (0,1)\,, \quad t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \sin(2\pi x) + \dfrac{1}{2}\sin(\pi x), \end{cases} \tag{1.1}$$

where ε is a physical parametric. According to the method of lines[1] for the PDE, a semi-discretization of Burgers' equation (1.1) using central finite differences in space is given by the following initial value problem of ODEs of M dimension

$$\begin{cases} \dfrac{du_1}{dt} = 0, \\ \dfrac{du_j}{dt} = \dfrac{\varepsilon}{(\Delta x)^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) - \dfrac{1}{4\Delta x}\left(u_{j+1}^2 - u_{j-1}^2\right), \\ \dfrac{du_M}{dt} = 0, \end{cases} \tag{1.2}$$

subject to the initial conditions

$$u_1(0) = 0, \quad u_j(0) = \sin(2\pi x_j) + \frac{1}{2}\sin(\pi x_j)\,, \quad u_M(0) = 0, \tag{1.3}$$

where $x_j = (j-1)\Delta x$ $(j = 1, ..., M)$ and $\Delta x = 1/(M-1)$, and $u_j(t)$ is an approximation to the solution $u = u(x,t)$ at $x = x_j$, i.e., $u_j(t) \approx u(x_j, t)$.

Let $\mathbf{u}(t) = [u_1(t), ..., u_M(t)]^T$ and $\mathbf{f}(t,\mathbf{u}) = [f_1(t,\mathbf{u}), ..., f_M(t,\mathbf{u})]^T$. Now we consider (1.2) as $\frac{d}{dt}\mathbf{u}(t) = \mathbf{f}\left(t, \mathbf{u}(t)\right)$ with $\mathbf{u}(0) = \mathbf{u}^0$ and $t \in \Lambda = (0,T)$, which can be rewritten as below:

$$\mathcal{L}[\mathbf{u}(t)] + \mathcal{N}[\mathbf{u}(t)] = \mathbf{g}(t) \tag{1.4}$$

The Chebyshev collocation parametric iteration method for (1.4) is to approximate $\mathbf{u}(t)$ by $\mathbf{u}^N \in (\mathcal{P}_N(\Lambda))^n$, such that [2, 3]

$$\begin{cases} \mathcal{L}\left[\mathbf{u}^{N,m+1}\left(t_{T,k}^N\right) - \mathbf{u}^{N,m}\left(t_{T,k}^N\right)\right] = hH\left(t_{T,k}^N\right)\mathcal{A}\left[\mathbf{u}^{N,m}\left(t_{T,k}^N\right)\right], \\ \mathbf{u}^{N,m+1}(0) = \mathbf{u}^0, \quad m \geq 0, \quad 0 \leq k \leq N, \end{cases} \tag{1.5}$$

[1]The PDE is first discretized in the spatial domain, and then the resulting system of ODEs can be integrated using an ODE solver.

where $\mathcal{P}_N(\Lambda)$ denotes the set of polynomials of degree at most N and

$$\mathcal{A}[\mathbf{u}^{N,m}(t)] = \mathcal{L}[\mathbf{u}^{N,m}(t)] + \mathcal{N}[\mathbf{u}^{N,m}(t)] - \mathbf{g}(t) \equiv \frac{d}{dt}\mathbf{u}^{N,m}(t) - \mathbf{f}\left(t, \mathbf{u}^{N,m}(t)\right). \tag{1.6}$$

Assume that $u_j(t),\ j = 1, ..., M$ is the jth component of the solution $\mathbf{u}^N(t)$. Therefore, we will have the following explicit piecewise-spectral PIM (PSP) [4] formula for solving the system (1.5):

$$\mathbf{L}_j \mathbf{u}_{j,s+1}^{N,m+1} = (I + h\mathbf{H}_{j,s})\mathbf{L}_j \mathbf{u}_{j,s+1}^{N,m} + h\mathbf{H}_{j,s}\left(\mathbf{N}_j \mathbf{u}_{j,s+1}^{N,m} - \mathbf{g}_{j,s}\right), \tag{1.7}$$

where $\mathbf{L}$, $\mathbf{N}$, $\mathbf{H}$ and $\mathbf{g}$ are as mentioned in [4].

It is obvious that the best PSP method can be achieved by using a variable number of N and a variable step size Δ in the solution to obtain a specified tolerance. Therefore, the following adaptive strategy is proposed for the PSP method. The new step size of the PSP algorithm

$$\Delta_{new} = \Delta_{m+1} = \Delta_m . min\left\{fac_{max}, max\left\{fac_{min}, fac.\left(\frac{1}{err}\right)^{\alpha}\right\}\right\}, \tag{1.8}$$

is obtained by using $err = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{Est}{Tol_i}\right)^2}$ with N_{m+1} as order of polynomial where $Est = \widehat{\mathbf{u}}^{N,m+1} - \mathbf{u}^{N,m+1}$ and $Tol_i = Atol + Rtol.|u_i^{N,m+1}|$.

Here, for $err \leq fac_{err}\,(fac_{err} \in (0, 1])$, we use $\alpha = \frac{1}{N_m+1}$ and $N_{m+1} = max\{N_m, N_{min}\}$, and for $err > fac_{err}$, $\alpha = \frac{1}{N_m}$ and $N_{m+1} = min\{N_m + 1, N_{max}\}$. The integration of the growth factors fac_{max} and fac_{min} to relation (1.8) prevents for too large step increase and contribute to the safety of the code. Additionally, using the safety factor fac makes sure that err will be accepted in the next step with high probability. The step is accepted, in case that $err \leq fac_{err}$ otherwise it is rejected and then the procedure is redone.

2. Main results

In this section, in order to show the efficiency of the proposed algorithm, we solve the system (1.2) using the adaptive PSP (APSP) algorithm. We note that all tests here are performed in Matlab 7 with double precision in a Toshiba Tecra A8 (Windows XP Professional with SP2): Intel(R) Core(TM)2 Duo Processor T7200 (2.00GHz, 4MB Cache, 997 MHz, 0.99 GB of RAM). In the framework of the PSP algorithm, for simplicity, we now put the auxiliary parameter $h = -1$, the auxiliary function $\mathbf{H} = I$, the auxiliary linear operator $\mathcal{L} = \frac{d}{dt}\mathbf{u}$, the nonlinear operator $\mathcal{N} = -\mathbf{f}\,(t, \mathbf{u})$ and the source term $g(t) = 0$. The numerical results can be observed in Table 1. In Table 1, we listed

the costed number of steps (labeled as No. of steps) for some different values of T, $Atol$ and $Rtol$, and the corresponding costed CPU elapsed time (labeled as CPU time). We can observe, from Table 1, that the PSP algorithm costs both less computational time and very smaller steps than the Matlab `ode45` solver with the same tolerances for this system. Besides, Table 1 reveals that the Matlab `ode45` solver can not solve the system (1.2) on some intervals, while the scheme developed here readily solve it on the same interval. In fact, this kind of methods (orthogonal approximations) permits bigger step size and so could save total computational time.

TABLE 1. The numerical results obtained from solving (1.2) when $\varepsilon = 10$ using the APSP algorithm when $N_{min} = 5$, $N_{max} = 25$, $fac_{err} = 0.1$, $fac = 0.9$, $fac_{min} = 0.5$ and $fac_{max} = 1.5$

Algorithm	M	T	$Atol$	$Rtol$	No. of steps	CPU time (s)
APSP	5	1000	10^{-13}	10^{-13}	47	0.30
ODE45	5	1000	10^{-13}	10^{-13}	661481	132.43
APSP	10	1000	10^{-12}	10^{-12}	119	0.95
ODE45	10	1000	10^{-12}	10^{-12}	–	> 3000
APSP	20	100	10^{-11}	10^{-11}	406	5.29
ODE45	20	100	10^{-11}	10^{-11}	–	> 5000

REFERENCES

1. A. Ghorbani, *Toward a new analytical method for solving nonlinear fractional differential equations*, Comput. Meth. Appl. Mech. Engrg. 197 (2008), 4173–4179.
2. J. Saberi-Nadjafi and A. Ghorbani, *Piecewise-truncated parametric iteration method: a promising analytical method for solving Abel differential equations*, Z. Naturforsch. 65a (2010), 529–539.
3. A. Ghorbani, M. Gachpazan and J. Saberi-Nadjafi, *A modified parametric iteration method for solving nonlinear second order BVPs*, Comput. Appl. Math. 30 (2011), 499–515.
4. A. Ghorbani and J. Saberi-Nadjafi, *A piecewise–spectral parametric iteration method for solving the nonlinear chaotic Genesio system*, Math. Comput. Modelling 54 (2011), 131–139.