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On Extension of Bi-Derivations to the Bidual of Banach Algebras

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Abstract. We present some necessary and sufficient conditions such that the (Arens) extensions of a bi-derivation on Banach algebras are again bi-derivations. We then examine our results for some Banach algebras. In particular, we show that the (Arens) extensions of a bi-derivation on C^*-algebras are bi-derivations. Some results on extensions of an inner bi-derivation are also included.

1. Introduction

Let $\mathscr{A}$ be a Banach algebra and $\mathscr{X}$ be a Banach $\mathscr{A}-$module. A bounded bilinear mapping $\mathscr{D} : \mathscr{A}\times\mathscr{A} \longrightarrow \mathscr{X}$ is called a bi-derivation if it is a derivation in each argument. For example, for every $x \in Z(\mathscr{X}) := \{x \in \mathscr{X}; \quad ax = xa \quad \text{for all} \quad a \in \mathscr{A}\}$, the mapping $(a, b) \mapsto x[a, b]$ is a bi-derivation ($[\cdot, \cdot]$ stands for the Lie bracket), which is called an inner bi-derivation implemented by x. To give a non-inner bi-derivation, it is easy to verify that for every non-zero (bounded) derivation $\delta : \mathscr{A} \to \mathscr{A}$ on a commutative Banach algebra $\mathscr{A}$, the map $(a, b) \mapsto \delta(a)\delta(b) : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$, defines a non-inner bi-derivation. Bi-derivations are studied from many different approaches, the main one is based on the study of commuting maps. A bounded linear map $h : \mathscr{A} \to \mathscr{X}$ is said to be commuting if $[h(a), a] = 0$ for every $a \in \mathscr{A}$. A commuting map h gives rise to a bi-derivation $(a, b) \mapsto [h(a), b]$. For more applications of bi-derivations in some other field, see the survey article [5, Section 3]. Some algebraic aspects of bi-derivations on certain algebras were investigated by many authors; see for example [4, 10], where the structures of bi-derivations on triangular algebras and generalized matrix algebras are discussed, and particularly the question of whether bi-derivations on these algebras are inner, was considered.

In this paper we study the (Arens) extensions of a bi-derivation. The main motivation comes from the papers [6, 9, 13], where the extension of derivations from a Banach algebra $\mathscr{A}$ to certain Banach modules have been investigated. Let $\delta : \mathscr{A} \longrightarrow \mathscr{X}$ be a derivation, the question "is the second adjoint δ^{**} of δ a derivation?" was studied by Dales et al. in [9], for the special case $\mathscr{X} = \mathscr{A}^*$. They showed that δ is a derivation if and only if $\delta^{**}(\mathscr{A}^{**}).\mathscr{A}^{**} \subseteq \mathscr{A}^*$ (see [9, Theorem 7.1]). Then Mohammadzadeh and Vishki answered the question for a general dual Banach $\mathscr{A}-$module $\mathscr{X}^*$, (see [13, Theorem 4.2]). Barootkoob and Vishki [3] have also generalized the results of [6, 9, 13] for derivations into the various duals of a Banach $\mathscr{A}-$module $\mathscr{X}$.

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The question which is of interest is under what conditions the Arens extensions (i.e. the third adjoints $\mathscr{D}^{***}, \mathscr{D}^{t***t}$) of a bounded bi-derivation $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}$ (or $\mathscr{X}^*$) is again a bi-derivation? In this respect, we shall give some necessary and sufficient conditions such that the extensions $\mathscr{D}^{***}$ and $\mathscr{D}^{t***t}$ are again bi-derivations (Theorems 3.2, 4.2). We conclude that this is the case for Banach algebras with property (WC), which include the C^*−algebras (Corollaries 5.2, 5.3). We also provide some conditions under which the extensions of an inner bi-derivation is again an inner bi-derivation.

2. Preliminaries

In [2] Arens showed that every bounded bilinear map $f : \mathscr{X} \times \mathscr{Y} \longrightarrow \mathcal{Z}$ on normed spaces has two natural but, in general, different extensions f^{***} and f^{t***t} from $\mathscr{X}^{**} \times \mathscr{Y}^{**}$ to $\mathcal{Z}^{**}$. The third adjoint f^{***} is defined recursively by $f^{***} = (f^{**})^*$ and $f^{**} = (f^*)^*$, where the adjoint $f^* : \mathcal{Z}^* \times \mathscr{X} \longrightarrow \mathscr{Y}^*$ of f is defined by

$$\langle f^*(\tau, x), y\rangle = \langle \tau, f(x, y)\rangle \qquad (x \in \mathscr{X}, y \in \mathscr{Y} \text{ and } \tau \in \mathcal{Z}^*).$$

Then $f^{***} : \mathscr{X}^{**} \times \mathscr{Y}^{**} \longrightarrow \mathcal{Z}^{**}$ is the unique extension of f such that $f^{***}(\cdot, n)$ and $f^{***}(x, \cdot)$ are w^*−continuous for every $n \in \mathscr{Y}^{**}, x \in \mathscr{X}$. Similarly $f^{t***t} : \mathscr{X}^{**} \times \mathscr{Y}^{**} \longrightarrow \mathcal{Z}^{**}$ is the unique extension of f such that $f^{t***t}(m, \cdot)$ and $f^{t***t}(\cdot, y)$ are w^*−continuous for every $m \in \mathscr{X}^{**}, y \in \mathscr{Y}$; where the flip map $f^t : \mathscr{Y} \times \mathscr{X} \to \mathcal{Z}$ is defined by $f^t(y, x) = f(x, y) \quad (x \in \mathscr{X}, y \in \mathscr{Y})$. If $f^{***} = f^{t***t}$ then f is said to be Arens regular. Some characterizations for the Arens regularity of f are proved in [13].

For the multiplication $\pi : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ of a Banach algebra $\mathscr{A}$, π^{***} and π^{t***t} are called the first and second Arens products of $\mathscr{A}$ and will be denoted by $\Box$ and $\Diamond$, respectively. A Banach algebra $\mathscr{A}$ is said to be Arens regular if the multiplication map π is Arens regular, or equivalently $\Box = \Diamond$ on the whole of $\mathscr{A}^{**}$. For example, every C^*−algebra is Arens regular; to see a proof for this fact and also for the complete account on Arens regularity of Banach algebras one may refer to [7, 8] and references therein.

Let $\mathscr{A}$ be a Banach algebra, $\mathscr{X}$ be a Banach space and let $\pi_\ell : \mathscr{A} \times \mathscr{X} \longrightarrow \mathscr{X}$, $\pi_r : \mathscr{X} \times \mathscr{A} \longrightarrow \mathscr{X}$ be bounded bilinear maps. Then the triple $(\pi_\ell, \mathscr{X}, \pi_r)$ is said to be a Banach $\mathscr{A}$−module if $\pi_\ell(ab, x) = \pi_\ell(a, \pi_\ell(b, x)), \pi_r(x, ab) = \pi_r(\pi_r(x, a), b)$ and $\pi_\ell(a, \pi_r(x, b)) = \pi_r(\pi_\ell(a, x), b)$ for every $a, b \in \mathscr{A}$ and $x \in \mathscr{X}$. Throughout the paper, when there is no ambiguity, we usually write $\mathscr{X}$ instead of $(\pi_\ell, \mathscr{X}, \pi_r)$.

For a Banach $\mathscr{A}$−module $(\pi_\ell, \mathscr{X}, \pi_r)$ it is obvious that $(\pi_r^{t*t}, \mathscr{X}^*, \pi_\ell^*)$ is a Banach $\mathscr{A}$−module which is called the dual of (π_ℓ, X, π_r). A direct verification also reveals that $(\pi_\ell^{***}, \mathscr{X}^{**}, \pi_r^{***})$ is a Banach $(\mathscr{A}^{**}, \Box)$−module while $(\pi_\ell^{t***t}, \mathscr{X}^{**}, \pi_r^{t***t})$ is a Banach $(\mathscr{A}^{**}, \Diamond)$−module. Arens regularity of certain module operations are studied in [12].

3. Extending of a General Bi-derivation

Let $\mathscr{X}$ be a Banach $\mathscr{A}$−module. A direct verification reveals that the second adjoint $\delta^{**} : \mathscr{A}^{**} \longrightarrow \mathscr{X}^{**}$ of a derivation $\delta : \mathscr{A} \longrightarrow \mathscr{X}$ is always a derivation. However, as we have illustrated in the next example, this is not the case for a bi-derivation, in general.

Example 3.1. *Let $\delta : \mathscr{A} \longrightarrow \mathscr{A}$ be a derivation on a Banach algebra $\mathscr{A}$. Then it can be easily verified that the map $\mathscr{D}_\delta : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ defined by $\mathscr{D}_\delta(a, b) = \delta(a)\delta(b)$ is a bi-derivation in the case where $\delta(A) \subseteq Z(\mathscr{A})$; for example this is satisfied if $\mathscr{A}$ is commutative. More precisely, one may show that $\mathscr{D}_\delta$ is a bi-derivation if and only if $[\delta(a), b]\delta(c) = 0 = \delta(c)[\delta(a), b]$ for every $a, b, c \in \mathscr{A}$. Moreover it is easy to verify that ${\mathscr{D}_\delta}^{***}(m, n) = \delta^{**}(m)\Box\delta^{**}(n)$; so ${\mathscr{D}_\delta}^{***}$ is a bi-derivation if and only if $[\delta^{**}(m), n]\delta^{**}(p) = 0 = \delta^{**}(p)[\delta^{**}(m), n]$ for all $m, n, p \in \mathscr{A}^{**}$. In particular, ${\mathscr{D}_\delta}^{***}$ is a bi-derivation if $(\mathscr{A}^{**}, \Box)$ is commutative. We recall that the bidual $(\mathscr{A}^{**}, \Box)$ of a commutative Banach algebra $\mathscr{A}$ is commutative if and only if $\mathscr{A}$ is Arens regular, [7].*

Therefore it is quite easy to find examples of bi-derivations whose third adjoints are not bi-derivations. Indeed, by choosing a suitable nonzero derivation δ on a commutative non-Arens regular Banach algebra $\mathscr{A}$ we can have a bi-derivation $\mathscr{D}_\delta : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ for which the third adjoint ${\mathscr{D}_\delta}^{***} : (\mathscr{A}^{**}, \Box) \times (\mathscr{A}^{**}, \Box) \longrightarrow (\mathscr{A}^{**}, \Box)$ may not be a bi-derivation.

In the next result we provide a necessary and sufficient condition such that the third adjoint $\mathscr{D}^{***}$ of a bi-derivation $\mathscr{D}:\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{X}$ is again a bi-derivation.

Theorem 3.2. *Let $(\pi_\ell,\mathscr{X},\pi_r)$ be a Banach $\mathscr{A}-$module and let $\mathscr{D}:\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{X}$ be a bi-derivation. Then $\mathscr{D}^{***}:(\mathscr{A}^{**},\Box)\times(\mathscr{A}^{**},\Box)\longrightarrow\mathscr{X}^{**}$ is a bi-derivation if and only if for every $a\in\mathscr{A},\tau\in\mathscr{X}^*$ the bilinear maps*

$$\begin{array}{rcl}\varphi_a:\mathscr{A}\times\mathscr{A} & \to & \mathscr{X}\\ (c,b) & \mapsto & \pi_r(\mathscr{D}(a,b),c)\end{array}\quad \text{and}\quad \begin{array}{rcl}\psi_\tau:\mathscr{A}\times\mathscr{A} & \to & \mathscr{A}^*\\ (c,b) & \mapsto & \mathscr{D}^*(\pi_\ell{}^*(\tau,b),c)\end{array}$$

are Arens regular.

Proof. For arbitrary elements $m,n,p\in\mathscr{A}^{**}$ put $\{a_\alpha\},\{b_\beta\},\{c_\gamma\}$ be bounded nets in $\mathscr{A}$, converging in w^*-topology to m,n,p, respectively. Note that $\mathscr{X}^{**}$ as an $(\mathscr{A}^{**},\Box)-$module is equipped with the module operations given by $(\pi_\ell^{***},\mathscr{X}^{**},\pi_r^{***})$. Then

$$\begin{aligned}\mathscr{D}^{***}(m\Box p,n) &= w^*-\lim_\alpha\lim_\gamma\lim_\beta \mathscr{D}(a_\alpha c_\gamma,b_\beta)\\ &= w^*-\lim_\alpha\lim_\gamma\lim_\beta\Big(\pi_\ell(a_\alpha,\mathscr{D}(c_\gamma,b_\beta))+\pi_r(\mathscr{D}(a_\alpha,b_\beta),c_\gamma)\Big)\\ &= \pi_\ell{}^{***}(m,\mathscr{D}^{***}(p,n))+w^*-\lim_\alpha \pi_r{}^{t***t}(\mathscr{D}^{***}(a_\alpha,p),n).\end{aligned}$$

On the other hand,

$$\pi_\ell{}^{***}(m,\mathscr{D}^{***}(p,n))+\pi_r{}^{***}(\mathscr{D}^{***}(m,n),p)=\pi_\ell{}^{***}(m,\mathscr{D}^{***}(p,n))+w^*-\lim_\alpha\pi_r{}^{***}(\mathscr{D}^{***}(a_\alpha,p),n).$$

Therefore, $\mathscr{D}^{***}(\cdot,n)$ is a derivation if and only if

$$w^*-\lim_\alpha\pi_r{}^{***}(\mathscr{D}^{***}(a_\alpha,p),n)=w^*-\lim_\alpha\pi_r{}^{t***t}(\mathscr{D}^{***}(a_\alpha,p),n);$$

or equivalently, for all $a\in\mathscr{A}$,

$$\pi_r^{***}(\mathscr{D}^{***}(a,p),n)=\pi_r^{t***t}(\mathscr{D}^{***}(a,p),n).$$

And the latter is equivalent to the Arens regularity of φ_a.

Similarly we shall prove that $\mathscr{D}^{***}(m,\cdot)$ is a derivation if and only if ψ_τ is Arens regular for all $\tau\in\mathscr{X}^*$. To do this, first we have

$$\begin{aligned}\langle\mathscr{D}^{***}(m,n\Box p),\tau\rangle &= \lim_\alpha\lim_\beta\lim_\gamma\langle\tau,\mathscr{D}(a_\alpha,b_\beta c_\gamma)\rangle\\ &= \lim_\alpha\lim_\beta\lim_\gamma\langle\tau,\Big(\pi_\ell(b_\beta,\mathscr{D}(a_\alpha,c_\gamma))+\pi_r(\mathscr{D}(a_\alpha,b_\beta),c_\gamma)\Big)\rangle\\ &= \lim_\alpha\lim_\beta\langle p,\mathscr{D}^*(\pi_\ell{}^*(\tau,b_\beta),a_\alpha)\rangle+\langle\pi_r{}^{***}(\mathscr{D}^{***}(m,n),p),\tau\rangle\\ &= \langle\mathscr{D}^{*t***t}(\pi_\ell{}^{****}(\tau,n),m),p\rangle+\pi_r{}^{***}(\mathscr{D}^{***}(m,n),p),\tau\rangle.\end{aligned}$$

On the other hand,

$$\langle\Big(\pi_\ell{}^{***}(n,\mathscr{D}^{***}(m,p))+\pi_r{}^{***}(\mathscr{D}^{***}(m,n),p)\Big),\tau\rangle \quad=\quad \langle\mathscr{D}^{****}(\pi_\ell{}^{****}(\tau,n),m),p\rangle+\langle\pi_r{}^{***}(\mathscr{D}^{***}(m,n),p),\tau\rangle.$$

Therefore $\mathscr{D}^{***}(m,\cdot)$ is a derivation if and only if for every $\tau\in\mathscr{X}^*$,

$$\mathscr{D}^{*t***t}(\pi_\ell^{****}(\tau,n),m)=\mathscr{D}^{****}(\pi_\ell^{****}(\tau,n),m),$$

and this is equivalent to the Arens regularity of ψ_τ, and this completes the proof. $\Box$

Remark 3.3. *It should be remarked, however, that $(\pi_\ell^{***},\mathscr{X}^{**},\pi_r^{***})$ is a Banach $(\mathscr{A}^{**},\Box)-$module, but in the proof of Theorem 3.2 we merely used the fact that $(\pi_\ell^{***},\mathscr{X}^{**})$ is a left $(\mathscr{A}^{**},\Box)-$module and that $(\mathscr{X}^{**},\pi_r^{***})$ is a right $(\mathscr{A}^{**},\Box)-$module. As $(\pi_\ell^{t***t},\mathscr{X}^{**})$ and $(\mathscr{X}^{**},\pi_r^{t***t})$ are left and right Banach $(\mathscr{A}^{**},\Diamond)-$module, respectively, similar to what was presented in Theorem 3.2, one may prove the next statements for a bi-derivation $\mathscr{D}:\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{X}$.*

(1) *$\mathscr{D}^{***} : (\mathscr{A}^{**}, \Diamond) \times (\mathscr{A}^{**}, \Box) \longrightarrow \mathscr{X}^{**}$ is a bi-derivation if and only if for every $a \in \mathscr{A}, \tau \in \mathscr{X}^*$ the maps*

$$\begin{array}{rcl} \omega_a : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{X} \\ (c, b) & \mapsto & \pi_\ell(c, \mathscr{D}(a, b)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi_\tau : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{A}^* \\ (c, b) & \mapsto & \mathscr{D}^*({\pi_\ell}^*(\tau, b), c) \end{array}$$

are Arens regular.

(2) *$\mathscr{D}^{***} : (\mathscr{A}^{**}, \Box) \times (\mathscr{A}^{**}, \Diamond) \longrightarrow \mathscr{X}^{**}$ is a bi-derivation if and only if for every $a \in \mathscr{A}, \tau \in \mathscr{X}^*$ the maps*

$$\begin{array}{rcl} \varphi_a : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{X} \\ (c, b) & \mapsto & \pi_r(\mathscr{D}(a, b), c) \end{array} \quad \text{and} \quad \begin{array}{rcl} \chi_\tau : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{A}^* \\ (c, b) & \mapsto & \mathscr{D}^*({\pi_r}^{t*}(\tau, b), c) \end{array}$$

are Arens regular.

(3) *$\mathscr{D}^{***} : (\mathscr{A}^{**}, \Diamond) \times (\mathscr{A}^{**}, \Diamond) \longrightarrow \mathscr{X}^{**}$ is a bi-derivation if and only if for every $a \in \mathscr{A}, \tau \in \mathscr{X}^*$ the maps*

$$\begin{array}{rcl} \omega_a : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{X} \\ (c, b) & \mapsto & \pi_\ell(c, \mathscr{D}(a, b)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \chi_\tau : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{A}^* \\ (c, b) & \mapsto & \mathscr{D}^*({\pi_r}^{t*}(\tau, b), c) \end{array}$$

are Arens regular.

Hereafter we only work with the first Arens product.

Parallel to the results [9, Theorem 7.1] and [13, Theorem 4.2] about the extension of a derivation, in the next result we present the same characterization for the extension of a bi-derivation.

Theorem 3.4. *Let $\mathscr{A}$ be a Banach algebra and let $\mathscr{X}$ be a Banach $\mathscr{A}$−module. Then the third adjoint $\mathscr{D}^{***} : (\mathscr{A}^{**}, \Box) \times (\mathscr{A}^{**}, \Box) \longrightarrow \mathscr{X}^{**}$ of a bi-derivation $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}$ is a bi-derivation if and only if*

$$ {\pi_\ell}^{*****}(\mathscr{D}^{*t****}(\mathscr{A}^{**}, \mathscr{A}^{**}), \mathscr{X}^*) \subseteq \mathscr{A}^* \qquad \text{and} \qquad \mathscr{D}^{****}({\pi_r}^{t****}(\mathscr{X}^*, \mathscr{A}^{**}), \mathscr{A}) \subseteq \mathscr{A}^*.$$

Proof. By Theorem 3.2, $\mathscr{D}^{***}$ is a bi-derivation if and only if the maps φ_a and ψ_τ are Arens regular for every $a \in \mathscr{A}$ and $\tau \in \mathscr{X}^*$. On the other hand, by [13, Theorem 2.1], φ_a and ψ_τ are Arens regular if and only if ${\varphi_a}^{****}(\mathscr{X}^*, \mathscr{A}^{**}) \subseteq \mathscr{A}^*$ and ${\psi_\tau}^{****}(\mathscr{A}^{**}, \mathscr{A}^{**}) \subseteq \mathscr{A}^*$. Now the conclusion follows from the fact that $\mathscr{D}^{****}({\pi_r}^{t****}(\mathscr{X}^*, \mathscr{A}^{**}), a) = {\varphi_a}^{****}(\mathscr{X}^*, \mathscr{A}^{**})$ and ${\pi_\ell}^{*****}(\mathscr{D}^{*t****t}(\mathscr{A}^{**}, \mathscr{A}^{**}), \tau) = {\psi_\tau}^{****}(\mathscr{A}^{**}, \mathscr{A}^{**})$. □

In the next result, we focus on the inner bi-derivation and study with some conditions under which the third adjoint of an inner bi-derivation is again an inner bi-derivation.

Theorem 3.5. *Let $\mathscr{X}$ be a Banach $\mathscr{A}$−module and let $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}$ be an inner bi-derivation implemented by $x \in Z(\mathscr{X})$. Then the following assertions are equivalent.*

(a) *$\mathscr{D}^{***} : (\mathscr{A}^{**}, \Box) \times (\mathscr{A}^{**}, \Box) \longrightarrow \mathscr{X}^{**}$ is an inner bi-derivation implemented by x.*

(b) *$\mathscr{D}^{***} : (\mathscr{A}^{**}, \Diamond) \times (\mathscr{A}^{**}, \Diamond) \longrightarrow \mathscr{X}^{**}$ is an inner bi-derivation implemented by x.*

(c) *$m \Box n = m \Diamond n$ on $\pi_r^*(\mathscr{X}^*, x)$, for all $m, n \in \mathscr{A}^{**}$.*

(d) *The bilinear map $(a, b) \longmapsto \pi_r((\pi_r(x, b), a) : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}$ is Arens regular.*

Proof. First note that the condition $x \in Z(\mathscr{X})$ implies that $x \in Z(\mathscr{X}^{**})$. That (c) is equivalent to (d) follows trivially. For the equivalency (a)⇔(c), let $m, n \in \mathscr{A}^{**}$ and let $\{a_\alpha\}, \{b_\beta\}$ be nets in $\mathscr{A}$, w^*−converging to m and n, respectively. Then for every $\tau \in \mathscr{X}^*$,

$$\begin{aligned} \langle \mathscr{D}^{***}(m, n), \tau \rangle &= \lim_\alpha \lim_\beta \langle \tau, D(a_\alpha, b_\beta) \rangle \\ &= \lim_\alpha \lim_\beta \langle \tau, \pi_r(x, [a_\alpha, b_\beta]) \rangle \\ &= \lim_\alpha \lim_\beta \langle \tau, \pi_r(x, a_\alpha b_\beta) - \pi_r(x, b_\beta a_\alpha) \rangle \\ &= \langle \pi_r^{***}(x, m \Box n) - \pi_r^{***}(x, n \Diamond m), \tau \rangle \\ &= \langle (m \Box n - n \Diamond m), \pi_r^*(\tau, x) \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \pi_r^{***}(x,[m,n]),\tau\rangle &= \langle \pi_r^{***}(x,(m\Box n - n\Box m)),\tau\rangle\\ &= \langle (m\Box n - n\Box m), \pi_r^{*}(\tau,x)\rangle.\end{aligned}$$

Therefore (a) holds if and only if $\mathscr{D}^{***}(m,n) = \pi_r^{***}(x,[m,n])$ and the latter is valid if and only if $m\Box n = m\Diamond n$ on $\pi_r^*(\mathscr{X}^*, x)$. The equivalency (b)⇔(c) follows from a similar argument. □

As an immediate consequence we get the next result.

Corollary 3.6. *Let $\mathscr{X}$ be a Banach $\mathscr{A}$−module. If $\mathscr{A}$ is Arens regular then every Arens extension of an inner bi-derivation $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}$ is an inner bi-derivation implemented by the same element.*

4. Extending a Dual Module Valued Bi-derivation

We recall that for a Banach $\mathscr{A}$−module $(\pi_\ell, \mathscr{X}, \pi_r)$, the third dual $\mathscr{X}^{***}$ of $\mathscr{X}$ as a Banach $(\mathscr{A}^{**},\Box)$−module can be equipped with two, in general, different module operations. Indeed, $(\pi_r^{t*t***}, \mathscr{X}^{***}, \pi_\ell^{****})$ and also $(\pi_r^{***t*t}, \mathscr{X}^{***}, \pi_\ell^{****})$ are both $(\mathscr{A}^{**},\Box)$−modules, which the first one comes as the bidual of $(\pi_r^{t*t}, \mathscr{X}^*, \pi_\ell^*)$ and the second one is the dual of $(\pi_\ell^{***}, \mathscr{X}^{**}, \pi_r^{***})$. Similarly there are two, in general, different $(\mathscr{A}^{**},\Diamond)$−module operations for $\mathscr{X}^{***}$; one of them is $(\pi_r^{t****t}, \mathscr{X}^{***}, \pi_\ell^{t***t*})$ and the other one is $(\pi_r^{t****r}, \mathscr{X}^{***}, \pi_\ell^{*t***t})$. We commence with the next lemma.

Lemma 4.1. *Let $\mathscr{X}$ be a Banach $\mathscr{A}$−module and let $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}^*$ be a bi-derivation such that the map $\Psi_q : \mathscr{A} \times \mathscr{A} \to \mathscr{A}^* : (a,b) \mapsto \mathscr{D}^*({\pi_r}^{t*t*}(q,b),a)$ is Arens regular for each $q \in \mathscr{X}^{**}$. Then the following assertions are equivalent.*

(i) *For every $m,n,p \in \mathscr{A}^{**}$,*

$$\pi_r^{t*t***}(n,\mathscr{D}^{***}(m,p)) = \pi_r^{***t*t}(n,\mathscr{D}^{***}(m,p)).$$

(ii) *For every $a \in \mathscr{A}, n,p \in \mathscr{A}^{**}$*

$$\pi_r^{t*t***}(n,\mathscr{D}^{***}(a,p)) = \pi_r^{***t*t}(n,\mathscr{D}^{***}(a,p)).$$

Proof. We only need to show that (ii) implies (i). For arbitrary elements $m,n,p \in \mathscr{A}^{**}$ let $\{a_\alpha\}, \{b_\beta\}, \{c_\gamma\}$ be bounded nets in $\mathscr{A}$, w^*−converging to m,n,p, respectively. By (ii) and the fact Ψ_q is Arens regular we get

$$\begin{aligned}\langle \pi_r^{t*t***}(n,\mathscr{D}^{***}(m,p)),q\rangle &= \lim_\beta \lim_\alpha \lim_\lambda \langle q, {\pi_r}^{t*t}(b_\beta, \mathscr{D}(a_\alpha, c_\lambda))\rangle\\ &= \lim_\beta \lim_\alpha \langle \mathscr{D}^*(\pi_r^{t*t*}(q,b_\beta),a_\alpha),p)\rangle\\ &= \lim_\beta \lim_\alpha \langle \Psi_q(a_\alpha,b_\beta),p\rangle\\ &= \lim_\alpha \lim_\beta \langle \Psi_q(a_\alpha,b_\beta),p\rangle\\ &= \lim_\alpha \lim_\beta \lim_\lambda \langle q, {\pi_r}^{t*t}(b_\beta, \mathscr{D}(a_\alpha, c_\lambda))\rangle\\ &= \lim_\alpha \langle {\pi_r}^{t*t***}(n,\mathscr{D}^{***}(a_\alpha,p)),q\rangle\\ &= \lim_\alpha \langle {\pi_r}^{***t*t}(n,\mathscr{D}^{***}(a_\alpha,p)),q\rangle\\ &= \langle {\pi_r}^{***t*t}(n,\mathscr{D}^{***}(m,p)),q\rangle;\end{aligned}$$

as claimed. □

In the next result, we are concerned with some necessary and sufficient conditions for the (Arens) extension $\mathscr{D}^{***}$ of a dual valued bi-derivation $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}^*$ to be a bi-derivation.

Theorem 4.2. *Let $\mathscr{X}$ be a Banach $\mathscr{A}$−module and let $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}^*$ be a bi-derivation. Then $\mathscr{D}^{***} : (\mathscr{A}^{**}, \square) \times (\mathscr{A}^{**}, \square) \longrightarrow \mathscr{X}^{***}$ is a bi-derivation if and only if the following assertions hold.*

(i) *For every $a \in \mathscr{A}, q \in \mathscr{X}^{**}$ the maps*

$$\begin{array}{rcl} \Phi_a : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{X}^* \\ (c, b) & \mapsto & {\pi_\ell}^*(\mathscr{D}(a, b), c) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi_q : \mathscr{A} \times \mathscr{A} & \rightarrow & \mathscr{A}^* \\ (c, b) & \mapsto & \mathscr{D}^*({\pi_r}^{t*t*}(q, b), c) \end{array}$$

are Arens regular.

(ii) *For every $a \in \mathscr{A}, n, p \in \mathscr{A}^{**}$,*

$$\pi_r^{t*t***}(n, \mathscr{D}^{***}(a, p)) = \pi_r^{***t*t}(n, \mathscr{D}^{***}(a, p)). \tag{4.1}$$

Proof. Applying Theorem 3.2 for $\mathscr{D} : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{X}^*$, condition (i)–that is the Arens regularity Φ_a and Ψ_q for all $a \in \mathscr{A}, q \in \mathscr{X}^{**}$– is equivalent to the fact that $\mathscr{D}^{***} : (\mathscr{A}^{**}, \square) \times (\mathscr{A}^{**}, \square) \longrightarrow (\mathscr{X}^*)^{**}$ is a bi-derivation, or equivalently, for every m, n, p in $\mathscr{A}^{**}$,

$$\mathscr{D}^{***}(m\square n, p) = {\pi_\ell}^{****}(\mathscr{D}^{***}(m, p), n) + \pi_r^{t*t***}(m, \mathscr{D}^{***}(n, p)), \tag{4.2}$$

and

$$\mathscr{D}^{***}(m, n\square p) = \pi_\ell^{****}(\mathscr{D}^{***}(m, n), p) + \pi_r^{t*t***}(n, \mathscr{D}^{***}(m, p)). \tag{4.3}$$

On the other hand $\mathscr{D}^{***} : (\mathscr{A}^{**}, \square) \times (\mathscr{A}^{**}, \square) \longrightarrow (\mathscr{X}^{**})^*$ is a bi-derivation if and only if for every m, n, p in $\mathscr{A}^{**}$,

$$\mathscr{D}^{***}(m\square n, p) = {\pi_\ell}^{****}(\mathscr{D}^{***}(m, p), n) + \pi_r^{***t*t}(m, \mathscr{D}^{***}(n, p)), \tag{4.4}$$

and

$$\mathscr{D}^{***}(m, n\square p) = \pi_\ell^{****}(\mathscr{D}^{***}(m, n), p) + \pi_r^{***t*t}(n, \mathscr{D}^{***}(m, p)). \tag{4.5}$$

By comparing (4.2) with (4.4) and by applying Theorem 3.2, we can deduce that $\mathscr{D}^{***} : (\mathscr{A}^{**}, \square) \times (\mathscr{A}^{**}, \square) \longrightarrow \mathscr{X}^{***}$ is a bi-derivation if and only (i) and (ii) hold. □

As remarked in Remark 3.3, Theorem 4.2 can also be obtained by replacing the first Arens product with the second Arens product.

5. Extension of Bi-derivations and Property (WC)

Here, we examine our results for Banach algebras with property (WC). We say that a Banach space $\mathscr{X}$ has property (WC) if every bounded linear operator from $\mathscr{X}$ to $\mathscr{X}^*$ is weakly compact. For example, every C^*−algebra has property (WC). This is a consequence of a well-known result of Akemann [1, Corollry II9], stating that every bounded linear operator from a C^*− algebra to the predual of a W^*−algebra is weakly compact. Godefroy and Iochum [11] have also studied property (WC) and they established some useful links to the geometry of Banach spaces. It should be also reminded that if a Banach algebra has the property (WC) then it is Arens regular (see for example, [11, Remarks I.4]). However, this can also be deduced as a consequence of the next more general result which will be used in the sequel.

Proposition 5.1. *Let $\mathscr{X}, \mathcal{Z}$ be Banach spaces. If $\mathscr{X}$ has the property (WC) then every bounded bilinear mapping $f : \mathscr{X} \times \mathscr{X} \longrightarrow \mathcal{Z}$ is Arens regular.*

Proof. It follows from [13, Theorem 2.1] and the fact that (by the (WC) property of $\mathscr{X}$) the linear operator $x \mapsto f^*(\tau, x) : \mathscr{X} \rightarrow \mathscr{X}^*$ is weakly compact for every $\tau \in \mathcal{Z}^*$. □

Combining Proposition 5.1 and Theorem 3.2 we get the next result for extension of a general bi-derivation.

Corollary 5.2. *Let $\mathscr{X}$ be a Banach $\mathscr{A}-$module. If $\mathscr{A}$ has property (WC) then every Arens extension of a bi-derivation $\mathscr{D}:\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{X}$ is a bi-derivation.*

Proof. By Theorem 3.2 it is enough to show that the maps $\varphi_a:\mathscr{A}\times\mathscr{A}\to\mathscr{X}$ and $\psi_\tau:\mathscr{A}\times\mathscr{A}\to\mathscr{A}^*$ are Arens regular for all $a\in\mathscr{A}$ and $\tau\in\mathscr{X}^*$; and these follow from Proposition 5.1 and the hypothesis that $\mathscr{A}$ possesses property (WC). □

Following Theorem 4.2 we arrive at the next result on the extension of a dual valued bi-derivation on algebras with property (WC).

Corollary 5.3. *Let $(\pi_\ell,\mathscr{X},\pi_r)$ be a Banach $\mathscr{A}-$module such that π_r and ${\pi_r}^{t*}$ are Arens regular. If $\mathscr{A}$ has property (WC) then the extension $\mathscr{D}^{***}:(\mathscr{A}^{**},\Box)\times(\mathscr{A}^{**},\Box)\longrightarrow\mathscr{X}^{***}$ of a bi-derivation $\mathscr{D}:\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{X}^*$ is a bi-derivation.*

Proof. By Theorem 4.2, $\mathscr{D}^{***}$ is a bi-derivation if and only if $\Phi_a:\mathscr{A}\times\mathscr{A}\to\mathscr{X}^*$, $\Psi_q:\mathscr{A}\times\mathscr{A}\to\mathscr{A}^*$ are Arens regular for all $a\in\mathscr{A},q\in\mathscr{X}^{**}$, and

$$\pi_r^{t*t***}(n,\mathscr{D}^{***}(a,p))=\pi_r^{***t*t}(n,\mathscr{D}^{***}(a,p)), \tag{5.1}$$

for every $a\in\mathscr{A},n,p\in\mathscr{A}^{**}$. The Arens regularity of Φ_a and Ψ_q follows from (WC) property of $\mathscr{A}$ (see Proposition 5.1) and the equality (5.1) follows from the Arens regularity of π_r and ${\pi_r}^{t*}$. Indeed, as it has been shown in [9, Proposition 4.1] (see also [13, Corollary 2.2]) the Arens regularity of π_r and ${\pi_r}^{t*}$ is equivalent to the equality $\pi_r^{t*t***}=\pi_r^{***t*t}$. □

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References

[1] C. A. Akemann, The dual space of an operator algebra, Trans. Amer. Math. Soc. **126** (1967), 286-302.
[2] A. Arens, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. **2** (1951), 839-848.
[3] S. Barootkoob and H. R. Ebrahimi Vishki, Lifting derivations and $n-$weak amenability of the second dual of Banach algebras, Bull. Austral. Math. Soc. **83** (2011), 122-129.
[4] D. Benkovič, Biderivations of triangular algebras, Linear Algebra Appl. **431** (2009), 1587-1602.
[5] M. Brešar, Commuting maps: A survey, Taiwanese J. Math. **8** (2004), 361-397.
[6] J. W. Bunce and W. L. Paschke, Derivations on a C^*-algebra and its double dual, J. Funct. Anal. **37** (1980), 253–247.
[7] H. G. Dales, Banach algebras and automatic continuity, London Math. Soc. Monographs 24 (Clarendon Press, Oxford, 2000).
[8] H. G. Dales and A. T.-M. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc. 177 (836) (2005).
[9] H. G. Dales, A. Rodriguez-Palacios and M. V. Velasco, The second transpose of a derivation, J. London Math. Soc. **64**(2) (2001), 707-721.
[10] Y. Du and Y. Wang, Biderivations of generalized matrix algebras, Linear Algebra. Appl. **438** (2013), 4483-4499.
[11] G. Godefroy and B. Iochum, Arens-regularity of Banach algebras and the geometry of Banach spaces, J. Funct. Anal. **80** (1988), 47-59.
[12] A. A. Khadem-Maboudi and H. R. Ebrahimi Vishki, Strong Arens irregularity of bilinear mappings and reflexivity, Banach J. Math. Anal. **6** (2012), 155-160.
[13] S. Mohammadzadeh and H. R. E. Vishki, Arens Regularity of module actions and the second adjoint of a derivation, Bull. Austral. Math. Soc. **77** (2008), 465-476.