Daws Conjecture on the Arens regularity of $B(X)$

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Abstract

We focus on a question raised by M. Daws [Bull. London Math. Soc. 36 (2004), 493-503] concerning the Arens regularity of $B(X)$, the algebra of operators on a Banach space X. In this respect, we introduce the notion of ultra-reflexivity for a Banach space and we characterize the Arens regularity of $B(X)$ in terms of the ultra-reflexivity of X.

2010 *Mathematics subject classification:* 47L10; 47L50; 46H25 .

Keywords and phrases: Arens products, algebra of operators, reflexive space, super-reflexive, weak operator topology, ultrapower.

1. Introduction

The second dual A^{**} of a Banach algebra A can be made into a Banach algebra with two, in general different, (Arens) products, each extending the original product of A [1]. A Banach algebra A is said to be Arens regular when the Arens products coincide. For example, every C^*-algebra is Arens regular [2]. For an explicit description of the properties of these products and the notion of Arens regularity one may consult with [3].

For the Banach algebra $B(X)$, bounded operators on a Banach space X, Daws showed that, if X is super-reflexive then $B(X)$ is Arens regular; [4, Theorem 1]. He also conjectured the validity of the converse. To the best of our knowledge, it seems that this has not be solved yet. It has been, however, known that the Arens regularity of $B(X)$ necessities the reflexivity of X, (for a proof see [3, Theorem 2.6.23]).

We introduce the notion of ultra-reflexive space and compare it with the super-reflexivity. Our main aim is to characterize the Arens regularity of $B(X)$ in terms of the ultra-reflexivity of X.

2. Preliminaries

Let X be a Banach space, I be an indexing set and let $\mathcal{U}$ be an ultrafilter on I. We define the ultrapower $X_{\mathcal{U}}$ of X with respect to $\mathcal{U}$, by the quotient space

$$X_{\mathcal{U}} = \ell^{\infty}(X, I)/\mathcal{N}_{\mathcal{U}},$$

* speaker

where $\ell^\infty(X, I)$ is the Banach space

$$\ell^\infty(X, I) = \{(x_\alpha)_{\alpha\in I} \subseteq X : \|(x_\alpha)\| = \sup_{\alpha\in I} \|x_\alpha\| < \infty\},$$

and $\mathcal{N}_\mathcal{U}$ is the closed subspace

$$\mathcal{N}_\mathcal{U} = \{(x_\alpha)_{\alpha\in I} \in \ell^\infty(X, I) : \lim_\mathcal{U} \|x_\alpha\| = 0\}.$$

Then the norm $\|(x_\alpha)\|_\mathcal{U} := \lim_\mathcal{U} \|x_\alpha\|$ coincides with the quotient norm. We can identify X with a closed subspace of $X_\mathcal{U}$ via the canonical isometric embedding $X \hookrightarrow X_\mathcal{U}$, sending $x \in X$ to the constant family (x). Ample information about ultrapowers can be found in [5].

A Banach space X is called super-reflexive if every finitely representable Banach space in X is reflexive. In the language of ultrapowers, it has been shown that Y is finitely representable in X if and only if Y is isometrically isomorphic to a subspace of $X_\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on X; [5, Theorem 6.3]. It follows that a Banach space is super-reflexive if and only if all of its ultrapowers are reflexive.

As it has been shown in [5, Section 7], there is a canonical isometry $J : (X^*)_\mathcal{U} \to (X_\mathcal{U})^*$ defined by the rule

$$\langle J((f_\alpha)_\mathcal{U}), (x_\alpha)_\mathcal{U}\rangle = \lim_\mathcal{U} \langle f_\alpha, x_\alpha\rangle \qquad ((f_\alpha)_\mathcal{U} \in (X^*)_\mathcal{U}, (x_\alpha)_\mathcal{U} \in X_\mathcal{U}),$$

which is a surjection if and only if $X_\mathcal{U}$ is reflexive (where $\mathcal{U}$ is countably incomplete). In particular, when X is super-reflexive then J is an isometric isomorphism.

As the $\mathrm{Ball}(X^{**})$ is w^*-compact, one can define a norm-decreasing map $\sigma : X_\mathcal{U} \to X^{**}$ by

$$\sigma((x_\alpha)_\mathcal{U}) = w^* - \lim_\mathcal{U} \kappa_X(x_\alpha), \qquad ((x_\alpha)_\mathcal{U} \in X_\mathcal{U}),$$

where κ_X is the canonical embedding of X into X^{**}. We quote the next result which establishes a useful connection between X^{**} and $X_\mathcal{U}$ for some ultrafilter $\mathcal{U}$.

Proposition 2.1 ([5, Proposition 6.7]). *Let X be a Banach space. Then there exist an ultrafilter $\mathcal{U}$ and a linear isometric embedding $K : X^{**} \to X_\mathcal{U}$ such that $\sigma \circ K$ is the identity on X^{**} and $K \circ \kappa_X$ is the canonical embedding of X into $X_\mathcal{U}$. Thus $K \circ \sigma$ is a norm-1 projection of $X_\mathcal{U}$ onto $K(X^{**})$.*

3. The main result

We recall that the super-reflexivity of X is equivalent to that of $\ell^2(X)$, the Banach space of all 2−summable sequences in X, (see [4, Proposition 4]). So X is super-reflexive if and only if $\mathrm{Ball}(\ell^2(X)_\mathcal{U})$ is weakly compact. This motivates to introduce the notion of ultra-reflexivity in the next definition.

Definition 3.1. *A Banach space X is called ultra-reflexive if* $\mathrm{Ball}(B(X))(x_\mathcal{U})$ *is weakly compact for every ultra-filter $\mathcal{U}$ and each $x_\mathcal{U} \in \ell^2(X)_\mathcal{U}$.*

It is obvious that every ultra-reflexive space X is reflexive. Indeed, for each non-zero $x \in X$, $B(X)(x) = X$. It is also worth to note that, if X is super-reflexive then X is ultra-reflexive. Therefore ultra-reflexivity lies between reflexivity and super-reflexivity.

In the following we present our main result characterizing the Arens regularity of $B(X)$ in terms of the ultra-reflexivity of X.

Theorem 3.2. *For a Banach space X the following assertions are equivalent.*

(a) *$B(X)$ is Arens regular.*
(b) *$f_{\mathcal{U}} \circ \mathrm{Ball}(B(X))$ is w^*−compact for every ultrafilter $\mathcal{U}$ and each $f_{\mathcal{U}} \in {\ell^2(X)_{\mathcal{U}}}^*$.*
(c) *$f_{\mathcal{U}} \circ \mathrm{Ball}(B(X))$ is w−compact for every ultrafilter $\mathcal{U}$ and each $f_{\mathcal{U}} \in {\ell^2(X)_{\mathcal{U}}}^*$.*
(d) *X is ultra-reflexive.*

Since for every super-reflexive space X the algebra $B(X)$ is Arens regular (see [4, Theorem 1]), it concludes that every super-reflexive space is ultra-reflexive. However, to the best of our knowledge, we do not know an ultra-reflexive space which is not super-reflexive! An example of a reflexive space which is not ultra-reflexive has been presented by Daws in [4, Corollary 2].

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