Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 109-114
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Product of derivations on C*-algebras

Sayed Khalil Ekrami[a,*], Madjid Mirzavaziri[b], Hamid Reza Ebrahimi Vishki[b]

[a]*Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran*
[b]*Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran*

(Communicated by M. Eshaghi)

Abstract

Let $\mathfrak{A}$ be an algebra. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ is called a *derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for each $a, b \in \mathfrak{A}$. Given two derivations δ and δ' on a C^*-algebra $\mathfrak{A}$, we prove that there exists a derivation Δ on $\mathfrak{A}$ such that $\delta\delta' = \Delta^2$ if and only if either $\delta' = 0$ or $\delta = s\delta'$ for some $s \in \mathbb{C}$.

Keywords: Derivation; C*-algebra.
2010 MSC: Primary 46L57; Secondary 47B47, 16W25.

1. Introduction

Let $\mathfrak{A}$ be an algebra. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ is called a *derivation* if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for each $a, b \in \mathfrak{A}$. When $\mathfrak{A}$ is a $*$-algebra, δ is called a $*$-*derivation* if $\delta(a^*) = \delta(a)^*$ for each $a \in \mathfrak{A}$.

Let δ be a $*$-derivation on a C*-algebra $\mathfrak{A}$, then δ^2 is a derivation if and only if $\delta = 0$. To see this, note that δ^2 is a derivation if and only if

$$\delta^2(x)y + 2\delta(x)\delta(y) + x\delta^2(y) = \delta^2(xy) = \delta^2(x)y + x\delta^2(y).$$

The latter is equivalent to the fact that $\delta(x)\delta(y) = 0$ for each $x, y \in \mathfrak{A}$. Thus $\delta(x)\delta(x)^* = \delta(x)\delta(x^*) = 0$ for each $x \in \mathfrak{A}$. Hence $\|\delta(x)\|^2 = \|\delta(x)\delta(x)^*\| = 0$. This shows that $\delta(x) = 0$ for each $x \in \mathfrak{A}$.

As a typical example of a non-zero derivation in a non-commutative algebra, we can consider the *inner derivation* δ_a *implemented by* an element $a \in \mathfrak{A}$ which is defined as $\delta_a(x) = xa - ax$ for each $x \in \mathfrak{A}$. Even for an inner derivation δ_a on an algebra $\mathfrak{A}$, it is very probable that δ_a^2 is *not* a derivation.

*Corresponding author
Email addresses: ekrami@pnu.ac.ir, khalil.ekrami@gmail.com (Sayed Khalil Ekrami), mirzavaziri@um.ac.ir, mirzavaziri@gmail.com (Madjid Mirzavaziri), vishki@um.ac.ir (Hamid Reza Ebrahimi Vishki)

Received: July 2016 *Revised:* December 2016

These considerations show that the set of derivations on an algebra $\mathfrak{A}$ is not in general closed under product. There are various researches seeking for some conditions under which the product of two derivations will be again a derivation. Posner [9] was the first one who studied the product of two derivations on a prime ring. He showed that if the product of two derivations on a prime ring, with characteristic not equal to 2, is a derivation then one of them must be equal to zero. The same question has been investigated by several authors on various algebras, see for example [1, 2, 3, 5, 6, 7, 8] and references therein. In the realm of C*-algebras, Mathieu [5] showed that, if the product of two derivations δ and δ' on a C*-algebra is a derivation then $\delta\delta' = 0$. The same result was proved by Pedersen [8] for unbounded densely defined derivations on a C*-algebra.

There are known algebras $\mathfrak{A}$ such that each derivation on $\mathfrak{A}$ is inner which is implemented by an element of the algebra $\mathfrak{A}$ or an algebra $\mathfrak{B}$ containing $\mathfrak{A}$. For example, each derivation on a von Neumann algebra $\mathfrak{M}$ is inner and is implemented by an element of $\mathfrak{M}$. Moreover, each derivation on a C*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathfrak{H}$ is inner and implemented by an element of the weak closure $\mathfrak{M}$ of $\mathfrak{A}$ in $\mathbf{B}(\mathfrak{H})$ (See [4, 10]).

In the present paper, we are concerned with the following problem: *"Given two derivations δ and δ' on a C*-algebra $\mathfrak{A}$, find necessary and sufficient condition under which there exists a derivation Δ on $\mathfrak{A}$ satisfying $\delta\delta' = \Delta^2$."*

We affirm that the condition is: either $\delta' = 0$ or $\delta = s\delta'$ for some $s \in \mathbb{C}$. We do this in two steps; for the matrix algebra $M_n(\mathbb{C})$ and for an arbitrary C*-algebra.

2. The equation for the case of matrix algebras

In this section we are mainly concerned with the structure of derivations on the matrix algebra $M_n(\mathbb{C})$. Let $A = [a_{ij}] \in M_n(\mathbb{C})$. We denote the diagonal matrix whose diagonal entries are a_{ii} by A^D.

Proposition 2.1. *Let $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{C})$. Then there exists a $C = [c_{ij}] \in M_n(\mathbb{C})$ such that $\delta_A\delta_B = {\delta_C}^2$ if and only if either $\delta_B = 0$ or $\delta_A = s\delta_B$ for some $s \in \mathbb{C}$.*

Proof . Let $\{E_{ij}\}_{1\leqslant i,j\leqslant n}$ be the standard system of matrix units for $M_n(\mathbb{C})$. First we show that $a_{ik}b_{\ell j} = b_{ik}a_{\ell j}$ for all $1 \leqslant i, k, \ell, j \leqslant n$ if and only if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$.

To see this, suppose that $a_{ik}b_{\ell j} = b_{ik}a_{\ell j}$ for all $1 \leqslant i, k, \ell, j \leqslant n$ then we can write

$$(E_{ii}AE_{k\ell})(E_{\ell\ell}BE_{jj}) = a_{ik}b_{\ell j}E_{ij} = b_{ik}a_{\ell j}E_{ij} = (E_{ii}BE_{k\ell})(E_{\ell\ell}AE_{jj}).$$

We thus have

$$(\sum_{i=1}^{n} E_{ii})AE_{k\ell}B(\sum_{j=1}^{n} E_{jj}) = (\sum_{i=1}^{n} E_{ii})BE_{k\ell}A(\sum_{j=1}^{n} E_{jj}).$$

This shows that $AE_{k\ell}B = BE_{k\ell}A$ for each $1 \leqslant k, \ell \leqslant n$. We can therefore deduce that $AXB = BXA$ for all $X \in M_n(\mathbb{C})$. On the other hand, if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$, then

$$a_{ij}b_{k\ell}E_{i\ell} = (E_{ii}AE_{jk})(E_{kk}BE_{\ell\ell}) = (E_{ii}BE_{jk})(E_{kk}AE_{\ell\ell}) = b_{ij}a_{k\ell}E_{i\ell}.$$

We can assume that $a_{11} = b_{11} = c_{11} = 0$. This is due to the fact that $\delta_{A-a_{11}I} = \delta_A$, $\delta_{B-b_{11}I} = \delta_B$ and $\delta_{C-c_{11}I} = \delta_C$. Then $\delta_A\delta_B = {\delta_C}^2$ if and only if

$$ABE_{k\ell} - AE_{k\ell}B - BE_{k\ell}A + E_{k\ell}BA = C^2E_{k\ell} - 2CE_{k\ell}C + E_{k\ell}C^2,$$

for each $1 \leqslant k, \ell \leqslant n$. This is equivalent to the fact that

$$E_{ii}(ABE_{k\ell} - AE_{k\ell}B - BE_{k\ell}A + E_{k\ell}BA)E_{jj} = E_{ii}(C^2E_{k\ell} - 2CE_{k\ell}C + E_{k\ell}C^2)E_{jj},$$

for each $1 \leqslant i, j, k, \ell \leqslant n$. Now for $i \neq k$ and $j \neq \ell$ we have

$$(0 - a_{ik}b_{\ell j} - b_{ik}a_{\ell j} + 0)E_{ij} = (0 - 2c_{ik}c_{\ell j} + 0)E_{ij}. \tag{2.1}$$

For $i \neq k$ and $j = \ell$ we have

$$(\sum_{m=1}^{n} a_{im}b_{mk} - a_{ik}b_{\ell\ell} - b_{ik}a_{\ell\ell} + 0)E_{i\ell} = (\sum_{m=1}^{n} c_{im}c_{mk} - 2c_{ik}c_{\ell\ell} + 0)E_{i\ell}. \tag{2.2}$$

For $i = k$ and $j \neq \ell$ we have

$$(0 - a_{kk}b_{\ell j} - b_{kk}a_{\ell j} + \sum_{m=1}^{n} b_{\ell m}a_{mj})E_{kj} = (0 - 2c_{kk}c_{\ell j} + \sum_{m=1}^{n} c_{\ell m}c_{mj})E_{kj}. \tag{2.3}$$

And finally for $i = k$ and $j = \ell$ we have

$$(\sum_{m=1}^{n} a_{km}b_{mk} - a_{kk}b_{\ell\ell} - b_{kk}a_{\ell\ell} + \sum_{m=1}^{n} b_{\ell m}a_{m\ell})E_{k\ell} = (\sum_{m=1}^{n} c_{km}c_{mk} - 2c_{kk}c_{\ell\ell} + \sum_{m=1}^{n} c_{\ell m}c_{m\ell})E_{k\ell}. \tag{2.4}$$

If $k \neq \ell$ then putting $i = \ell$ and $j = k$ in the equation (2.1) we have $c_{\ell k}^2 = a_{\ell k}b_{\ell k}$. Thus for $i \neq k$ and $j \neq \ell$ we have $(a_{ik}b_{\ell j} + b_{ik}a_{\ell j})^2 = 4c_{ik}^2c_{\ell j}^2 = 4a_{ik}b_{ik}a_{\ell j}b_{\ell j}$. This implies that

$$a_{ik}b_{\ell j} = b_{ik}a_{\ell j}, \text{ for } i \neq k, j \neq \ell. \tag{2.5}$$

Now, if $b_{\ell j} \neq 0$ for some $1 \leq \ell, j \leq n$ with $\ell \neq j$, then the equation

$$a_{ik} = \frac{a_{\ell j}}{b_{\ell j}} b_{ik}, \text{ for } i \neq k,$$

implies the existence of some α and β with $|\alpha| + |\beta| \neq 0$ such that

$$\alpha(A - A^D) = \beta(B - B^D). \tag{2.6}$$

If $b_{\ell j} = 0$ for all $1 \leq \ell, j \leq n$ with $\ell \neq j$, then $B = B^D$ and so the equation (2.6) holds for $\alpha = 0$ and any nonzero $\beta \in \mathbb{C}$.

Interchanging $\ell \leftrightarrow i, j \leftrightarrow k$ and $k \leftrightarrow \ell$ in (2.3) we have

$$\sum_{m=1}^{n} b_{im}a_{mk} - a_{\ell\ell}b_{ik} - b_{\ell\ell}a_{ik} = \sum_{m=1}^{n} c_{im}c_{mk} - 2c_{\ell\ell}c_{ik}, \text{ for } i \neq k. \tag{2.7}$$

It follows from (2.2) and (2.7) that

$$\sum_{m=1}^{n} a_{im}b_{mk} = \sum_{m=1}^{n} b_{im}a_{mk}, \text{ for } i \neq k.$$

Returning to the fact that $a_{im}b_{mk} = b_{im}a_{mk}$ for $m \neq i, k$, we have

$$a_{ii}b_{ik} + a_{ik}b_{kk} = b_{ii}a_{ik} + b_{ik}a_{kk}, \text{ for } i \neq k.$$

This implies that

$$a_{ik}(b_{ii} - b_{kk}) = b_{ik}(a_{ii} - a_{kk}). \tag{2.8}$$

Putting $k = \ell$ in (2.4) we get

$$\sum_{m=1}^{n} a_{km}b_{mk} - a_{kk}b_{kk} = \sum_{m=1}^{n} c_{km}c_{mk} - c_{kk}c_{kk}.$$

Thus it follows from (2.4) that

$$a_{kk}b_{kk} - a_{kk}b_{\ell\ell} - b_{kk}a_{\ell\ell} + b_{\ell\ell}a_{\ell\ell} = c_{kk}c_{kk} - 2c_{kk}c_{\ell\ell} + c_{\ell\ell}c_{\ell\ell}.$$

For $\ell = 1$ we have

$$c_{kk}^2 = a_{kk}b_{kk},$$

and then $a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell} = 2c_{kk}c_{\ell\ell}$. Thus for all $1 \leq k, \ell \leq n$ we have $(a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell})^2 = 4c_{kk}^2c_{\ell\ell}^2 = 4a_{kk}b_{kk}a_{\ell\ell}b_{\ell\ell}$. This implies that

$$a_{kk}b_{\ell\ell} = b_{kk}a_{\ell\ell}, \text{ for all } k, \ell.$$

A similar argument as about the equation (2.5) implies the existence of some α' and β' with $|\alpha'| + |\beta'| \neq 0$ such that

$$\alpha' A^D = \beta' B^D.$$

Using (2.8) we have

$$b_{jj}a_{ik}(b_{ii} - b_{kk}) = b_{ik}b_{jj}(a_{ii} - a_{kk}) = b_{ik}a_{jj}(b_{ii} - b_{kk}).$$

Now let $B^D \notin \mathbb{C}I$. Then $b_{ii} \neq b_{kk}$ for some i and k. This shows that $b_{jj}a_{ik} = a_{jj}b_{ik}$. So we have $\alpha = \alpha'$ and $\beta = \beta'$. By a similar argument we can say that if $A^D \notin \mathbb{C}I$ then $\alpha = \alpha'$ and $\beta = \beta'$. We therefore have

$$\text{if } A^D \notin \mathbb{C}I \text{ or } B^D \notin \mathbb{C}I \text{ then } \alpha A = \beta B \text{ for some } \alpha \text{ and } \beta \text{ with } |\alpha| + |\beta| \neq 0.$$

On the other hand, if $A^D = sI$ and $B^D = tI$ for some $s, t \in \mathbb{C}$ then

$$\alpha' A^D + \alpha(A - A^D) = s(\alpha' - \alpha)I + \alpha A,$$

and

$$\beta' B^D + \beta(B - B^D) = t(\beta' - \beta)I + \beta B.$$

Therefore $s(\alpha' - \alpha)I + \alpha A = t(\beta' - \beta)I + \beta B$. Summarizing these we can say that $\delta_A\delta_B = {\delta_C}^2$ if and only if $\alpha A = \beta B + rI$ for some $\alpha, \beta, r \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$. This is equivalent to the fact that either $\delta_B = 0$ or $\delta_A = s\delta_B$ for some $s \in \mathbb{C}$. □

A natural question is the following: Is it true in general that $\delta\delta' = \Delta^2$ on an algebra $\mathcal{A}$ is equivalent to either $\delta' = 0$ or $\delta = s\delta'$ for some $s \in \mathbb{C}$? In this case we of course have $\Delta = \sqrt{s}\delta'$. The following example shows that the answer is not affirmative in general.

Example 2.2. Let $\mathcal{A}$ be the subalgebra of $M_2(\mathbb{C})$ generated by E_{11} and E_{12}. If $\delta = \delta_{E_{12}}$ and $\delta' = \delta_{E_{11}}$ then for each $X = xE_{11} + yE_{12} \in \mathcal{A}$ we have

$$\delta\delta'(X) = \delta(xE_{11} + yE_{12} - xE_{11}) = \delta(yE_{12}) = 0.$$

Thus $\delta\delta' = \delta_0^2$. But $\delta' \neq 0$ and δ is not a multiple of δ'.

Lemma 2.3. *Let $\mathcal{A}$ be the subalgebra of $M_2(\mathbb{C})$ generated by E_{11} and E_{12}. Then each derivation on $\mathcal{A}$ is of the form $\delta = \delta_{cE_{12}-dE_{11}}$ for some $c, d \in \mathbb{C}$.*

Proof . Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation defined by $\delta(xE_{11} + yE_{12}) = f(x, y)E_{11} + g(x, y)E_{12}$. Since δ is linear,

$$f(x, y) = f(x, 0) + f(0, y) = xf(1, 0) + yf(0, 1).$$

We therefore have $f(x, y) = ax + by$ and $g(x, y) = cx + dy$ for some $a, b, c, d \in \mathbb{C}$. Moreover,

$$\begin{aligned} & \delta((xE_{11} + yE_{12})(x'E_{11} + y'E_{12})) \\ = \; & \delta(xE_{11} + yE_{12})(x'E_{11} + y'E_{12}) + (xE_{11} + yE_{12})\delta(x'E_{11} + y'E_{12}) \end{aligned}$$

implies

$$f(xx', xy')E_{11} + g(xx', xy')E_{12} = f(x, y)x'E_{11} + f(x, y)y'E_{12} + xf(x', y')E_{11} + xg(x', y')E_{12}.$$

We thus have

$$\begin{aligned} f(xx', xy') &= f(x, y)x' + xf(x', y'), \\ g(xx', xy') &= f(x, y)y' + xg(x', y'). \end{aligned}$$

By using the fact that $f(x, y) = ax + by$ and $g(x, y) = cx + dy$, we have $f(x, y) = 0$. Whence $\delta = \delta_{cE_{12}-dE_{11}}$. $\square$

Proposition 2.4. *Let $\mathcal{A}$ be the subalgebra of $M_2(\mathbb{C})$ generated by E_{11} and E_{12} and δ, δ' be two derivations on $\mathcal{A}$. Then $\delta\delta' = \Delta^2$ if and only if $\delta' = 0$ or $\delta' = \delta_{\alpha' E_{12}}$ for some $\alpha' \in \mathbb{C}$ implies $\delta = \delta_{\alpha E_{12}}$ for some $\alpha \in \mathbb{C}$, or equivalently $\delta' = 0$ or $\delta'^2 = 0$ implies $\delta^2 = 0$.*

Proof . Let $\delta = \delta_{\alpha E_{12}-\beta E_{11}}, \delta' = \delta_{\alpha' E_{12}-\beta' E_{11}}$ and $\Delta = \delta_{rE_{12}-sE_{11}}$. Then $\delta\delta' = \Delta^2$ if and only if $rs = \beta\alpha'$ and $s^2 = \beta\beta'$. The latter is equivalent to the fact that $\delta' = 0$ or $\delta' = \delta_{\alpha' E_{12}}$ for some $\alpha' \in \mathbb{C}$ implies $\delta = \delta_{\alpha E_{12}}$ for some $\alpha \in \mathbb{C}$. On the other hand, a derivation δ on $\mathcal{A}$ is of the form $\delta_{\lambda E_{12}}$ for some $\lambda \in \mathbb{C}$ if and only if $\delta^2 = 0$. $\square$

3. Derivations on C*-algebras

Theorem 3.1. *Let $\mathfrak{A}$ be a C^*-algebra and δ, δ' be two derivations on $\mathfrak{A}$. Then there exists a derivation Δ on $\mathfrak{A}$ such that $\delta\delta' = \Delta^2$ if and only if either $\delta' = 0$ or $\delta = s\delta'$ for some $s \in \mathbb{C}$.*

Proof . Let $\mathfrak{A}$ act faithfully on the Hilbert space $\mathfrak{H}$ with the orthonormal basis $\{\xi_i\}_{i\in\mathbb{I}}$. For a bounded operator $T \in B(\mathfrak{H})$, let $t_{ij} = \langle T\xi_j, \xi_i\rangle$ for $i, j \in \mathbb{I}$. We thus have $T\xi_j = \sum_{i\in\mathbb{I}} t_{ij}\xi_i$ and we can write $T = [t_{ij}]_{i,j\in\mathbb{I}}$. The latter is called the matrix representation of T. For $i, j \in \mathbb{I}$, let $E_{ij} \in B(\mathfrak{H})$ be the operator defined by $E_{ij}\xi_j = \xi_i$ and $E_{ij}\xi_k = 0$ for $k \neq j$. Then we have $T = \sum_{p\in\mathbb{I}} \sum_{q\in\mathbb{I}} t_{qp}E_{qp}$ for every $T \in B(\mathfrak{H})$.

By the Kadison-Sakai theorem [4, 10], $\delta = \delta_R, \delta' = \delta_S$ and $\Delta = \delta_T$ for some R, S and T in $B(\mathfrak{H})$. Thus $\delta\delta' = \Delta^2$ if and only if

$$RSE_{k\ell} - RE_{k\ell}S - SE_{k\ell}R + E_{k\ell}SR = T^2E_{k\ell} - 2TE_{k\ell}T + E_{k\ell}T^2,$$

for each $k, \ell \in \mathbb{I}$. This is equivalent to the fact that

$$E_{ii}(RSE_{k\ell} - RE_{k\ell}S - SE_{k\ell}R + E_{k\ell}SR)E_{jj} = E_{ii}(T^2E_{k\ell} - 2TE_{k\ell}T + E_{k\ell}T^2)E_{jj},$$

for each $i, j, k, \ell \in \mathbb{I}$. For $i \neq k$ and $j \neq \ell$ we have

$$r_{ik}s_{\ell j} + s_{ik}r_{\ell j} = 2t_{ik}t_{\ell j}.$$

Similarly, for $i \neq k$ and $j = \ell$ we have

$$\sum_{m\in\mathbb{I}} r_{im}s_{mk} - r_{ik}s_{\ell\ell} - s_{ik}r_{\ell\ell} = \sum_{m\in\mathbb{I}} t_{im}t_{mk} - 2t_{ik}t_{\ell\ell}.$$

Also, for $i = k$ and $j \neq \ell$ we have

$$- r_{kk}s_{\ell j} - s_{kk}r_{\ell j} + \sum_{m\in\mathbb{I}} s_{\ell m}r_{mj} = -2t_{kk}t_{\ell j} + \sum_{m\in\mathbb{I}} t_{\ell m}t_{mj}.$$

And finally for $i = k$ and $j = \ell$ we have

$$\sum_{m\in\mathbb{I}} r_{km}s_{mk} - r_{kk}s_{\ell\ell} - s_{kk}r_{\ell\ell} + \sum_{m\in\mathbb{I}} s_{\ell m}r_{m\ell} = \sum_{m\in\mathbb{I}} t_{km}t_{mk} - 2t_{kk}t_{\ell\ell} + \sum_{m\in\mathbb{I}} t_{\ell m}t_{m\ell}.$$

Now a similar verification as in Proposition 2.1 implies the result. □

References

[1] M. Barraa and S. Pedersen, *On the product of two generalized derivations*, Proc. Amer. Math. Soc. 127 (1999) 2679–2683.

[2] T. Creedon, *Products of derivations*, Proc. Edinburgh Math. Soc. 41 (1998) 407–410.

[3] C.K. Fong and A.R. Sourour, *On the operator identity* $\sum A_k X B_k \equiv 0$, Canad. J. Math. 31 (1979) 845–857.

[4] R.V. Kadison, *Derivations of operator algebras*, Ann. Math., 83 (1966) 280–293.

[5] M. Mathieu, *Properties of the product of two derivations of a C^*-algebra*, Canad. Math. Bull. 32 (1989) 490–497.

[6] M. Mathieu, *More properties of the product of two derivations of a C^*-algebra*, Bull. Austral. Math. Soc. 42 (1990) 115–120.

[7] M. Mathieu, *Posner's second theorem deduced from the first*, Proc. Amer. Math. Soc. 114 (1992) 601–602.

[8] S. Pedersen, *The product of two unbounded derivations*, Canad. Math. Bull. 33 (1990) 354–348.

[9] E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957) 1093–1100.

[10] S. Sakai, *Derivations of W^*-algebras*, Ann. Math. 83 (1966) 273–279.