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CHARACTERIZATION OF ADJOINTABLE OPERATORS ON HILBERT C^*-modules

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Abstract: Let $\mathcal{H}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. In this paper, we find the general form of the mappings $T : \mathcal{H} \to \mathcal{H}$ satisfing

$$2\langle T(x), T(y)\rangle = \langle T(x), y\rangle + \langle x, T(y)\rangle \quad (x, y \in \mathcal{H}),$$

as adjointable (bounded) $\mathcal{A}$-linear operators. The generalized Hyers-Ulam stability of the functional equation is discussed.

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Key Words: Hilbert C^*-module, C^*-algebra, $\mathcal{A}$-linear mapping

1. Introduction and Preliminaries

The notion of a Hilbert C^*-module initiated by Kaplansky [3] as a generalization of a Hilbert space in which the inner product takes its values in a C^*-algebra.

Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert $\mathcal{A}$-module or an inner product $\mathcal{A}$-module is a complex linear space $\mathcal{H}$ which is a left $\mathcal{A}$-module with compatible scalar multiplication $\lambda(ax) = (\lambda a)x = a(\lambda x) \quad (\lambda \in \mathbb{C}, x \in \mathcal{H}, a \in \mathcal{A})$, together with an $\mathcal{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that for each $x, y, z \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

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(i) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$.

(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

(iii) $\langle ax, y\rangle = a\langle x, y\rangle$,

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$.

The notion of a right Hilbert $\mathcal{A}$-module can be defined similarly. Note that the condition (i) is understood as a statement in the C^*-algebra $\mathcal{A}$, where an element a is called positive if it can be represented as bb^* for some $b \in \mathcal{A}$. The conditions (ii) and (iv) impliy the inner product to be conjugate-linear in its second variable. Validity of a useful version of the classical Cauchy-Schwartz inequality follows that $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathcal{H}$ making it into a normed left $\mathcal{A}$-module. An inner product $\mathcal{A}$-module $\mathcal{H}$ which is complete with respect to the norm $\|x\|$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$. Every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the $\mathcal{A}$-valued inner product $\langle a, b\rangle = a^*b \quad (a, b \in A)$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module.

One may define an $\mathcal{A}$-*valued norm* $|x| = \langle x, x\rangle^{\frac{1}{2}}$ (where, $|a|$ denotes the unique square root of the positive element a in $\mathcal{A}$). Clearly, $\||x|\| = \|x\|$, for each $x \in \mathcal{H}$. The $\mathcal{A}$-valued norm $|x|$ is a useful device but it needs to be handled with care. For example, it is known that $|.|$ does not satisfy the triangle inequality $|x + y| \leq |x| + |y|$ for each $x, y \in \mathcal{H}$; cf. [4].

Let $\mathcal{H}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. A system $(e_i)_{i\in I}$ in $\mathcal{H}$ is called orthogonal, if $\langle e_i, e_j\rangle = 0$ whenever $i \neq j$. An orthogonal system $(e_i)_{i\in I}$ in $\mathcal{H}$ is said to be an orthonormal, provided $\mathcal{A}$ is unital and for the inner squares it happens that $\langle e_i, e_i\rangle = 1$ for all $i \in I$. Let $(e_i)_{i\in I}$ be an orthonormal system in a Hilbert module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Landi and Pavlov showed in Theorem 2.10 of [5] that the following conditions are equivalent:

(i) For any x of $\mathcal{H}$ there are elements a_i of $\mathcal{A}$ such that

$$x = \sum_{i\in I} a_i e_i \tag{1}$$

where convergence in norm is meant and

$$\sum_{i\in I} a_i e_i = \lim_{F\in\mathcal{F}} \sum_{i\in F} a_i e_i$$

indicates the limit over the set $\mathcal{F}$ of all finite subsets of I, directed by inclusions.

(ii) The system $(e_i)_{i\in I}$ generates $\mathcal{H}$ over $\mathcal{A}$, that is to say, the closure of its $\mathcal{A}$-linear span coincides with $\mathcal{H}$.

(iii) The system $(e_i)_{i\in I}$ is closed, that is to say, for any $x \in \mathcal{H}$

$$\langle x, x\rangle = \sum_{i\in I} \langle x, e_i\rangle\langle e_i, x\rangle,$$

where the series converges in norm.

An orthonormal system $(e_i)_{i\in I}$ satisfying the equivalent conditions (i)-(iii) is called a *Schauder basis* for $\mathcal{H}$ over $\mathcal{A}$. If $(e_i)_{i\in I}$ is a Schauder basis for $\mathcal{H}$, then the coefficients in the decomposition (1) are unique for any vector x of $\mathcal{H}$. In fact for any $i \in I$, $a_i = \langle x, e_i\rangle$. Thus any vector of $\mathcal{H}$ is the limit in norm of its Fourier series. Any Schauder basis $(e_i)_{i\in I}$ is complete, i.e. there is no non-zero vector x of $\mathcal{H}$ such that $\langle x, e_i\rangle = 0$ for all $i \in I$. Note that by Proposition 3.1. of [5] any two closed orthonormal systems of a Hilbert module over a unital C^*-algebra have the same cardinality.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$. A mapping $T : \mathcal{H} \to \mathcal{K}$ is said to be $\mathcal{A}$-linear, if

$$T(ax + \lambda y) = aT(x) + \lambda T(y)$$

for all $x, y \in \mathcal{H}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A mapping $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable, if there exists a mapping $S : \mathcal{K} \to \mathcal{H}$ such that

$$\langle T(x), y\rangle = \langle x, S(y)\rangle$$

for all $x \in D_T \subseteq \mathcal{H}$, $y \in D_S \subseteq \mathcal{K}$. The unique mapping S is denoted by T^* and is called the adjoint of T. It is well known that any adjointable mapping $T : \mathcal{H} \to \mathcal{K}$ is $\mathcal{A}$-linear and bounded. $L(\mathcal{H}, \mathcal{K})$, the set of adjointable maps from $\mathcal{H}$ to $\mathcal{K}$ is a C^*-algebra [4]. The C^*-algebra of adjointable maps from $\mathcal{H}$ to $\mathcal{H}$ is denoted by $L(\mathcal{H})$.

In general, bounded $\mathcal{A}$-linear operator may fail to possess an adjoint (cf., [4]). However, if $\mathcal{H}$ is a Hilbert C^*-module over the C^*-algebra $\mathcal{K} = K(H)$ of all compact operators on a Hilbert space H, then (with another concept of orthonormal basis for Hilbert C^*-modules) D. Bakić and B. Guljaš ([1], Remark 5) showed that each bounded $\mathcal{K}$-linear operator on $\mathcal{H}$ is necessarily adjointable.

In 2003, Radu [6] employed the following result, due to Diaz and Margolis [2], to prove the stability of a Cauchy functional equation.

Proposition 1.1. *(The fixed point alternative principle). Let (X, d) be a generalized complete metric space and $J : X \to X$ be a strictly contractive mapping; that is*

$$d(J(x), J(y)) \leq Ld(x, y) \quad (x, y \in X)$$

for some (Lipschitz) constant $0 < L < 1$. Then, for a given element $x \in X$, exactly one of the following assertions is true: either

(a) $d(J^n x, J^{n+1} x) = \infty$ *for all* $n \geq 0$, *or*

(b) there exists some integer n_0 such that $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$.

Actually, if (b) holds, then

(b_1) the sequence $\{J^n x\}$ converges to a fixed point x^* of J,

(b_2) x^* is the unique fixed point of J in $X_0 := \{y \in X; d(J^{n_0} x, y) < \infty\}$;

(b_3) $d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in X_0$.

Let $\mathcal{H}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ and $T : \mathcal{H} \to \mathcal{H}$ be a mapping. In this paper, we introduce the new functional equation

$$2\langle T(x), T(y)\rangle = \langle T(x), y\rangle + \langle x, T(y)\rangle \quad (x, y \in \mathcal{H}). \tag{$\diamondsuit$}$$

In the first section, we show that T is a bounded $\mathcal{A}$-linear mapping and we find all of such bounded $\mathcal{A}$-linear mappings. Also we show that T is adjointable. We will show that the unique inner product preserving solution of the functional equation ($\diamondsuit$), is the identity function.

In the next section, we consider the generalized Hyers-Ulam stability for the above functional equation. We use the fixed point alternative theorem to show that if $\mathcal{H}$ is a Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$ and $f : \mathcal{H} \to \mathcal{H}$ satisfies the inequality

$$\|2\langle f(x), f(y)\rangle - \langle f(x), y\rangle - \langle x, f(y)\rangle\| \leq \varphi(x, y) \quad (x, y \in \mathcal{H})$$

then under suitable conditions on $\varphi : \mathcal{H} \times \mathcal{H} \to [0, \infty)$, there is a unique adjointable $\mathcal{A}$-linear mapping $T : \mathcal{H} \to \mathcal{H}$, which suitably approximates f.

2. General Solution

In this section, we show that a function $T : \mathcal{H} \to \mathcal{H}$ satisfies $(\diamondsuit)$, if and only if, T is an adjointable (bounded) $\mathcal{A}$-linear operator satisfying the equation $2T^*T = T + T^*$.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Any mapping $T : \mathcal{H} \to \mathcal{H}$ satisfying $(\diamondsuit)$ is a bounded $\mathcal{A}$-linear operator with $\|T\| \leq 1$.*

Proof. It follows from $(\diamondsuit)$ that

$$\begin{aligned}
&2\langle T(x+y) - T(x) - T(y), T(z)\rangle \\
&= 2\langle T(x+y), T(z)\rangle - 2\langle T(x), T(z)\rangle - 2\langle T(y), T(z)\rangle \\
&= \langle T(x+y), z\rangle + \langle x+y, T(z)\rangle \\
&- \langle T(x), z\rangle - \langle x, T(z)\rangle - \langle T(y), z\rangle - \langle y, T(z)\rangle \\
&= \langle T(x+y) - T(x) - T(y), z\rangle
\end{aligned}$$

for all $x, y, z \in \mathcal{H}$, which implies that

$$\langle T(x+y) - T(x) - T(y), T(z)\rangle = \Big\langle T(x+y) - T(x) - T(y), \frac{z}{2}\Big\rangle \tag{2}$$

for all $x, y, z \in \mathcal{H}$. It follows from (2) that

$$\begin{aligned}
&\langle T(x+y) - T(x) - T(y), T(x+y) - T(x) - T(y)\rangle \\
&= \langle T(x+y) - T(x) - T(y), T(x+y)\rangle \\
&- \langle T(x+y) - T(x) - T(y), T(x)\rangle \\
&- \langle T(x+y) - T(x) - T(y), T(y)\rangle \\
&= \Big\langle T(x+y) - T(x) - T(y), \frac{x+y}{2} - \frac{x}{2} - \frac{y}{2}\Big\rangle = 0
\end{aligned}$$

for all $x, y \in \mathcal{H}$. From the above equation, we get

$$T(x+y) = T(x) + T(y)$$

for all $x, y \in \mathcal{H}$. Hence T is an additive mapping.

From

$$\begin{aligned}&2\langle T(ax)-aT(x),T(ax)-aT(x)\rangle\\&=2\langle T(ax),T(ax)\rangle-2\langle aT(x),T(ax)\rangle\\&-2\langle T(ax),aT(x)\rangle+2\langle aT(x),aT(x)\rangle\\&=\langle T(ax),x\rangle a^*+a\langle x,T(ax)\rangle-a\langle T(x),x\rangle a^*-a\langle x,T(ax)\rangle\\&-\langle T(ax),x\rangle a^*-a\langle x,T(x)\rangle a^*+a\langle T(x),x\rangle a^*+a\langle x,T(x)\rangle a^*\\&=0\end{aligned}$$

for all $a\in\mathcal{A}$ and $x\in\mathcal{H}$, we deduce that $T(ax)=aT(x)$. In the same manner, we deduce that $T(\lambda x)=\lambda T(x)$ for all $x\in\mathcal{H}$ and $\lambda\in\mathbb{C}$. Thus

$$T(ax+\lambda y)=aT(x)+\lambda T(y)$$

for all $x,y\in\mathcal{H}$, $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$, i.e. T is an $\mathcal{A}$-linear operator.

Putting $x=y$ in $(\diamondsuit)$, we get

$$2\langle T(x),T(x)\rangle=\langle T(x),x\rangle+\langle x,T(x)\rangle\quad(x\in\mathcal{H}).\tag{3}$$

Thus

$$\begin{aligned}2\|T(x)\|^2&=2\||T(x)|^2\|=2\|\langle T(x),T(x)\rangle\|=\|\langle T(x),x\rangle+\langle x,T(x)\rangle\|\\&\le 2\|T(x)\|\|x\|\quad(x\in\mathcal{H}).\end{aligned}$$

and so

$$\|T\|^2=\sup_{\|x\|\le1}\|T(x)\|^2\le\sup_{\|x\|\le1}\|T(x)\|\|x\|=\|T\|$$

which implies that $\|T\|\le1$. This completes the proof. □

Lemma 2.2. *An $\mathcal{A}$-linear operator $T:\mathcal{H}\to\mathcal{H}$ satisfies $(\diamondsuit)$, if and only if T satisfies*

$$2\langle T(e_i),T(e_j)\rangle=\langle T(e_i),e_j\rangle+\langle e_i,T(e_j)\rangle\quad(i,j\in I).\tag{4}$$

Proof. If T satisfies $(\diamondsuit)$, then trivially T satisfies (4). Assume that T satisfies (4), and let $x,y\in\mathcal{H}$. Then $x=\sum_{i\in I}a_ie_i$ and $y=\sum_{j\in I}b_je_j$, where $a_i=\langle x,e_i\rangle$ and $b_j=\langle y,e_j\rangle$ for $i,j\in I$. It follows from Theorem 2.1 that

$$2\langle T(x),T(y)\rangle = 2\Big\langle T\big(\sum_{i\in I}a_ie_i\big),T\big(\sum_{j\in I}b_je_j\big)\Big\rangle$$

$$
\begin{aligned}
&= \sum_{i\in I}\sum_{j\in I} a_i(2\langle T(e_i), T(e_j)\rangle)b_j^* \\
&= \sum_{i\in I}\sum_{j\in I} a_i(\langle T(e_i), e_j\rangle + \langle e_i, T(e_j)\rangle)b_j^* \\
&= \sum_{i\in I}\sum_{j\in I} a_i\langle T(e_i), e_j\rangle b_j^* + \sum_{i\in I}\sum_{j\in I} a_i\langle e_i, T(e_j)\rangle b_j^* \\
&= \big\langle T\big(\sum_{i\in I} a_i e_i\big), \sum_{j\in I} b_j e_j\big\rangle + \big\langle \sum_{i\in I} a_i e_i, T\big(\sum_{j\in I} b_j e_j\big)\big\rangle \\
&= \langle T(x), y\rangle + \langle x, T(y)\rangle.
\end{aligned}
$$

□

Let $(e_i)_{i\in I}$ be a Schauder basis for Hilbert C^*-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $T : \mathcal{H} \to \mathcal{H}$ be an $\mathcal{A}$-linear operator. Let $a_{ij} = \langle T(e_j), e_i\rangle$ for all $i, j \in I$ and consider the matrix $A = [a_{ij}]$ corresponding to the $\mathcal{A}$-linear operator T. The next theorem characterizes the bounded $\mathcal{A}$-linear operators T satisfying $(\diamondsuit)$.

Theorem 2.3. *Let $\mathcal{H}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. An $\mathcal{A}$-linear operator $T : \mathcal{H} \to \mathcal{H}$ satisfies $(\diamondsuit)$, if and only if,*

$$2A^tA^* = A^t + A^*. \tag{5}$$

where A^t is the transpose of A and $A^* = [a_{ij}^*]$.

Proof. Suppose that T satisfies $(\diamondsuit)$, then from Lemma 2.2, T satisfies (4) for all $i, j \in I$. Since for all $i \in I$, $T(e_i) = \sum_{p\in I}\langle T(e_i), e_p\rangle e_p$, then we have

$$
\begin{aligned}
&2\big\langle \sum_{p\in I}\langle T(e_i), e_p\rangle e_p, \sum_{q\in I}\langle T(e_j), e_q\rangle e_q\big\rangle \\
&= \big\langle \sum_{p\in I}\langle T(e_i), e_p\rangle e_p, e_j\big\rangle + \big\langle e_i, \sum_{q\in I}\langle T(e_j), e_q\rangle e_q\big\rangle.
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&2\sum_{p\in I}\sum_{q\in I}\langle T(e_i), e_p\rangle\langle e_p, e_q\rangle\langle e_q, T(e_j)\rangle \\
&= \sum_{p\in I}\langle T(e_i), e_p\rangle\langle e_p, e_j\rangle + \sum_{q\in I}\langle e_i, e_q\rangle\langle e_q, T(e_j)\rangle.
\end{aligned}
$$

Therefore it follows from the last equation that

$$2\sum_{p\in I}\langle T(e_i),e_p\rangle\langle e_p,T(e_j)\rangle=\langle T(e_i),e_j\rangle+\langle e_i,T(e_j)\rangle\quad (i,j\in I).$$

Let $a_{ij}=\langle T(e_j),e_i\rangle$ for all $i,j\in I$, then the last equation implies that

$$2\sum_{p\in I}a_{pi}a_{pj}^{*}=a_{ji}+a_{ij}^{*}\quad (i,j\in I).$$

This means that $2A^tA^*=A^t+A^*$. Conversely, if $2A^tA^*=A^t+A^*$, it is easy to see that T satisfies $(\diamondsuit)$. □

Corollary 2.4. *Let $\mathcal{H}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. An $\mathcal{A}$-linear operator $T:\mathcal{H}\to\mathcal{H}$ satisfies $(\diamondsuit)$, if and only if T is adjointable and satisfies the equation $2T^*T=T+T^*$.*

Proof. Suppose that T satisfies $(\diamondsuit)$. From (5) we have

$$2(A^*)^tA=A+(A^*)^t. \tag{6}$$

Define the $\mathcal{A}$-linear operator $S:\mathcal{H}\to\mathcal{H}$ by the matrix $(A^*)^t$. Then trivially $\langle S(e_j),e_i\rangle=a_{ji}^*$ for all $i,j\in I$ and it follows that $\langle e_j,S(e_i)\rangle=a_{ij}=\langle T(e_j),e_i\rangle$ for all $i,j\in I$. From $\mathcal{A}$-linearity of T and S, we get

$$\langle T(x),y\rangle=\langle x,S(y)\rangle\quad (x,y\in\mathcal{H}).$$

So T is adjointable with $T^*=S$. From (6) we get $2T^*T=T+T^*$. Conversely, if T is an adjointable $\mathcal{A}$-linear operator such that $2T^*T=T+T^*$, then it satisfies $(\diamondsuit)$. □

Remark 2.5. Suppose that the mappings T and S satisfy $(\diamondsuit)$ and $\langle T(x),S(y)\rangle=0$ for all $x,y\in\mathcal{H}$, then for any complex number $\lambda=(r,\theta)$ with $r=2\cos\theta$, the adjointable $\mathcal{A}$-linear mapping $\lambda T+S$ satisfies $(\diamondsuit)$. Also if the mapping T satisfies $(\diamondsuit)$, then $I-T$ satisfies $(\diamondsuit)$.

Example 2.6. The Hilbert space ℓ^2 is a Hilbert $\mathbb{C}-$module. The $\mathbb{C}$-linear operator $T:\ell^2\to\ell^2$ defined by

$$T(x_1,x_2,x_3,\ldots)=(0,x_2,x_3,\ldots)$$

for all $x=(x_1,x_2,x_3,\ldots)\in\ell^2$, satisfies $(\diamondsuit)$, also $\|T\|<1$ and $T=T^*$.

Example 2.7. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a $\mathbb{C}$-linear operator, corresponding with the complex matrix $A = [a_{ij}]$. T satisfies $(\diamondsuit)$, if and only if the complex numbers a_{ij} satisfy the following equations:

$$\left|a_{11} - \frac{1}{2}\right|^2 + |a_{21}|^2 = \frac{1}{4}, \quad \left|a_{22} - \frac{1}{2}\right|^2 + |a_{12}|^2 = \frac{1}{4},$$

$$|a_{12}| = |a_{21}| \leq \frac{1}{2}, \quad \left|a_{11} - \frac{1}{2}\right| = \left|a_{22} - \frac{1}{2}\right| \leq \frac{1}{2}.$$

Remark 2.8. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be nonzero complex numbers and $T : \mathcal{H} \to \mathcal{H}$ be a mapping satisfing

$$\alpha\langle T(x), T(y)\rangle = \beta\langle T(x), y\rangle + \gamma\langle x, T(y)\rangle \quad (x, y \in \mathcal{H}). \qquad (\diamondsuit')$$

It is easy to see that T is an adjointable $\mathcal{A}$-linear mapping with $\|T\| \leq \frac{|\beta|+|\gamma|}{|\alpha|}$. Moreover it follows that $\alpha T^*T = \beta T + \gamma T^*$ or equivalently $(\frac{\alpha}{\beta}T^* - I)(\frac{\alpha}{\gamma}T - I) = I$. Thus if $\frac{\alpha}{\gamma}T - I$ is surjective, then it is invertible in $L(\mathcal{H})$ with $(\frac{\alpha}{\gamma}T - I)^{-1} = \frac{\alpha}{\beta}T^* - I$.

In particular, if the mapping T satisfies $(\diamondsuit')$ with $\alpha = \beta = \gamma = 1$, then $I - T$ is inner product preserving and so is an isometry.

3. Stability

In this section, we prove the generalized Hyers-Ulam stability of the equation $(\diamondsuit)$.

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ and $\varphi : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ be an control function such that*

$$\lim_{n\to\infty} 2^{2n}\varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0 \qquad (7)$$

for all $x, y \in \mathcal{H}$. Assume that there is $0 < L < 1$ such that

$$\psi\Big(\frac{x}{2}\Big) \leq \frac{L}{2}\psi(x) \qquad (8)$$

for all $x \in \mathcal{H}$, where

$$\psi(x) = \big(4\varphi(x, x) + 2\varphi(2x, x) + 2\varphi(x, 2x) + \varphi(2x, 2x)\big)^{\frac{1}{2}}. \qquad (9)$$

If a function $f:\mathcal{H}\to\mathcal{H}$ satisfies the inequality

$$\|2\langle f(x),f(y)\rangle-\langle f(x),y\rangle-\langle x,f(y)\rangle\|\leq\varphi(x,y) \tag{10}$$

for all $x,y\in\mathcal{H}$, then there exists a unique adjointable mapping $T:\mathcal{H}\to\mathcal{H}$ such that

$$\|f(x)-T(x)\|\leq\frac{\sqrt{2}L}{4(1-L)}\psi(x) \tag{11}$$

for all $x\in\mathcal{H}$.

Proof. Replacing x by $2x$ in (10), we get

$$\|2\langle f(2x),f(y)\rangle-\langle f(2x),y\rangle-2\langle x,f(y)\rangle\|\leq\varphi(2x,y). \tag{12}$$

From (10) and (12), we have

$$\|2\langle f(2x)-2f(x),f(y)\rangle-\langle f(2x)-2f(x),y\rangle\|\leq 2\varphi(x,y)+\varphi(2x,y). \tag{13}$$

Replacing y by $2y$ in (13), we get

$$\|2\langle f(2x)-2f(x),f(2y)\rangle-2\langle f(2x)-2f(x),y\rangle\|\leq 2\varphi(x,2y)+\varphi(2x,2y). \tag{14}$$

From (13) and (14), we have

$$\|2\langle f(2x)-2f(x),f(2y)-2f(y)\rangle\|\leq\psi(x)^2. \tag{15}$$

Letting $x=y$ in (15), we obtain

$$\|f(2x)-2f(x)\|^2=\||f(2x)-2f(x)|^2\|\leq\frac{1}{2}\psi(x)^2.$$

and so

$$\|f(2x)-2f(x)\|\leq\frac{\sqrt{2}}{2}\psi(x) \tag{16}$$

Let $X=\{g:\mathcal{H}\to\mathcal{H},g(0)=0\}$ and define $d:X\times X\to[0,\infty]$ by

$$d(g,h)=\inf\{\alpha\geq 0:\|g(x)-h(x)\|\leq\alpha\psi(x),\forall x\in\mathcal{H}\}\quad(g,h\in X).$$

Define $J:X\to X$ by $J(g)(x)=2g(\frac{x}{2})$ for each $x\in\mathcal{H}$. Then (X,d) is a complete generalized metric space and from (8) it follows that J is a strictly contractive mapping on X with the Lipschitz constant $L=\frac{1}{2}$. From (8) and (16) we have

$$\|J(f)(x)-f(x)\|=\big\|2f\big(\frac{x}{2}\big)-f(x)\big\|=2\big\|f\big(\frac{x}{2}\big)-\frac{f(x)}{2}\|\leq\frac{\sqrt{2}}{4}L\psi(x)$$

for each $x \in \mathcal{H}$. This means that $d(J(f), f) \le \frac{\sqrt{2}}{4}L$. Therefore, by Proposition 1.1, J has a unique fixed point in the set $X_0 = \{g \in X : d(f, g) < \infty\}$. Let $T : \mathcal{H} \to \mathcal{H}$ be the unique fixed point of J. We have $\lim_n (J^n(f), T) = 0$, so T is defined by

$$T(x) := \lim_{n\to\infty} 2^n f\big(\frac{x}{2^n}\big) \quad (x \in \mathcal{H}). \tag{17}$$

On the other hand, we have $d(f, J(f)) \le \frac{\sqrt{2}}{4}L$ and $J(T) = T$, then

$$d(f, T) \le d(f, J(f)) + d(J(f), J(T)) \le \frac{\sqrt{2}}{4}L + Ld(f, T).$$

So

$$d(f, T) \le \frac{\sqrt{2}L}{4(1-L)},$$

which implies the inequality (11).

It is easy to see that for all $x, y \in \mathcal{H}$, we have

$$\lim_{n\to\infty} \big\langle 2^n f\big(\frac{x}{2^n}\big), y\big\rangle = \langle T(x), y\rangle, \tag{18}$$

$$\lim_{n\to\infty} \big\langle x, 2^n f\big(\frac{y}{2^n}\big)\big\rangle = \langle x, T(y)\rangle. \tag{19}$$

Since for every $x \in \mathcal{H}$ the sequence $\{2^n f(\frac{x}{2^n})\}$ is convergent, so it is bounded. Therefore for every $x \in \mathcal{H}$ there exists $K_x > 0$ such that $\|2^n f(\frac{x}{2^n})\| \le K_x$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}
&\lim_{n\to\infty} \big\|\big\langle 2^n f\big(\frac{x}{2^n}\big), 2^n f\big(\frac{y}{2^n}\big)\big\rangle - \langle T(x), T(y)\rangle\big\| \\
&= \lim_{n\to\infty} \big\|\big\langle 2^n f\big(\frac{x}{2^n}\big), 2^n f\big(\frac{y}{2^n}\big)\big\rangle - \big\langle 2^n f\big(\frac{x}{2^n}\big), T(y)\big\rangle \\
&+ \big\langle 2^n f\big(\frac{x}{2^n}\big), T(y)\big\rangle - \langle T(x), T(y)\rangle\big\| \\
&= \lim_{n\to\infty} \big\|\big\langle 2^n f\big(\frac{x}{2^n}\big), 2^n f\big(\frac{y}{2^n}\big) - T(y)\big\rangle + \big\langle 2^n f\big(\frac{x}{2^n}\big) - T(x), T(y)\big\rangle\big\| \\
&\le \lim_{n\to\infty} \big(\big\|2^n f\big(\frac{x}{2^n}\big)\big\|\big\|2^n f\big(\frac{y}{2^n}\big) - T(y)\big\| + \big\|2^n f\big(\frac{x}{2^n}\big) - T(x)\big\|\|T(y)\|\big) \\
&\le \lim_{n\to\infty} \big(K_x\big\|2^n f\big(\frac{y}{2^n}\big) - T(y)\big\| + \big\|2^n f\big(\frac{x}{2^n}\big) - T(x)\big\|\|T(y)\|\big) = 0
\end{aligned}$$

for all $x, y \in \mathcal{H}$. This shows that

$$\lim_{n\to\infty} \big\langle 2^n f\big(\frac{x}{2^n}\big), 2^n f\big(\frac{y}{2^n}\big)\big\rangle = \langle T(x), T(y)\rangle \quad (x, y \in \mathcal{H}). \tag{20}$$

It follows from (7), (18), (19) and (20) that

$$\begin{aligned}
&\|2\langle T(x),T(y)\rangle - \langle T(x),y\rangle - \langle x,T(y)\rangle\| \\
&= \lim_{n\to\infty} 2^{2n}\big\|2\big\langle f\big(\frac{x}{2^n}\big), f\big(\frac{y}{2^n}\big)\big\rangle - \big\langle f\big(\frac{x}{2^n}\big), \frac{y}{2^n}\big\rangle - \big\langle \frac{x}{2^n}, f\big(\frac{y}{2^n}\big)\big\rangle\big\| \\
&\leq \lim_{n\to\infty} 2^{2n}\varphi\big(\frac{x}{2^n}, \frac{y}{2^n}\big) = 0
\end{aligned}$$

for all $x, y \in \mathcal{H}$. Whence

$$2\langle T(x),T(y)\rangle = \langle T(x),y\rangle + \langle x,T(y)\rangle \tag{21}$$

for all $x, y \in \mathcal{H}$ and so by Corollary (2.4), T is a adjointable $\mathcal{A}$-linear mapping.

To see the uniqueness of T, let $T' : \mathcal{H} \to \mathcal{H}$ be another adjointable $\mathcal{A}$-linear mapping satisfying (11). Then

$$\begin{aligned}
\|T(x) - T'(x)\| &= 2^n\big\|T\big(\frac{x}{2^n}\big) - T'\big(\frac{x}{2^n}\big)\big\| \\
&\leq 2^n\big(\big\|T\big(\frac{x}{2^n}\big) - f\big(\frac{x}{2^n}\big)\big\| + \big\|f\big(\frac{x}{2^n}\big) - T'\big(\frac{x}{2^n}\big)\big\|\big) \\
&\leq 2^n\frac{\sqrt{2}L}{2(1-L)}\psi\big(\frac{x}{2^n}\big) \\
&\leq \frac{\sqrt{2}L}{2(1-L)}L^n\psi(x)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathcal{H}$. This completes the proof. □

The following Theorem can be proved in a similar way as Theorem 3.1.

Theorem 3.2. *Let $\mathcal{H}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ and $\varphi : \mathcal{H} \times \mathcal{H} \to [0,\infty)$ be an control function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{2^{2n}} = 0$$

for all $x, y \in \mathcal{H}$. Assume that there is $0 < L < 1$ such that

$$\psi(2x) \leq 2L\psi(x)$$

for all $x \in \mathcal{H}$, where

$$\psi(x) = \big(4\varphi(x,x) + 2\varphi(2x,x) + 2\varphi(x,2x) + \varphi(2x,2x)\big)^{\frac{1}{2}}.$$

If a function $f:\mathcal{H}\to\mathcal{H}$ *satisfies the inequality*

$$\|2\langle f(x),f(y)\rangle-\langle f(x),y\rangle-\langle x,f(y)\rangle\|\leq\varphi(x,y)$$

for all $x,y\in\mathcal{H}$, *then there exists a unique adjointable mapping* $T:\mathcal{H}\to\mathcal{H}$ *such that*

$$\|f(x)-T(x)\|\leq\frac{\sqrt{2}}{4(1-L)}\psi(x)\quad(x\in\mathcal{H}).$$

The next result follows from Theorem 3.1, where $\varphi(x,y)=\|x-y\|^p$ for all $x,y\in\mathcal{H}$.

Corollary 3.3. *Let* $\mathcal{H}$ *be a Hilbert* C^**-module over a unital* C^**-algebra* $\mathcal{A}$. *If a function* $f:\mathcal{H}\to\mathcal{H}$ *satisfies the inequality*

$$\|2\langle f(x),f(y)\rangle-\langle f(x),y\rangle-\langle x,f(y)\rangle\|\leq\|x-y\|^p$$

for all $x,y\in\mathcal{H}$ *and some* $p>2$, *then there exists a unique adjointable mapping* $T:\mathcal{H}\to\mathcal{H}$ *such that*

$$\|f(x)-T(x)\|\leq\frac{\sqrt{2}}{\sqrt{2^p}-2}\|x\|^{\frac{p}{2}}\quad(x\in\mathcal{H}).$$

The next result follows from Theorem 3.2, where $\varphi(x,y)=\varepsilon$.

Corollary 3.4. *Let* $\mathcal{H}$ *be a Hilbert* C^**-module over a unital* C^**-algebra* $\mathcal{A}$. *If a function* $f:\mathcal{H}\to\mathcal{H}$ *satisfies the inequality*

$$\|2\langle f(x),f(y)\rangle-\langle f(x),y\rangle-\langle x,f(y)\rangle\|\leq\varepsilon$$

for all $x,y\in\mathcal{H}$, *then there exists a unique adjointable mapping* $T:\mathcal{H}\to\mathcal{H}$ *such that*

$$\|f(x)-T(x)\|\leq\frac{3\sqrt{2\varepsilon}}{2}\quad(x\in\mathcal{H}).$$

Example 3.5. Consider the Hilbert $\mathbb{C}-$module ℓ^2. Let $\phi:\ell^2\to\mathbb{C}$ be an arbitrary bounded function such that $|\phi(x)|\leq\|x\|$ for all $x\in\ell^2$, where $\|x\|=(\sum_{n=1}^{\infty}|x_n|^2)^{\frac{1}{2}}$. Define the mapping $f:\ell^2\to\ell^2$ by

$$f(x_1,x_2,x_3,\ldots)=(\phi(x)x_1,x_2,x_3,\ldots)$$

for all $x = (x_1, x_2, x_3, \dots) \in \ell^2$. Then

$$\begin{aligned}
&|2\langle f(x), f(y)\rangle - \langle f(x), y\rangle - \langle x, f(y)\rangle| \\
&= \big|2\phi(x)x_1\overline{\phi(y)y_1} + 2\sum_{n=2}^{\infty} x_n\overline{y_n} \\
&- \phi(x)x_1\overline{y_1} - \sum_{n=2}^{\infty} x_n\overline{y_n} - x_1\overline{\phi(y)y_1} - \sum_{n=2}^{\infty} x_n\overline{y_n}\big| \\
&= \big|(x_1\overline{y_1})\big(2\phi(x)\overline{\phi(y)} - \phi(x) - \overline{\phi(y)}\big)\big| \\
&\leq |x_1||y_1|\big(2|\phi(x)||\phi(y)| + |\phi(x)| + |\phi(y)|\big) \\
&\leq \|x\|\|y\|\big(2\|x\|\|y\| + \|x\| + \|y\|\big) \\
&= 2\|x\|^2\|y\|^2 + \|x\|^2\|y\| + \|x\|\|y\|^2
\end{aligned}$$

for all $x, y \in \ell^2$. Thus if the control function is defined by

$$\varphi(x, y) = 2\|x\|^2\|y\|^2 + \|x\|^2\|y\| + \|x\|\|y\|^2 \quad (x, y \in \ell^2),$$

then we have $\lim_{n\to\infty} 2^{2n}\varphi(2^{-n}x, 2^{-n}y) = 0$ and $\psi(x) = (72\|x\|^4 + 48\|x\|^3)^{\frac{1}{2}}$. Also for constant $0 < L = \frac{\sqrt{2}}{2} < 1$, we have $\psi(\frac{x}{2}) \leq \frac{L}{2}\psi(x)$ for all $x \in \ell^2$. Then the function $T : \ell^2 \to \ell^2$ defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

for all $x = (x_1, x_2, x_3, \dots) \in \ell^2$, is the unique adjointable mapping fulfilling the condition (11) in Theorem 3.1.

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