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Modeling and control of inverted pendulum based on PWA-FUZZY approach

Mostafa Mobara
Department of Electrical Engineering
Ferdowsi University of Mashhad
Mashhad, Iran
mostafa.mobaragosh@stu.um.ac.ir

Ali Karimpour
Department of Electrical Engineering
Ferdowsi University of Mashhad
Mashhad, Iran
karimpor@um.ac.ir

Hamed MollaAhamadian
Electrical Eng Group
Khorasan Institute ofHigher Education
Mashhad, Iran
Mollaahmadian@khorasan.ac.ir

Amirhossein Pahnabi
Department of Electrical Engineering
Ferdowsi University of Mashhad
Mashhad, Iran
Amirhossein.pahnabi@stu.um.ac.ir

***Abstract*—The inverted pendulum is the most popular benchmark for teaching and researches in control theory and robotics. Modeling and control of inverted pendulum on a cart is as one of the most important case in control theory. In this paper a new method in Piecewise Affine (PWA) models identification based on numerical analysis is proposed. The evaluation of this algorithm is investigated on a real system. Also the PWA-FUZZY approach as a solution to obviate the problem of hard switching in PWA system is used.A patched LQR and LQG controller in order to stabilize open-loop unstable equilibrium point of this system is proposed.**

Keywords-Piecewise Affine Systems; Tsk Fuzzy model; Inverted pendulum on a cart; PWA-Fuzzy approach

I. INTRODUCTION

Inverted pendulum on a cart is an example of classical control which is widely use in military industry, aerospace, humanoid robot. Since 1950s, inverted pendulum is used to teaching linear feedback control theory to stabilized open-loop unstable systems. Almost all control techniques has been investigated for inverted pendulum, techniques like Bang-Bang control, Fuzzy logic control, PID adaptive control, robust control, predictive control [1].

Piecewise affine systems are an importantsubclass ofcomplex systems. Piecewise affine systems are a very important and powerful modeling class for practical applications involving nonlinear dynamics. There are many nonlinearities which are either piecewise affine or can be approximated as a piecewise affine model. PWA modeling of inverted pendulum on a cart is introduced in[2].

One of the advantages of PWA system is simplicity in controller design, whereas one of the main drawbacks of this system is discontinuity in control signal. To overcome this problem PWA-FUZZY approach is proposed. PWA-FUZZY approach base on Takagi and Sugeno (TSK) [3] fuzzy model has been used for modeling of PFC rectifiers in[4].

The main contributions of this paper are as follows 1) PWA model identification of a real inverted pendulum on a cart based on numerical methods, 2) modeling and control of inverted pendulum on a cart by PWA-FUZZY approach.

This paper is organized as follows: in section II the inverted pendulum on a cartis introduced. PWA and PWA-FUZZY approach are introduced in section III. In section IV the piecewise affine system identification based on numerical method is introduced.In section V designing a patched LQR/LQG controller for PWA-FUZZY model is presented. Simulation results for verification of proposed identification and control method on a real system are given in section VI, and the conclusion is presented in section VII.

II. INVERTED PENDULUM ON A CART

The problem of controlling an inverted pendulum to its open-loop unstable equilibrium point is considered. Because of assuming that pendulum can start from anywhere in ± 45 degrees of the vertical, the full nonlinear dynamic must be used.

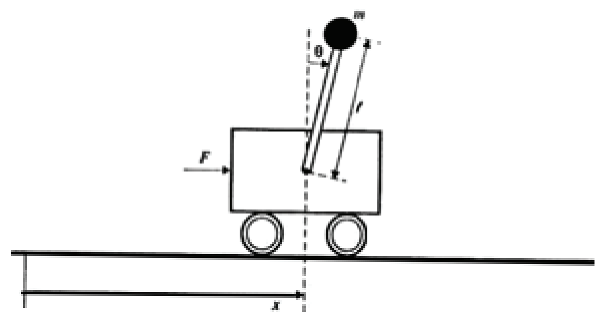

Figure 1. Inverted pendulum on a cart[5].

As we can see in Fig.1 there is an inverted pendulum with mass m and length l. The pendulum is located on a cart with mass M which is moving in horizontal surface. Control force F move cart and it makes pendulum to stabilize in vertical direction. By considering control force F as input, pendulum angle and cart position as outputs, system equations would be in form of following equations[5]:

$$\begin{cases} \ddot{x} = \dfrac{F + ml\dot{\theta}^2 sin\theta - mg\, sin\theta cos\theta}{M + m\left(1 - cos^2\theta\right)} \\ \ddot{\theta} = \dfrac{-Fcos\theta - ml\dot{\theta}^2 sin\theta cos\theta + \left(M + m\right) g\, sin\theta}{l\left[M + m\left(1 - cos^2\theta\right)\right]} \end{cases} \quad (1)$$

Where x is cart position and θ is pendulum angle. By considering $\left(x \quad \theta \quad \dot{x} \quad \dot{\theta}\right)$ as state variables, state equations would be in form of following equation:

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_1(\theta,\dot{\theta}) \\ f_2(\theta,\dot{\theta}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{M + m\left(1 - cos^2\theta\right)} \\ \dfrac{-cos\theta}{l\left[M + m\left(1 - cos^2\theta\right)\right]} \end{bmatrix} F \quad (2)$$

Where

$$\begin{cases} f_1(\theta,\dot{\theta}) = \dfrac{ml\dot{\theta}^2 sin\theta - mg\, sin\theta cos\theta}{M + m\left(1 - cos^2\theta\right)} \\ f_2(\theta,\dot{\theta}) = \dfrac{-ml\dot{\theta}^2 sin\theta cos\theta + \left(M + m\right) g\, sin\theta}{l\left[M + m\left(1 - cos^2\theta\right)\right]} \end{cases} \quad (3)$$

By linearization equation (3) at operating points of each subsystems, state equations would be as follow:

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \alpha_{1i} & 0 & \alpha_{2i} \\ 0 & \alpha_{3i} & 0 & \alpha_{4i} \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \beta_{1i} \\ \beta_{2i} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \lambda_{1i} \\ \lambda_{2i} \end{bmatrix} \quad (4)$$

In PWA approximation, state equation like equation (4)would be specify to each region.

III. PWA AND PWA-FUZZY APPROACHS

Inverted pendulum system which is considered in this paper has nonlinear dynamic that can be form in framework of PWA systems. PWA approximation is constructed by meshing state space and input into multiple areas and assign a differential equation to each areain form of equation (5)[2, 6]:

$$\begin{cases} \dot{x}\left(t\right) = A_i x\left(t\right) + B_i u\left(t\right) + b_i \\ y = Cx\left(t\right) \end{cases} \quad (5)$$

Accordingly, each mesh can be specified by following inequality:

$$H_i^T x - g_i < 0 \quad (6)$$

In equation (5) and equation (6) A_i, B_i, b_i, C, H_i and g_i are constant matrixes with appropriate dimension. The u, x and y are input, state variable and output of corresponding system. PWA systems structure has affine dynamics and regions with fix borders. Describing block diagram of PWA systems is shown in Fig. 2.

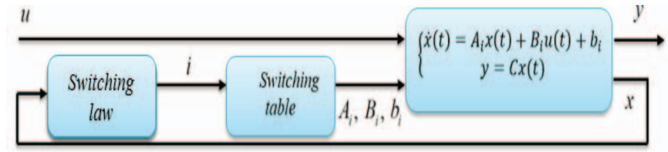

Figure2. Block diagram of piecewise affine systems[7].

PWA model of nonlinear systems has advantages such as simplicity in designing controller, however it has problem of discontinuity in control signal. Takagi and Sugeno (TSK)[3] fuzzy model which define with IF and THEN rules use to assign local input-output equation for a nonlinear system. The main property of TSK fuzzy model is to express local dynamics with a linear model of system. Fuzzy model of the system will express with weighted averaging of local linear models[4]:

$$\dot{x}\left(t\right) = \frac{\sum_{i=1}^{r} h_i \left\{A_i x\left(t\right) + B_i u\left(t\right)\right\}}{\sum_{i=1}^{r} h_i} = \sum_{i=1}^{r} w_i \left\{A_i x\left(t\right) + B_i u\left(t\right)\right\} \quad (7)$$

Where

$$h_i = \prod_{j=1}^{n} M_{ij}\left(x_j(t)\right) \quad (8)$$

$$w_i = \frac{h_i}{\sum_{i=1}^{r} h_i} \quad (9)$$

To obviate the problem of discontinuity in control signal PWA-FUZZY approach is proposed. Describing block diagram of PWA-FUZZY systems is shown in Fig. 3.

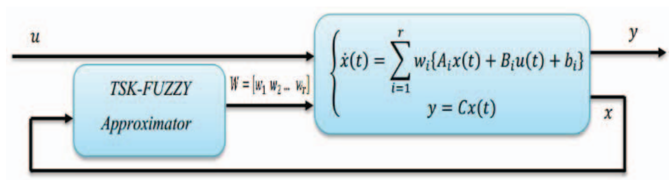

Figure 3. Block diagram of piecewise affine systems with fuzzy supervisory block.

IV. PIECEWISE AFFINE SYSTEM IDENTIFICATION

In this paper controller design for real system is considered. For this purpose we need to estimate the real system parameters. Here we use piecewise affine model instead of nonlinear model, in order to achieve this goal we should give the system specific input in each region and obtain the output while the system is still in that region, by this we are able to get the system dynamics in different subsystems. As it mentioned in equation (4), piecewise affine approximation of inverted pendulum is in form of following equation:

$$\begin{cases} \ddot{x}(t)=\alpha_{1i}\theta(t)+\alpha_{2i}\dot{\theta}(t)+\beta_{1i}u(t)+\gamma_{1i} \\ \ddot{\theta}(t)=\alpha_{3i}\theta(t)+\alpha_{4i}\dot{\theta}(t)+\beta_{2i}u(t)+\gamma_{2i} \end{cases} \quad (10)$$

Now discrete form of equation (10) would be in form of equation (11):

$$\begin{cases} x(k)=2x(k-1)-x(k-2)+\alpha_{1i}T^2\theta(k-2)+\ldots \\ \alpha_{2i}T\,\theta(k-1)-\alpha_{2i}T\,\theta(k-2)+\beta_{1i}T^2u(k-2)+\gamma_{1i}T^2 \\ \theta(k)=2\theta(k-1)-\theta(k-2)+\alpha_{3i}T^2\theta(k-2)+\cdots \\ \alpha_{4i}T\,\theta(k-1)-\alpha_{4i}T\,\theta(k-2)+\beta_{2i}T^2u(k-2)+\gamma_{2i}T^2 \end{cases} \quad (11)$$

Where T is sampling time. As we can see there are some known parameters in equation (11).This known parameters put constrained on identification algorithm to consider it in computing others.To achieve this goal, we used linear least square method[8] in order to identify unknown parameters. In the following equation, the parameters would be estimated by minimizing the sum of the square of the difference of measured and expected response:

$$J=\sum_{k=3}^{n}\left\|\theta_m(k)-\theta(k)\right\|_2+\left\|X_m(k)-X(k)\right\|_2 \quad (12)$$

Where index m is refer to measured response. To estimate the eight unknown parameter mentioned, we need to solve following eight equations:

$$\begin{cases} \dfrac{\partial J}{\partial\alpha_{1i}}=0,\dfrac{\partial J}{\partial\alpha_{1i}}=0,\dfrac{\partial J}{\partial\alpha_{1i}}=0,\dfrac{\partial J}{\partial\alpha_{1i}}=0 \\ \dfrac{\partial J}{\partial\alpha_{1i}}=0,\dfrac{\partial J}{\partial\alpha_{1i}}=0,\dfrac{\partial J}{\partial\alpha_{1i}}=0,\dfrac{\partial J}{\partial\alpha_{1i}}=0 \end{cases} \quad (13)$$

V. PATCHED LQR/LQG CONTROLLER DESIGN

In this section by assuming that all states are accessible, a patched state feedback controller for piecewise affine system design and control law of the form $u=-K_i x$ use in each region R_i. The close-loop state equations in each region R_i is in form of following equation:

$$\dot{x}=(A_i+B_iK_i)x+b_i \quad (14)$$

Now the gain matrix K_i can be designed using LQR or other design methods. In the following by removing the assumption that full state is accessible we will design a patched LQG output feedback controller for piecewise affine systems with dynamics (4). The controller to be design for each region on R_i is in form of following state space equation:

$$\begin{cases} \dot{x}_c=A_{ci}x_c+L_iy \\ u=K_ix_c \end{cases} \quad (15)$$

Base on LQG framework, assume the plant dynamic has Gaussian process noise η(t) and measurement noise v(t) with zero mean and uncorrelated ($E[\eta]=E[v]=0,\ E[\eta v^T]=0$).

Now the state space equation is:

$$\begin{cases} \dot{x}=A_ix+B_iu+b_i+\eta \\ y=Cx+v \end{cases} \quad (16)$$

The close-loop state space equation in each region R_i is in form of equation (16):

$$\begin{pmatrix}\dot{x}\\ \dot{x}_c\end{pmatrix}=\begin{pmatrix}A_i & B_iK_i\\ L_iC & A_{ci}\end{pmatrix}\begin{pmatrix}x\\ x_c\end{pmatrix}+\begin{pmatrix}b_i+\eta\\ L_iv\end{pmatrix} \quad (17)$$

As we can see in Fig.4 there is a schematic of system with patched LQG controller in presence of process and measurement noises.

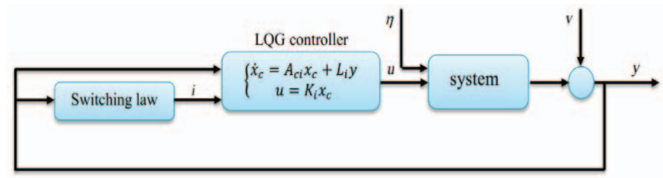

Figure 4. Schematic of LQG controller.

In order to obviate the problem of discontinuity in control signal, we need to consider PWA-Fuzzy controller. The equation of new controller is shown in following equation:

$$\begin{cases} \dot{x}_c=\sum_{i=1}^{r}w_i\{A_{ci}x_c+L_iy\} \\ u=\sum_{i=1}^{r}w_i\{K_ix_c\} \end{cases} \quad (18)$$

The schematic ofpatched LQG controller with Fuzzy supervisory block shown in Fig.5.

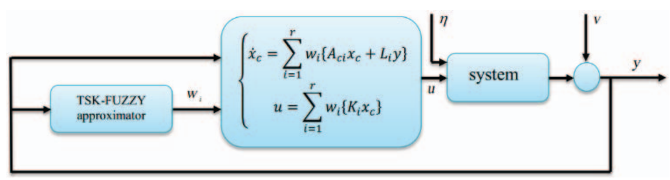

Figure 5. Schematic of LQG controller with FUZZY supervisory block.

VI. SIMULATION

In this section the effectiveness of proposed method will assess by presenting simulation results. First of all the PWA system identification of a real inverted pendulum on a cart which is shown in Fig.6 is investigated.

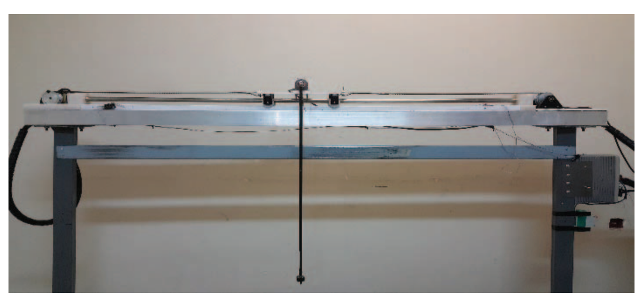

Figure 6. Inverted pendulum on a cart.

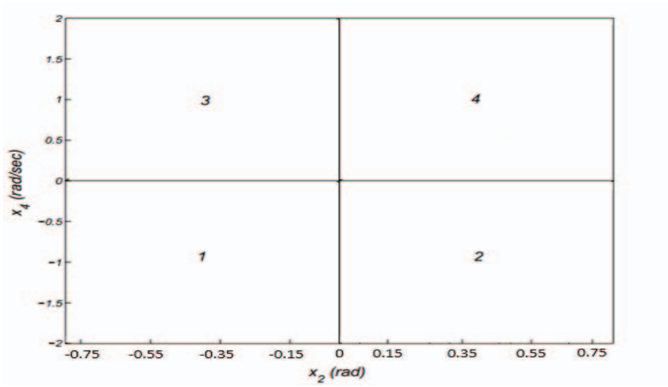

Figure 7. Graphical representation of PWA system for inverted pendulum on a cart.

System shown in Fig.6 is an inverted pendulum located on a cart. The cart is linked to a servo motor. The pendulum angle and cart position of this system sense with two incremental rotatory encoders.

Consider partitioning the area around open-loop unstable equilibrium point into four cells. The Graphical representation of PWA system for inverted pendulum on a cart is shown in Fig.7.The inputs are applied to motor in different region, pendulum angle and cart position recorded. Notice that the output should recorded while system states are still in that region.The input and output of system has been recorded by data acquisition card. The Sampling time of identification is T=0.01s. The identified state space matrixes are defined in Table I. The results of model matching for PWA system identification are shown from Fig.8 to Fig.11. Input signals which applied to each subsystems are step voltage of 5 volts.

Table I. Estimated state matrixes of different subsystems.

i	A_i	B_i	b_i	Condition
1	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.292 & 0 & -1.154 \\ 0 & 283.973 & 0 & -1084.8 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -0.01 \\ -12.52 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -0.0018 \\ -0.35 \end{bmatrix}$	$-0.8 < x_2 < 0$ $-2 < x_4 < 0$
2	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.658 & 0 & 3.535 \\ 0 & 187.36 & 0 & -1036.34 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.034 \\ -9.15 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -0.0019 \\ -0.034 \end{bmatrix}$	$0 < x_2 < 0.8$ $-2 < x_4 < 0$
3	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.215 & 0 & 1.16 \\ 0 & 174.49 & 0 & -1029.57 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.012 \\ -8.9 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.002 \\ 0.27 \end{bmatrix}$	$-0.8 < x_2 < 0$ $0 < x_4 < 2$
4	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.388 & 0 & -1.837 \\ 0 & 218.7 & 0 & -1032.59 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -0.01 \\ -7 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.0015 \\ 0.24 \end{bmatrix}$	$0 < x_2 < 0.8$ $0 < x_4 < 2$

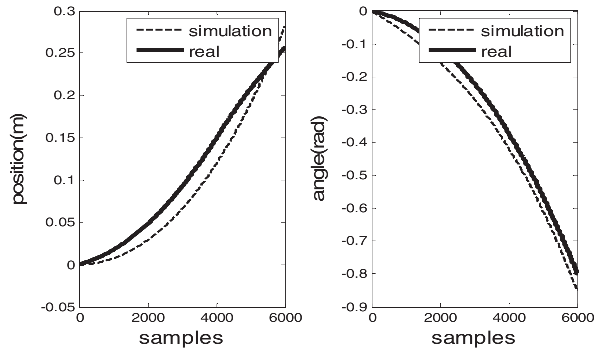

Figure 8. Model matching of real and simulation model for region 1.

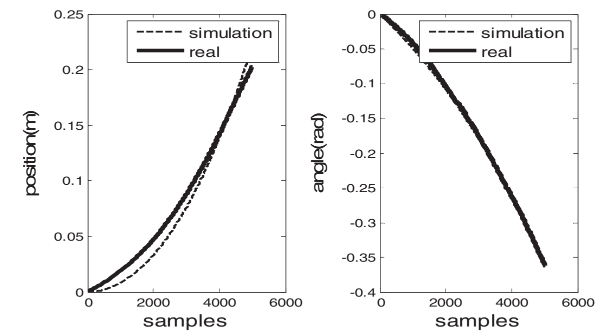

Figure 9.Model matching of real and simulation model for region 2.

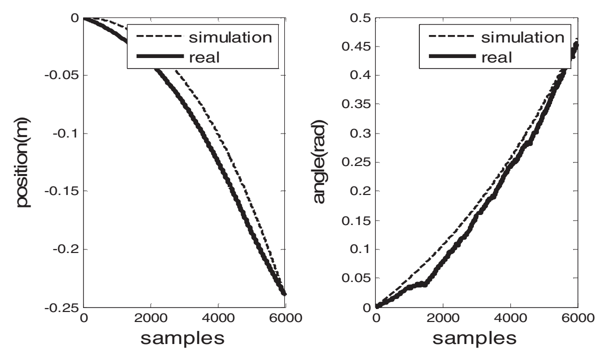

Figure 10.Model matching of real and simulation model for region 3.

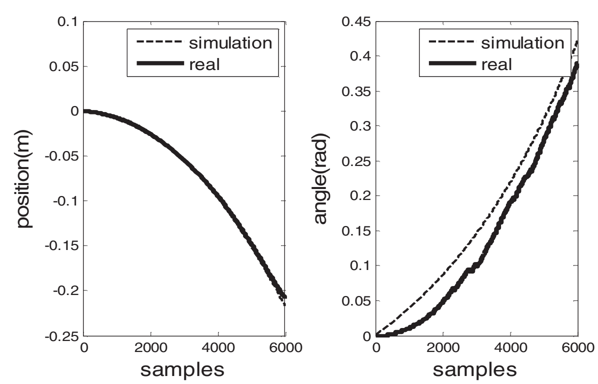

Figure 11. Model matching of real and simulation model for region 4.

As we can see the proposed algorithm has acceptable matching of simulation and real models. In the following the results of designing controller for identified model is presented.The parameters of patched LQR controller is shown in Table II. The state and input weighting matrixes for design of LQR controllers were Q=diag([62.5 2.04 4 4]) and R=10.The Fuzzy weighting functions (equation (9)) can define for subsystems for any point of state space. It has reverse relation to distance of state and center of cells.

In Fig.12 we can see response of PWA and PWA-FUZZY with patched LQR controller. The results shows the better settling time and less overshoot for PWA-FUZZY model. It also shows smoother control signal for PWA-FUZZY model.

By removing the assumption that full states are accessible the LQG controller design for each cells are investigated. The controller parameters are designed based on input state weighting matrix Q_{xu}=diag([0.1 0.1 1 1 100]) and noise covariance matrix$Q_{\eta v} = diag([10^{-5} * eye(5)10^{-6}])$. The parameters of LQG controllers are described by:

$$K_1 = [\text{-0.03} \quad \text{-47.83} \quad \text{-5.19} \quad \text{-0.038}], K_2 = [\text{-0.03} \quad \text{-44.5} \quad \text{-5.41} \quad \text{-0.06}]$$

$$K_3 = [\text{-0.03} \quad \text{-42.06} \quad \text{-6.56} \quad \text{-0.048}], K_4 = [\text{-0.03} \quad \text{-66.3} \quad \text{-5.27} \quad \text{-0.05}]$$

$$L_1 = \begin{bmatrix} 1.73 & -0.0018 \\ -0.0001 & 3.43 \\ 1.0000 & -0.0089 \\ 0 & 0.89 \end{bmatrix}, L_2 = \begin{bmatrix} 1.73 & -0.004 \\ -0.0004 & 3.34 \\ 1.0000 & -0.02 \\ 0 & 0.6 \end{bmatrix}, L_3 = \begin{bmatrix} 1.73 & -0.003 \\ -0.0003 & 3.33 \\ 1.0000 & -0.018 \\ 0 & 0.563 \end{bmatrix}, L_4 = \begin{bmatrix} 1.73 & 0 \\ 0 & 3.38 \\ 1.0000 & 0.0003 \\ 0 & 0.713 \end{bmatrix}$$

Table II. LQR controller parameters for each subsystem.

i	K_i	i	K_i
1	$k1 = [\text{-2.7} \ \text{-68} \ \text{-68} \ 0.001]$	3	$k3 = [\text{-2.7} \ \text{-82} \ \text{-100} \ \text{-0.15}]$
2	$k2 = [\text{-2.7} \ \text{-82} \ \text{-82} \ \text{-0.33}]$	4	$k4 = [\text{-2.7} \ \text{-100} \ \text{-68} \ 0.01]$

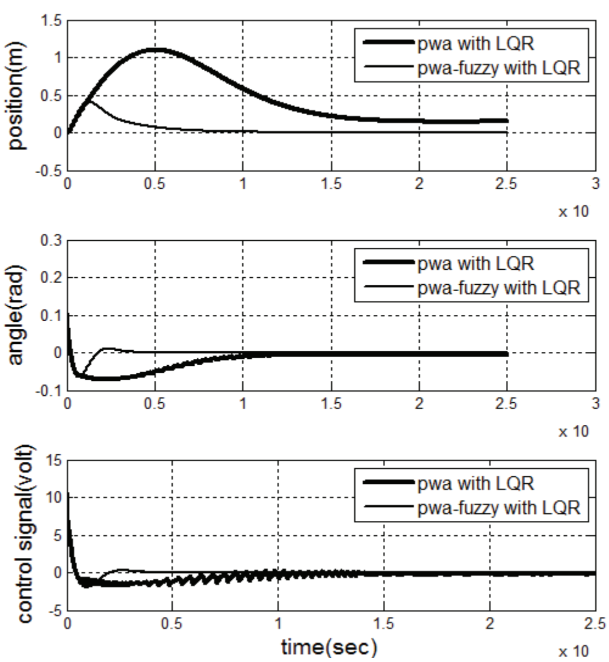

Figure 12. PWA and PWA-FUZZY with patched LQR controller

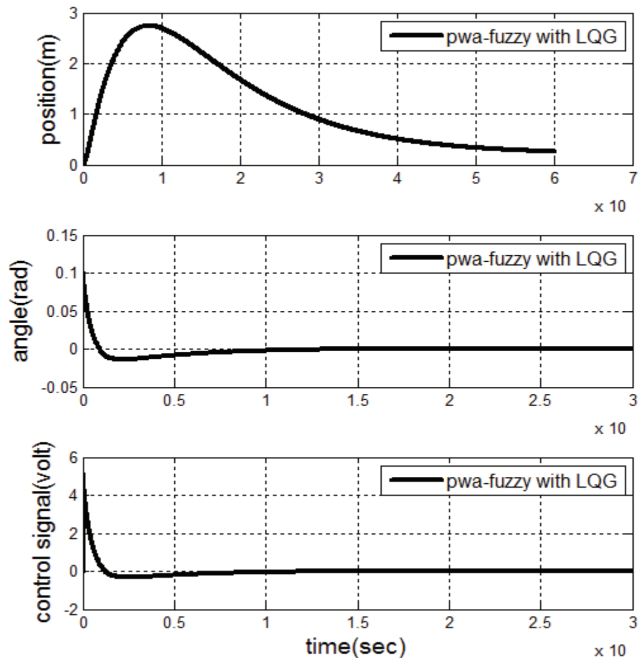

Figure 13. PWA and PWA-FUZZY with patched LQG controller

The results of patched LQG controller for PWA-FUZZY model is shown in Fig.12. Result in Fig.12 confirm the fast convergence in pendulum angle and soft movement of control signal for patched LQG controller.

VII. CONCLUSION

The result in modeling simulations shows the effectiveness of PWA-FUZZY method as proposed method for modeling of inverted pendulum on a cart. Also controller design results shows smoother control signal and faster convergence of system when we use FUZZY supervisory block in our control loop.

REFERENCES

[1] O. Boubaker, "The inverted pendulum: a fundamental benchmark in control theory and robotics," in Education and e-Learning Innovations (ICEELI), 2012 international conference on, 2012, pp. 1-6.

[2] L. Rodrigues, Dynamic output feedback controller synthesis for piecewise-affine systems: Stanford University, 2002.

[3] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," Systems, Man and Cybernetics, IEEE Transactions on, pp. 116-132, 1985.

[4] H. M. A. Kaseb and M. Momeni, "Modeling of PFC rectifiers by PWA-fuzzy method," in Power Electronics, Drive Systems and Technologies Conference (PEDSTC), 2011 2nd, 2011, pp. 507-511.

[5] H. Taghirad, "An Introduction to modern control," KN Toosi University Publication, 2003.

[6] A. Bemporad, "Modeling, control, and reachability analysis of discrete-time hybrid systems," Lecture Notes-DISC School on Hybrid Systems, 2003.

[7] H. M. A. Kaseb, "Modeling and Control of Resonant Converters by Hybrid and Switched Systems," PHD thesis, Faculty of Engineering, Ferdowsi University of Mashhad, 2012.

[8] R. L. Burden and J. D. Faires, "Numerical analysis PWS," ed: Kent Publishing Co. Boston, 1989.