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More on operator monotone and operator convex functions of several variables

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Let $C_1, C_2, \ldots, C_k$ be positive matrices in M_n and f be a continuous real-valued function on $[0, \infty)$. In addition, consider Φ as a positive linear functional on M_n and define

$$\phi(t_1, t_2, t_3, \ldots, t_k) = \Phi\left(f(t_1C_1 + t_2C_2 + t_3C_3 + \ldots + t_kC_k)\right),$$

as a k variables continuous function on $[0, \infty) \times \ldots \times [0, \infty)$. In this paper, we show that if f is an operator convex function of order mn, then ϕ is a k variables operator convex function of order $(n_1, \ldots, n_k)$ such that $m = n_1n_2 \ldots n_k$. Also, if f is an operator monotone function of order n^{k+1}, then ϕ is a k variables operator monotone function of order n. In particular, if f is a non-negative operator decreasing function on $[0, \infty)$, then the function $t \to \Phi\left(f(A + tB)\right)$ is an operator decreasing and can be written as a Laplace transform of a positive measure.

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1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and let I be the identity operator. An operator $A \in \mathbb{B}(\mathscr{H})$ is called *positive* if $\langle Ax, x\rangle \geq 0$ holds for every $x \in \mathscr{H}$ and then we can write $A \geq 0$. We say, $A \leq B$ if $B - A \geq 0$; see [1] for other possible orders.

For a continuous real-valued function f and a self adjoint operator A with spectrum in the domain of f, the operator $f(A)$ is defined by the continuous functional calculus. In particular, if $\mathscr{H}$ is a Hilbert space of finite dimension n and $A \in M_n(= \mathbb{B}(\mathscr{H}))$ has the spectral decomposition $A = \sum_{i=1}^n \lambda_i P_i$, where P_i is the projection corresponding to the eigenspace of the eigenvalue λ_i of A, then

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) P_i.$$

A continuous function $f : J \to \mathbb{R}$ defined on an interval J is said to be matrix monotone (or matrix increasing) of order n if $A \leq B$ implies that $f(A) \leq f(B)$ for any pair of self adjoint $n \times n$ matrices A, B with spectra in J. A function f is called matrix decreasing of order n if $-f$ is a matrix monotone function of order n. Also, we say that f is a matrix convex function of order n, if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B),$$

for all self adjoint matrices A, B in M_n with spectra in J and all $\lambda \in [0, 1]$. In the general case, a function $f : J \to \mathbb{R}$ is said to be operator monotone (operator convex) if it is a matrix monotone function (matrix convex function) of any arbitrary order. For more details, we refer readers to [3,6].

In [7], Hansen used the functional calculus developed by Koranyi [12] and presented an extension of one variable operator convex functions to multivariable functions. In his approach, Hansen considered $A = (A_1, ..., A_k)$ to be a k-tuple of self adjoint matrices of order $(n_1, \ldots, n_k)$ such that the spectra of $n_i \times n_i$ matrix A_i is contained in J_i for each $i = 1, \ldots, k$. If

$$A_i = \sum_{i_m}^{n_i} \lambda_{i_m}(A_i) P_{i_m}(A_i),$$

is the spectral decomposition of A_i and f is a k variable continuous function on the real interval $J_1 \times \ldots \times J_k$ of $\mathbb{R}^k$, then

$$f(A) = \sum_{i_1}^{n_1} \ldots \sum_{i_k}^{n_k} f\left(\lambda_{i_1}(A_1), \ldots, \lambda_{i_k}(A_k)\right)\ P_{i_1}(A_1) \otimes \ldots \otimes P_{i_m}(A_k),$$

can be defined as a self adjoint matrix in $M_{n_1} \otimes \ldots \otimes M_{n_k}$. Now, we say that a k variable continuous function f on a real interval $J_1 \times \ldots \times J_k$ of $\mathbb{R}^k$ is matrix convex of order $(n_1, \ldots, n_k)$, if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B),$$

for any $0 \leq \lambda \leq 1$ and for any two k-tuples $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ of self adjoint matrices such that the spectra of A_i and B_i are contained in J_i for each $i = 1, \ldots, k$. For more properties, see [20].

There are many approaches for extending the definition of one variable operator monotone functions to functions of several variables [2,15,17,19]. In particular, Agler, McCarthy and Young [2] worked on k-tuples of commuting self adjoint operators on a Hilbert space $\mathscr{H}$ and expressed an operator monotonicity for functions of several variables. In this approach, for k-tuples $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ of commutating self adjoint operators, we say that $A \leq B$ if $A_i \leq B_i$ for every $1 \leq i \leq k$. A k variable continuous function f on a real interval $J_1 \times \ldots \times J_k$ is called operator monotone of order n if for any two k tuples $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ of $n \times n$ commuting matrices such that the spectra of A_i and B_i are contained in J_i, the inequality $A \leq B$ implies that $f(A) \leq f(B)$.

Similar to this approach, we define k variable operator monotone functions of order $(n_1, \ldots, n_k)$. In particular, for k-tuples $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ of self adjoint matrices of order $(n_1, \ldots, n_k)$, we say $A \leq B$ if $A_i \leq B_i$ for all $1 \leq i \leq k$.

Definition 1.1. A k variable continuous function f on a cell $J_1 \times \ldots \times J_k$ of $\mathbb{R}^k$ is called operator monotone of order $(n_1, \ldots, n_k)$ if for any two k-tuples $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ of self adjoint matrices of order $(n_1, \ldots, n_k)$ such that the spectra of A_i and B_i are contained in J_i for $i = 1, \ldots, k$, the inequality $A \leq B$ implies that $f(A) \leq f(B)$.

A linear map Φ between two C^* algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be positive, if $a \geq 0$ implies $\Phi(a) \geq 0$, for each $a \in \mathcal{A}$. Also, Φ is said to be completely positive if for each $n \in \mathbb{N}$, the linear map $\Phi^n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by

$$\Phi^n([a_{i,j}]) = [\phi(a_{i,j})]$$

is positive. Any positive linear map is bounded and $||\Phi|| = ||\Phi(I)||$. In particular, if $\mathcal{B} = \mathbb{C}$ or Φ is a positive linear functional, then Φ is continuous.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. We say that a function $f : \mathfrak{X} \to \mathfrak{Y}$ defined on a subset $\mathfrak{G}$ of $\mathfrak{X}$ is Fréchet differentiable at an inner point $x \in \mathscr{G}$, if there exists a bounded linear operator $f^{[1]}(x) \in \mathbb{B}(\mathfrak{X}, \mathfrak{Y})$ such that

$$\lim_{h \to 0} ||h||_{\mathscr{X}}^{-1} ||f(x+h) - f(x) - f^{[1]}(x)(h)||_{\mathscr{Y}} = 0.$$

In this paper, we first investigate the operator convexity and the monotonicity of some functions of k variables by using one variable operator monotone functions. In particular, we show that if Φ is a positive linear functional on M_n, f is a matrix convex function of order mn on $[0,\infty)$ and

$$\phi(t_1,t_2,t_3,\ldots,t_k)=\Phi\left(f(t_1C_1+t_2C_2+\ldots+t_kC_k)\right),$$

for positive matrices $C_1,\ldots,C_n$ in M_n, then ϕ is a k variable operator convex function of order $(n_1,\ldots,n_k)$ such that $m=n_1\ldots n_k$. Similar results are valid for operator monotone functions.

In addition, by using a different approach, we extend some results of Hansen [10] (section 3) and prove that for any arbitrary Hilbert space $\mathscr{H}$ and any positive linear functional Φ on $\mathbb{B}(\mathscr{H})$, if f is an operator decreasing function on $(0,\infty)$, then the function $\phi(t)=\Phi(f(A+tB))$ has a Laplace transform of a positive measure, for positive operators $A,B\in\mathbb{B}(\mathscr{H})$. Moreover, the famous equivalent statement for Bessis–Moussa–Villani theorem [18] which states for each $p\le 0$ and for all positive semi-definite matrices A and B, the function $h_p(t)=Tr\ (A+tB)^p$ is completely monotone [13], is partially extended ($f:(0,\infty)\to\mathbb{R}$ is called a completely monotone function if $(-1)^nf^n(t)\ge 0$, for each $n=0,1,2,\ldots$ and each $t>0$). Indeed, we show that for a positive linear functional Φ on $\mathbb{B}(\mathscr{H})$, the function $\phi_p(t)=\Phi\left((A+tB)^p\right)$ is completely monotone for each $-1\le p\le 0$ and all positive operators $A,B\in\mathbb{B}(\mathscr{H})$.

Finally, some inequalities in relation to the sub-additivity of operator monotone functions are described in section 4.

2. Several variable operator functions

We begin this section with introducing some notation being used in the paper. Let $\mathscr{H}_i$ be a Hilbert space for each $1\le i\le k$ and $\overset{k}{\underset{i=1}{\otimes}}\mathscr{H}_i$ denotes the tensor product Hilbert space $\mathscr{H}_1\otimes\mathscr{H}_2\otimes\ldots\otimes\mathscr{H}_k$. Suppose that $Y_i^k:B(\mathscr{H}_i)\to B\left(\overset{k}{\underset{i=1}{\otimes}}\mathscr{H}_j\right)$ denotes the isometry

$$Y_i^k(A_i)=I\otimes\ldots I\otimes A_i\otimes\ldots\otimes I,$$

for $A_i\in B(\mathscr{H}_i)$. Also, for any operator C in $B(\mathscr{H}_0)$, consider $X_{i,C}^k:B\left(\overset{k}{\underset{i=1}{\otimes}}\mathscr{H}_i\right)\to B\left(\overset{k}{\underset{i=0}{\otimes}}\mathscr{H}_i\right)$ such that

$$X_{i,C}^k(A)=C\otimes Y_i(A),$$

for $A \in B\left(\overset{k}{\underset{i=1}{\otimes}} \mathscr{H}_i\right)$. Note that if C is a positive operator, then $X^k_{i,A}$ is a completely positive linear map for each $1 \leq i \leq n$. Now, let $x \in \mathscr{H}_0$. Assume that $T_{k,x} : \overset{k}{\underset{i=1}{\otimes}} \mathscr{H}_i \longrightarrow \overset{k}{\underset{i=0}{\otimes}} \mathscr{H}_i$ is the linear map defined by

$$T_{k,x}(y_1 \otimes y_2 \otimes \ldots \otimes y_k) = x \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_k,$$

for any $y_1 \otimes y_2 \otimes \ldots \otimes y_k$ in $\overset{k}{\underset{i=1}{\otimes}} \mathscr{H}_i$. If $T^*_{k,x} : \overset{k}{\underset{i=0}{\otimes}} \mathscr{H}_i \longrightarrow \overset{k}{\underset{i=1}{\otimes}} \mathscr{H}_i$ denotes the adjoint of $T_{k,x}$, we can conclude that

$$\begin{aligned}\langle y'_1 \otimes \ldots \otimes y'_k, T^*_{k,x}(y_0 \otimes y_1 \otimes \ldots \otimes y_k)\rangle &= \langle T_{k,x}(y'_1 \otimes \ldots \otimes y'_k), y_0 \otimes \ldots \otimes y_k\rangle \\ &= \langle x, y_0\rangle \ \langle y'_1, y_1\rangle \ldots \langle y'_k, y_k\rangle,\end{aligned}$$

for each $y'_1 \otimes \ldots \otimes y'_k \in \overset{k}{\underset{i=1}{\otimes}} \mathscr{H}_i$ and $y_0 \otimes y_1 \otimes \ldots \otimes y_k \in \overset{k}{\underset{i=0}{\otimes}} \mathscr{H}_i$. Therefore

$$T^*_{k,x}(y_0 \otimes y_1 \otimes \ldots \otimes y_k) = \langle y_0, x\rangle(y_1 \otimes y_2 \otimes \ldots \otimes y_k),$$

for $y_0 \otimes y_1 \otimes \ldots \otimes y_k \in \overset{k}{\underset{i=0}{\otimes}} \mathscr{H}_i$.

Let $\sigma_m(1, \ldots, k)$ be a word of length m over the set $\{1, 2, \ldots, k\}$. Put $\sigma_m(C_1, \ldots, C_k)$ to be the multiplication of operators $C_1, C_2, \ldots, C_k$ corresponded to $\sigma_m(1, \ldots, k)$. For example, if $m = 5$, $k = 3$ and $\sigma_5(1, 2, 3) = (12113)$, then

$$\sigma_5(C_1, C_2, C_3) = C_1C_2C_1C_1C_3 = C_1C_2C_1^2C_3.$$

The set of all words of length m over set $\{1, 2, \ldots, k\}$ is denoted by $S(m, k)$.

Lemma 2.1. *Let $C_1, \ldots, C_k$ be positive matrices in M_n and f be a continuous function on $[0, \infty)$. Let $x \in \mathbb{C}^n$ and*

$$\phi_x(t_1, t_2, t_3, \ldots, t_k) = \langle f(t_1C_1 + t_2C_2 + \ldots + t_kC_k)x, x\rangle.$$

Then, the following statements hold.

(i) *Let $A = (A_1, \ldots, A_k)$ be a k-tuple of positive matrices of order $(n_1, \ldots, n_k)$. Then*

$$\phi_x(A_1, \ldots, A_k) = T^*_{k,x}\left(f(X^k_{1,C_1}(A_1) + X^k_{2,C_2}(A_2) + \ldots + X^k_{k,C_k}(A_k))\right) T_{k,x},$$

as a matrix in $M_{n_1} \otimes \ldots \otimes M_{n_k}$.

(ii) *Let $A = (A_1, \ldots, A_k)$ be a k-tuple of commutating positive matrices of order n. Then*

$$\phi_x(A_1, \dots, A_k) = T_{1,x}^* \left(f(X_{1,C_1}^2(A_1) + \dots + X_{1,C_k}^2(A_k)) \right) T_{1,x},$$

as a matrix in M_n*.*

Proof. By the Stone–Weierstrass theorem, it is sufficient to prove the theorem for $f(t) = t^m$ for each $m \geq 0$. If $m = 0$ the proof is clear. Let $m > 0$. Then

$$\begin{aligned}\phi_x(t_1, t_2, \dots, t_k) &= \langle (t_1 C_1 + t_2 C_2 + \dots + t_k C_k)^m x, x \rangle \\ &= \sum_{\sigma \in S(m,k)} \sigma(t_1, \dots, t_k) \ \langle \sigma(C_1, \dots, C_k) x, x \rangle.\end{aligned}$$

(i) First, note that by a remark of Hansen [8, Page 1], if a k variable continuous function g can be separated as a product $g(t_1, \dots, t_k) = g_1(t_1) \dots g_k(t_k)$ of k functions each of which depends on only one variable, then

$$g(A) = g_1(A_1) \otimes \dots \otimes g_k(A_k).$$

So, we can conclude that

$$\phi_x(A_1, A_2, \dots, A_k) = \sum_{\sigma \in S(m,k)} \langle \sigma(C_1, \dots, C_k) x, x \rangle \ \sigma(Y_1^k(A_1), \dots, Y_k^k(A_k)).$$

On the other hand, we have

$$\begin{aligned}&T_{k,x}^* \left(\left(X_{1,C_1}^k(A_1) + \dots + X_{k,C_k}^k(A_k) \right)^m \right) T_{k,x} \\ &\quad = T_{k,x}^* \left(\sum_{\sigma \in S(m,k)} \sigma \left(X_{1,C_1}^k(A_1), \dots, X_{k,C_k}^k(A_k) \right) \right) T_{k,x} \\ &\quad = \sum_{\sigma \in S(m,k)} T_{k,x}^* \sigma \left(X_{1,C_1}^k(A_1), \dots, X_{k,C_k}^k(A_k) \right) T_{k,x} \\ &\quad = \sum_{\sigma \in S(m,k)} T_{k,x}^* \left(X_{1,\sigma(C_1,\dots,C_k)}^k(I) \otimes (I \otimes \sigma(Y_1^k(A_1), \dots, Y_k^k(A_k))) \right) T_{k,x}.\end{aligned}$$

Let $y_1 \otimes y_2 \otimes \dots \otimes y_k \in \bigotimes_{i=1}^{k} \mathbb{C}^{n_i}$. Then,

$$\begin{aligned}&T_{k,x}^* \left(X_{1,\sigma(C_1,\dots,C_k)}^k(I) \otimes (I \otimes \sigma(Y_1^k(A_1), \dots, Y_k^k(A_k))) \right) T_{k,x}(y_1 \otimes y_2 \otimes \dots \otimes y_k) \\ &= T_{k,x}^* \left(X_{1,\sigma(C_1,\dots,C_k)}^k(I) \otimes (I \otimes \sigma(Y_1^k(A_1), \dots, Y_k^k(A_k))) \right) (x \otimes y_1 \otimes y_2 \otimes \dots \otimes y_k) \\ &= T_{k,x}^*(\sigma(C_1, \dots, C_k)(x) \otimes (\sigma(Y_1(A_1), \dots, Y_k(A_k))(y_1 \otimes y_2 \otimes \dots \otimes y_k)) \\ &= \langle \sigma(C_1, \dots, C_k) x, x \rangle \sigma(Y_1(A_1), \dots, Y_k(A_k))(y_1 \otimes y_2 \otimes \dots \otimes y_k),\end{aligned}$$

for each $\sigma \in S(m,k)$. Therefore,

$$\phi_x(A_1,\ldots,A_k) = T_{k,x}^* \left(\left(X_{1,C_1}^k(A_1) + \ldots + X_{k,C_k}^k(A_k)\right)^m\right) T_{k,x}.$$

(ii) It is clear that

$$\phi_x(A_1,A_2,\ldots,A_k) = \sum_{\sigma\in S(m,k)} \langle \sigma(C_1,\ldots,C_k)x,x\rangle\ \sigma(A_1,\ldots,A_k).$$

Moreover,

$$\begin{aligned} & T_{1,x}^* \left(\left(X_{1,C_1}^2(A_1) + \ldots + X_{1,C_k}^2(A_k)\right)^m\right) T_{1,x} \\ &\quad = \sum_{\sigma\in S(m,k)} T_{1,x}^* \sigma\left(X_{1,C_1}^2(A_1),\ldots,X_{1,C_k}^2(A_k)\right) T_{1,x} \\ &\quad = \sum_{\sigma\in S(m,k)} T_{1,x}^* \left(\sigma\left(C_1,\ldots,C_k\right) \otimes \sigma\left(A_1,\ldots,A_k\right)\right) T_{1,x}. \end{aligned}$$

An argument similar to Part (i) implies that

$$\phi_x(A_1,A_2,\ldots,A_k) = T_{1,x}^* \left(\left(X_{1,C_1}^2(A_1) + \ldots + X_{1,C_k}^2(A_k)\right)^m\right) T_{1,x}. \qquad \square$$

Theorem 2.2. *Let $C_1,C_2,\ldots,C_k$ be positive matrices in M_n and $f:[0,\infty)\to\mathbb{R}$ be a continuous function. Let Φ be a positive linear functional on M_n and*

$$\phi(t_1,t_2,t_3,\ldots,t_k) = \Phi\left(f(t_1C_1+t_2C_2+t_3C_3+\ldots+t_kC_k)\right)$$

be a k variable continuous function. Then the following statements hold:

i) *If f is an operator convex function of order mn, then ϕ is a k variable operator convex function of order $(n_1,\ldots,n_k)$ such that $m=n_1\ldots n_k$.*
ii) *If f is an operator monotone function of order mn and $m=n_1\ldots n_k$, then ϕ is a k variable operator monotone function of order $(n_1,\ldots,n_k)$ in the sense of Definition 1.1.*
iii) *If f is an operator monotone function of order n^{k+1}, then ϕ is a k variable operator monotone function of order n in the sense of Agler, McCarthy, Young.*

Proof. By the Riesz–Fischer theorem [11, Page 13] there exists a positive matrix T such that $\Phi(X) = Tr(XT) = \sum_{i=1}^n \langle XT^{1/2}e_i, T^{1/2}e_i\rangle$ for each matrix $X \in M_n$. As the set of operator monotone functions and the set of operator convex functions are both convex cones, it is sufficient to prove the theorem for functions of the form

$$\phi_x(t_1,t_2,t_3,\ldots,t_k) = \langle \left(f(t_1C_1+t_2C_2+t_3C_3+\ldots+t_kC_k)\right)x,x\rangle,$$

for each $x \in \mathbb{C}^n$.

i) Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ be two k-tuples of positive matrices of order $(n_1, \dots, n_k)$ and $0 \le \lambda \le 1$. Then,

$$\begin{aligned}
&\phi_x(\lambda A_1 + (1-\lambda)B_1, \dots, \lambda A_k + (1-\lambda)B_k)\\
&= T^*_{k,x}\left(f(X^k_{1,C_1}(\lambda A_1 + (1-\lambda)B_1) + \dots + X^k_{k,C_k}(\lambda A_k + (1-\lambda)B_k))\right) T_{k,x}\\
&= T^*_{k,x}\left(f(\lambda X^k_{1,C_1}(A_1) + (1-\lambda)X^k_{1,C_1}(B_1) + \dots + \lambda X^k_{k,C_k}(A_k) + (1-\lambda)X^k_{k,C_k}(B_k))\right) T_{k,x}\\
&= T^*_{k,x}\left(f(\lambda(X^k_{1,C_1}(A_1) + \dots + X^k_{1,C_1}(A_k)) + (1-\lambda)(X^k_{1,C_1}(B_1) + \dots + X^k_{k,C_k}(B_k)))\right) T_{k,x}\\
&\le \lambda T^*_{k,x}\left(f((X^k_{1,C_1}(A_1) + \dots + X^k_{1,C_1}(A_k))\right) T_{k,x}\\
&+ (1-\lambda)T^*_{k,x}\left(f(X^k_{1,C_1}(B_1) + \dots + X^k_{1,C_k}(B_k)))\right) T_{k,x}\\
&= \lambda\phi_x(A_1, \dots, A_k) + (1-\lambda)\phi_x(B_1, \dots, B_k).
\end{aligned}$$

ii) Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ be two k-tuples of positive matrices of order $(n_1, \dots, n_k)$ and $A_i \le B_i$ for each $1 \le i \le k$. Since C_i is positive, X^k_{1,C_i} is a positive linear map and $X^k_{1,C_i}(A_i) \le X^k_{1,C_i}(B_i)$ for each $1 \le i \le k$. Therefore

$$\begin{aligned}
\phi_x(A_1, \dots, A_k) &= T^*_{k,x}\left(f(X^k_{1,C_1}(A_1) + \dots + X^k_{k,C_k}(A_k))\right) T_{k,x}\\
&\le T^*_{k,x}\left(f(X^k_{1,C_1}(B_1) + \dots + X^k_{k,C_k}(B_k))\right) T_{k,x}\\
&= \phi_x(B_1, \dots, B_k).
\end{aligned}$$

iii) Using Part (ii) of Lemma 2.1 and the same reasoning as in Part (ii) we obtain the result. □

3. Laplace transform of operator decreasing functions

The Bessis–Moussa–Villani conjecture [4] states that for a self adjoint matrix A and a positive matrix B, the function $f(t) = Tr\left(\exp^{A-tB}\right)$ can be represented as the Laplace transform

$$f(t) = \int_0^\infty \exp^{-tx}\, d\mu(x), \tag{1}$$

for a positive measure μ on $[0, \infty)$ [4]. This conjecture has attracted a lot of attention. Despite a lot of attempts to prove the conjecture, it remained open until 2012. Eventually, Stahl [18] proved this conjecture. Hansen also got similar results [10].

Theorem 3.1. [10] *If f is a non-negative operator decreasing function on $[0, \infty)$, then for positive matrices A, B, the map $t \to Trf(A + tB)$ can be written as the Laplace transform of a positive measure.*

In his proof of this theorem, Hansen employed the theory of the Fréchet differentials and the Bernstein's theorem highlighting the measure μ in (1) exists if and only if f is completely monotone or $(-1)^n f^n(t) \geq 0$, for each $n = 0, 1, 2, \ldots$ and $t > 0$. We can extend Theorem 3.1 as the following form.

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive operators and f be an operator decreasing function on $[0, \infty)$. For any positive linear functional Φ on $\mathbb{B}(\mathscr{H})$, the function $\phi(t) = \Phi(f(A + tB))$ is operator decreasing. In particular, if f is non-negative, then ϕ can be written as the Laplace transform of a positive measure.*

Proof. By [3, Theorem 2.3], if f is an operator decreasing function on the $(0, \infty)$, then f on $[0, \infty)$ can be represented as

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{1}{\lambda + t} - \frac{\lambda}{\lambda^2 + 1}\, d\mu(\lambda) \tag{2}$$

where $\beta \leq 0$ and μ is a positive measure on $(0, \infty)$ such that $\int_0^\infty \frac{\lambda}{\lambda^2+1}\, d\mu(\lambda)$ is finite; see [6, Chapter II]. Since $\int_0^\infty (\lambda I + A + tB)^{-1} - \frac{\lambda}{\lambda^2+1} I\, d\mu(\lambda)$ exists and Φ is a bounded linear functional, we have

$$\begin{aligned}\Phi\left(\int_0^\infty (\lambda I + A + tB)^{-1} - \frac{\lambda I}{\lambda^2 + 1} d\mu(\lambda)\right) &= \int_0^\infty \Phi\left((\lambda I + A + tB)^{-1} - \frac{\lambda I}{\lambda^2 + 1}\right) d\mu(\lambda) \\ &= \int_0^\infty \Phi\left((\lambda I + A + tB)^{-1}\right) - \frac{\lambda \Phi(I)}{\lambda^2 + 1}\, d\mu(\lambda).\end{aligned}$$

Hence

$$\phi(t) = \Phi(f(A + tB)) = \alpha\Phi(I) + \beta\Phi(I)t + \int_0^\infty \Phi\left((\lambda I + A + tB)^{-1}\right) - \frac{\Phi(I)\lambda}{\lambda^2 + 1} d\mu(\lambda).$$

As the set of operator decreasing functions on $[0, \infty)$ is a closed cone, it is sufficient to prove the theorem for the function $f_\lambda(t) = \frac{1}{\lambda+t}$ where $\lambda > 0$. Indeed, we show that the function $\phi(t) = \Phi\left((\lambda I + A + tB)^{-1}\right)$ is operator decreasing on $[0, \infty)$. First, assume that B be invertible. Note that $\Phi_B(X) = \Phi(B^{\frac{-1}{2}} X B^{\frac{-1}{2}})$ is a positive linear functional and

$$\begin{aligned}\phi(t) = \Phi(f_\lambda(A + Bt)) &= \Phi\left((\lambda I + A + tB)^{-1}\right) \\ &= \Phi\left(B^{\frac{-1}{2}}(\lambda B^{-1} + B^{\frac{-1}{2}} A B^{\frac{-1}{2}} + tI)^{-1} B^{\frac{-1}{2}}\right) \\ &= \Phi_B\left((\lambda B^{-1} + B^{\frac{-1}{2}} A B^{\frac{-1}{2}} + tI)^{-1}\right).\end{aligned}$$

Consider $H = \lambda B^{-1} + B^{\frac{-1}{2}} A B^{\frac{-1}{2}}$. Note that H is a positive operator and ϕ has analytic continuation to the whole upper half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$ as

$$\phi(z) = \Phi_B\left((H + zI)^{-1}\right),$$

see [16, Lemma 1.2.4]. Now, if $z = x + iy \in \mathbb{C}$ and $y = \mathrm{Im}(z) > 0$, then

$$\begin{aligned}
\mathrm{Im}((\phi(z)) &= \mathrm{Im}\left(\Phi_B\left((H + (x+iy)I)^{-1}\right)\right) \\
&= \mathrm{Im}\left(\Phi_B\left((H + (xI + iyI)^{-1}\right)\right) \\
&= \mathrm{Im}\left(\Phi_B\left((H + xI - iyI)((A + xI)^2 + y^2 I)^{-1}\right)\right) \\
&= -\Phi_B\left(y((H + xI)^2 + y^2 I)^{-1}\right) < 0.
\end{aligned}$$

The last equality is valid. Indeed, because Φ is a positive linear functional, $\Phi(T)$ is real for each self adjoint operator $T \in \mathbb{B}(\mathscr{H})$. Also, as $y((H+xI)^2+y^2I)^{-1}$ is positive invertible, the last inequality holds. Therefore, ϕ is analytic on $(0, \infty)$ and can be continued analytically to the whole upper half-plane and represents an analytic function whose imaginary part is negative. Hence, by Löwner's theorem [14], ϕ is operator decreasing on $(0, \infty)$ and, by the continuity, on $[0, \infty)$. If B is an arbitrary positive operator, then $\phi_n(t) = \Phi\left((\lambda + A + t(B + \frac{1}{n}))^{-1}\right)$ is operator decreasing on $[0, \infty)$ for each n. Since $\phi_n \to \phi$ in point-wise convergence topology, by [5, Proposition V.4.2], ϕ is operator decreasing on $[0, \infty)$.

If f is non-negative, then ϕ is non-negative and has an integral representation of the form (2). This representation shows that ϕ is an integral sum of functions that have Laplace transform. Therefore, ϕ can be written as the Laplace transform of a positive measure. $\square$

By replacing f by $-f$, a similar statement can be obtained. However, we can extend it as follows.

Corollary 3.3. *Let A and B be positive matrices in M_n. Let f be an n^3 operator monotone (convex) function on $[0, \infty)$ and Φ be a positive linear functional on M_n. Then $\phi(t) = \Phi\left(f(A + tB)\right)$ is an n-monotone (convex) function on $[0, \infty)$.*

Proof. Let f be an n^3 operator monotone function on $[0, \infty)$. By Theorem 2.2, the function $\phi_0(s, t) = \Phi\left(f(sA + tB)\right)$ of two variables is operator monotone on M_n. If $S \le T$, then (I, S) and (I, T) are 2-tuples of commutating matrices and $(I, S) \le (I, T)$. Therefore,

$$\phi(S) = \phi_0(I, S) \le \phi_0(I, T) = \phi(T). \qquad \square$$

The function $f(t) = t^p$ is operator increasing for $0 \le p \le 1$ and operator decreasing for $-1 \le p \le 0$. Therefore, as an example the following corollary can be given.

Corollary 3.4. *Let A, B be positive operators in $\mathbb{B}(\mathscr{H})$ and $-1 \leq p \leq 1$. For a positive linear functional Φ on $\mathbb{B}(\mathscr{H})$, the function $\phi(t) = \Phi\left((A + tB)^p\right)$ is operator increasing if $0 \leq p \leq 1$ and operator decreasing if $-1 \leq p \leq 0$.*

4. Sub-additivity type inequalities for operator monotone functions

If f is a non-negative operator monotone function on $[0, \infty)$ and A, B are positive operators in $\mathbb{B}(\mathscr{H})$, then the inequality

$$f(A + B) \leq f(A) + f(B),$$

does not hold in the general case, for more details, see [15]. In the next corollaries, several sub-additivity type inequalities for operator monotone functions are described.

Proposition 4.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a non-negative operator monotone function of order n^3 for some $n > 1$. Let A, B be positive operators in M_n. Then*

$$f(A + (s + t)B) \leq f(A + sB) + f(A + tB),$$

for each $s, t \in [0, \infty)$.

Proof. First note that by [9, Theorem 2.1] f is a concave function on $[0, \infty)$. Let Φ be an arbitrary positive linear functional on M_n. Then $\phi(x) = \Phi(f(A+xB))$ is a non-negative operator monotone function on $[0, \infty)$. Since ϕ is sub-additive on the real line, we have

$$\begin{aligned}\Phi(f(A + (s + t)B)) &= \phi(t + s) \leq \phi(t) + \phi(s)\\ &= \Phi(f(A + sB) + f(A + tB))\end{aligned}$$

for each $s, t \in [0, \infty)$. Since Φ is arbitrary, we obtain

$$f(A + (s + t)B) \leq f(A + sB) + f(A + tB),$$

for each $s, t \in [0, \infty)$. $\square$

Corollary 4.2. *Let $f : [0, \infty) \to \mathbb{R}$ be a non-negative operator monotone function of order n^3 for some $n > 1$. Let A, B be positive matrices in M_n. Then*

$$\int_0^1 f(A + \lambda B)\, d\lambda \leq f(A + B) \leq 2 \int_0^1 f(A + \lambda B)\, d\lambda.$$

Proof. Proposition 4.1 implies that

$$f(A + B) \leq f(A + \lambda B) + f(A + (1 - \lambda)B),$$

for each $0 \leq \lambda \leq 1$. Moreover, since $A + B \geq A + \lambda B$ we have

$$f(A + \lambda B) \leq f(A + B),$$

for each $0 \leq \lambda \leq 1$. Taking integral with respect to parameter λ and using the equality

$$\int_0^1 f(A + \lambda B)\ d\lambda = \int_0^1 f(A + (1 - \lambda)B)\ d\lambda,$$

we conclude the result. $\square$

Assume that $f^1[A]$ denotes the Fréchet derivative of f at A, in which f is a differential function on an interval I and A is a Hermitian matrix with eigenvalues in I. Our next result reads as follows.

Corollary 4.3. *Let $f \in C^1([0, \infty))$ be an operator monotone function of order n^3 for some $n > 1$. If $A, B \in M_n$ are positive matrices, then*

$$f(A + B) \leq f(A) + f^{[1]}(A) \circ B,$$

where $\circ$ stands for the Schur-product of two matrices with respect to the basis in which A can be represented as a diagonal matrix.

Proof. By [5, Theorem V.3.3], we have

$$f^{[1]}(A) \circ (B) = \frac{d}{dt}\Big|_{t=0} f(A + tB).$$

Suppose that Φ is an arbitrary positive linear functional on M_n. As $\phi(t) = \Phi(A + tH)$ is an operator monotone function on $[0, \infty)$, so it is concave. Hence

$$\begin{aligned}
\Phi\left(f^{[1]}(A) \circ (B)\right) &= \Phi\left(\frac{d}{dt}\Big|_{t=0} f(A + tB)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \Phi\left(f(A + tB)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \phi(t) \\
&\geq \phi(1) - \phi(0) \\
&= \Phi\left(f(A + B) - f(A)\right).
\end{aligned}$$

In the last inequality, we utilize the concavity of ϕ. Since Φ is arbitrary, we get

$$f(A + B) \leq f(A) + f^{[1]}(A) \circ B. \qquad \square$$

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