A Note On The Stahl's Theorem

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Abstract

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$ and A be a bounden operator in $\mathcal{H}$. Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators and Φ be a positive linear functional on $\mathbb{B}(\mathcal{H})$. We show that, if f is a non-negative operator decreasing function, then the function $t \to \Phi(f(A+tB))$ can be written as a Laplace transform of a positive measure.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$ and A be a bounden operator in $\mathcal{H}$. An operator A is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in \mathcal{H}$ and then we writ $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ we say $A \geq B$ if $A - B \geq 0$. For a continuous function f and a self adjoint operator A with spectra in domain of f, the operator $f(A)$ is defined by standard functional calculus. In particular, if $\mathcal{H}$ is a finite dimensional Hilbert space and A has the spectral decomposition $A = \sum_{i=1}^{n} \mu_i P_i$, where P_i are the projections corresponding to the eigenspaces of eigenvalue μ_i, then

$$f(A) = \sum_{i=1}^{n} f(\mu_i) P_i.$$

A function f on the interval I is called operator increasing if $A \geq B$ implies that $f(A) \geq f(B)$, for each self adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in I. Also, f is called operator decreasing function if $-f$ is operator increasing function.
The Bessis-Moussa-Villani conjecture states that for a self adjoint Matrix A and positive Matrix B, the function $f(t) = Tr\left(\exp^{A-tB}\right)$ can be represented as the Laplace transform

$$f(t) = \int_0^\infty \exp^{-tx} d\mu(x), \tag{1}$$

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of a positive measure μ on $[0,\infty)$ [1]. This conjecture has attracted a lot of attention in mathematics and physics. Despite a lot of afford to prove the conjecture, it remained open until 2012. Eventually, Stahl [5] proved this conjecture. In [3], Hansen got a similar results.

Theorem 1.1. *[3] If f is a non-negative operator decreasing function on $[0,\infty)$, then for positive matrices A, B the map $t \to Trf(A+tB)$ can be written as the Laplace transform of a positive measure.*

In proof of this theorem, Hansen used the theory of Frechet differentials and the Bernsteins theorem highlighting the measure μ in (1) exists if and only if f is completely monotone or $(-1)^n f^n(t) \geq 0$, for each $n = 0, 1, 2, \ldots$ and $t > 0$.
In this note, we extended the results of Hansen and prove that for an arbitrary Hilbert space $\mathcal{H}$ and any positive linear functional Φ on $\mathbb{B}(\mathcal{H})$.
Also, in a special case, we obtained a partial extension of a famous equivalent statement for Bessis-Moussa-Villani conjecture which state for each $p \leq 0$ and positive semi-definite matrices A and B, the function $h_p(t) = Tr\,(A+tB)^p$ is completely monotone [4]. Indeed, we show that for a positive linear functional Φ on $\mathbb{B}(\mathcal{H})$, the function $\phi_p(t) = \Phi\left((A+tB)^p\right)$ is completely monotone for each $-1 \leq p \leq 0$ and positive operators $A, B \in \mathbb{B}(\mathcal{H})$.

2. Main Results

The following theorem states the main results.

Theorem 2.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators and f be an operator decreasing function on $(0,\infty)$. For any positive linear functional Φ on $\mathbb{B}(\mathcal{H})$ the function $\phi(t) = \Phi\left(f(A+tB)\right)$ is operator decreasing. In particular, if f is non-negative then ϕ is completely monotone.*

By replacing f by $-f$ we obtain the following corollary.

Corollary 2.2. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators and f be an operator increasing function on $(0,\infty)$. Then, for any positive linear functional Φ on $\mathbb{B}(\mathcal{H})$ the function $\phi(t) = \Phi\left(f(A+tB)\right)$ is an operator increasing function.*

Example 2.3. *The functions $\ln(t+1)$ is a non-negative operator increasing function on $[0,\infty)$. Hence, for positive operators A, B and positive linear functional Φ, the function $t \to \Phi\left(f(A+tB)\right)$ is a non-negative operator increasing function on $[0,\infty)$.*

The function $f(t) = t^p$ is operator increasing for $0 \leq p \leq 1$ and operator decreasing for $-1 \leq p \leq 0$. Therefore, As an example we the following corollary is given.

Corollary 2.4. *Let A, B be positive operators in $\mathbb{B}(\mathcal{H})$ and $-1 \leq p \leq 1$. For a positive linear functional Φ on $\mathbb{B}(\mathcal{H})$, the function $\phi(t) = \Phi\left((A+tB)^p\right)$ is operator increasing if $0 \leq p \leq 1$ and operator decreasing if $-1 \leq p \leq 0$.*

References

[1] D. Bessis, P. Moussa and M. Villani, *Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics*, J. Math. Phys., **16** (1975) 2318-2385.

[2] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[3] F. Hansen, *Trace functions as Laplace transforms*, J. Math. Phys., **47** (2006) , 1-11.

[4] E. Lieb and R. Steiringer, *Equivalent forms of the Bessis-Moussa-Villani conjecture*, J. Stat. Phys., **115** (2004) 185-190.

[5] R., H., Stahl, *Proof of the BMV conjecture*, Acta Math, **211** (2013), 255-290.

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