The Extended Abstracts of
The 3rd Seminar on Operator Theory and its Applications
8-9th March 2017, Ferdowsi University of Mashhad, Iran

# POSITIVE BLOCK MATRICES

HAMED NAJAFI

*Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran;.*
*hamednajafi20@gmail.com*

Abstract. Let $C$ and $D$ be positive operators such that $C \leq D$ and $D$ be invertible. We show that there exists a trace preserving unital completely positive map $\Phi_{C,D} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ such that the block operator matrices

$$\begin{pmatrix} \Phi_{C,D}(A) & C \\ C & \Phi_{C,D}(B) \end{pmatrix}$$

are positive, for all positive operators $A$ and $B$ such that $D = A \sharp B$.

## 1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A$ is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in \mathcal{H}$ and then we writ $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ we say $A \geq B$ if $A - B \geq 0$.
Choi in [3], showed that for operators $A, X$, and invertible operator $B$ in $\mathbb{B}(\mathcal{H})$, the block matrix

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 47A63.; Secondary 47B65.
*Key words and phrases.* Geometric mean, Positive block matrix, Completely positive linear map.

in $M_2(\mathbb{B}(\mathcal{H}))$ is positive if and only if $A$ and $B$ are positive operator and

$$A \geq XB^{-1}X^*. \tag{1.2}$$

For positive operators $A$ and $B$, assume that

$$A\sharp B = \max\left\{ C \geq 0 \mid \begin{pmatrix} A & C \\ C & B \end{pmatrix} \geq 0 \right\}. \tag{1.3}$$

It is known that $A\sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ for two positive invertible operators $A$ and $B$ and in general case if $A$ and $B$ are positive, then

$$A\sharp B = \lim_{\text{SOT}}(A + \frac{1}{n})\sharp(B + \frac{1}{n}).$$

We recall that, a linear map $\Phi$ between two $C^*$ algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be completely positive if for each $n \in \mathbb{N}$, the linear map $\Phi^n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by

$$\Phi^n([a_{i,j}]) = [\phi(a_{i,j})]$$

is positive.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let $X$ and $Y$ be invertible operators in $\mathbb{B}(\mathcal{H})$ such that $||X||, ||Y|| \leq 1$. Then there exists unital completely positive linear maps $\Phi_X, \Phi_Y : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{B}(\mathcal{H})$ such that for all positive operators $A$ and $B$, the following operator matrix is positive.*

$$\begin{pmatrix} \Phi_Y(A) & Y(A\sharp B)X^* \\ X(A\sharp B)Y^* & \Phi_X(B) \end{pmatrix}.$$

*Also, if $\mathcal{H}$ is finite dimensional then $\Phi_X$ and $\Phi_Y$ are trace preserving.*

**Theorem 2.2.** *Let $C$ and $D$ be positive operators such that $C \leq D$ and $D$ be invertible. Then there exists a unital completely positive map $\Phi_{C,D} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ such that the following statements are hold:*
*(i) If $C$ is invertible, $\Phi$ is normal and faithful.*
*(ii) If $\mathcal{H}$ is finite dimensional, then $\Phi_{C,D}$ is a trace preserving map.*
*(iii) If $T$ commute with $C$ and $D$, then $\Phi(T) = T$.*
*(v) For any positive operators $A$ and $B$ such that $D = A\sharp B$, the block matrix*

$$\begin{pmatrix} \Phi_{C,D}(A) & C \\ C & \Phi_{C,D}(B) \end{pmatrix}$$

*are positive.*

**Theorem 2.3.** *Let $A, B$, and $C$ be positive operators such that $C \leq A \sharp B$. Then there exists a unital completely positive map $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ such that the block matrix*

$$\begin{pmatrix} \Phi(A) & C \\ C & \Phi(B) \end{pmatrix},$$

*is positive. Moreover, if $\mathcal{H}$ is finite dimensional then $\Phi$ is trace preserving.*

In Theorem 2.2, if we assumed that $D = 1$, then we have the following corollary.

**Corollary 2.4.** *Let $0 \leq C \leq 1$. Then there exists a unital completely positive map $\Phi_C : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ such that for each positive invertible operator $A$, the block matrix*

$$\begin{pmatrix} \Phi_C(A) & C \\ C & \Phi_C(A^{-1}) \end{pmatrix},$$

*is positive. Moreover, if $\mathcal{H}$ is finite dimensional then $\Phi$ is trace preserving.*

## References

1. R. Bhatia, *Positive Definite Matrices*, Princeton University Press, Princeton (2007).
2. N. P. Brown, N. Ozawa, *$C^*$-algebras and finite-dimensional approximation properties,* Graduate studies in Mathematics, (2008).
3. M.D. Choi, *Some Assorted Inequalities for Positive linear maps on $C^*$-algebras*, J. Oper. Th., **4** (1980), 271-285.
4. F. Hansen and G. Pedersen, *Jensen's inequality for operators and Lowner's theorem*, Math. Ann. **258** (1981/82), no. 3, 229-241.