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Weak coarse shape equivalences and infinite dimensional Whitehead theorem in coarse shape theory

Fateme Ghanei [a], Hanieh Mirebrahimi [a,*], Tayyebe Nasri [b]

[a] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran
[b] Department of Pure Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran

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ABSTRACT

In this paper, we study the weak coarse shape equivalences. First, we define paradominations and then we give a characterization of them, for uniformly movable pointed continuum spaces. Also, we show that a weak coarse shape equivalence to a pointed movable space is a paradomination. Finally, we prove that a weak coarse shape equivalence $F^* : (X, x) \to (Y, y)$ between pointed continuum spaces is a coarse shape equivalence, if (X, x) and (Y, y) are simultaneously movable according to F^*.

1. Introduction and motivation

Bilan and Uglešić in [2], generalized the Whitehead theorem for the coarse shape theory. They proved that, for $m \in \mathbb{N}$, if a pointed coarse shape morphism $F^* : (X, x) \to (Y, y)$ between spaces with $sd\ X \leq m-1$ and $sd\ Y \leq m$, is a coarse shape m-equivalence, then F^* is a pointed coarse shape isomorphism. We recall from [2], that F^* is a coarse shape m-equivalence, if the induced morphism

$$F_k^* \equiv pro^* - \pi_k(F^*) : pro^* - \pi_k(X, x) \to pro^* - \pi_k(Y, y)$$

is an isomorphism of pro*-Group for $k = 1, 2, \ldots, m-1$, an isomorphism of pro*-Set for $k = 0$ and an epimorphism of pro*-Group for $k = m$. Also, they defined a weak coarse shape equivalence as a coarse

* Corresponding author.
E-mail addresses: fateme.ghanei@mail.um.ac.ir (F. Ghanei), h_mirebrahimi@um.ac.ir (H. Mirebrahimi), tnasri72@yahoo.com (T. Nasri).

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shape morphism $F^* : (X, x) \to (Y, y)$ which is a coarse shape m-equivalence for all $m \in \mathbb{N}$; and mentioned that by considering the Adams's example (see [7]), one can conclude that the infinite dimensional Whitehead theorem does not hold in the coarse shape category, in general. That means a weak coarse shape equivalence need not be a coarse shape equivalence, in general.

In this paper, we study weak coarse shape equivalences for pointed continua (metric compact connected spaces) and similar to the methods of [9], we prove that the infinite dimensional Whitehead theorem holds for coarse shape theory in some conditions.

In [9], Morón and Portal, established an infinite dimensional Whitehead theorem in shape category. Using the topology on the set of shape morphisms $Sh(X, Y)$ defined in [3], they obtained a characterization of weak shape dominations. Also, they introduced a pointed movable triple (X, F, Y), for a shape morphism $F : X \to Y$ and pointed spaces X and Y. In particular, they proved that for pointed movable triple (X, F, Y), if X and Y are compact connected and F is a weak shape equivalence, then F is a shape equivalence.

Mashayekhy and the authors [6], defined a topology on the set of coarse shape morphisms $Sh^*(X, Y)$, for every topological spaces X and Y. Here, we define a paradomination, similar to [4], and then we use this topology to give a characterization of paradominations.

Morita in [8], stated and proved equivalent conditions for isomorphisms in the category pro-$\mathcal{T}$. Bilan [1], generalized this theorem for the category pro*-$\mathcal{T}$ and now, using methods similar to [5], we obtain another characterization of isomorphisms in the category pro*-HPol$_0$. Also, we prove that if $F^* : (X, x) \to (Y, y)$ is a weak coarse shape equivalence, in which (Y, y) is movable, then F^* is a paradomination and by using this fact, we show that F^* is an epimorphism in the category Sh^*_0. Finally, we define the simultaneously movability of (X, x) and (Y, y) according to a coarse shape morphism $F^* : (X, x) \to (Y, y)$ and then we prove that a weak coarse shape equivalence $F^* : (X, x) \to (Y, y)$ between pointed continuum spaces (X, x) and (Y, y) is a coarse shape equivalence provided (X, x) and (Y, y) are simultaneously movable according to F^*.

2. Preliminaries

Recall from [1] some of the main notions about the coarse shape category and pro*-category. Let $\mathcal{T}$ be a category and let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in the category $\mathcal{T}$. An *S^*-morphism* of inverse systems, $(f, f^n_\mu) : \mathbf{X} \to \mathbf{Y}$, consists of an index function $f : M \to \Lambda$ and of a set of $\mathcal{T}$-morphisms $f^n_\mu : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, such that for every related pair $\mu \leq \mu'$ in M, there exist a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$, and an $n \in \mathbb{N}$ so that for every $n' \geq n$,

$$q_{\mu\mu'} f^{n'}_{\mu'} p_{f(\mu')\lambda} = f^{n'}_\mu p_{f(\mu)\lambda}.$$

If $M = \Lambda$ and $f = 1_\Lambda$, then $(1_\lambda, f^n_\lambda)$ is said to be a *level S^*-morphism*.

Let $(f, f^n_\mu) : \mathbf{X} \to \mathbf{Y}$ and $(g, g^n_\nu) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be S*-morphisms of inverse systems. The *composition* of S*-morphisms (f, f^n_μ) and (g, g^n_ν) is an S*-morphism $(h, h^n_\nu) = (g, g^n_\nu)(f, f^n_\mu) : \mathbf{X} \to \mathbf{Z}$, where $h = fg$ and $h^n_\nu = g^n_\nu f^n_{g(\nu)}$, $n \in \mathbb{N}$, $\nu \in N$. For an inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, the S*-morphism $(1_\Lambda, 1^n_{X_\lambda}) : \mathbf{X} \to \mathbf{X}$, where 1_Λ is the identity function and $1^n_{X_\lambda} = 1_{X_\lambda}$ in $\mathcal{T}$, for all $n \in \mathbb{N}$ and every $\lambda \in \Lambda$, called the *identity* S*-morphism on $\mathbf{X}$.

An S*-morphism $(f, f^n_\mu) : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to an S*-morphism $(f', f'^n_\mu) : \mathbf{X} \to \mathbf{Y}$, denoted by $(f, f^n_\mu) \sim (f', f'^n_\mu)$, if for every $\mu \in M$ there exist a $\lambda \in \Lambda$ and $n \in \mathbb{N}$ such that $\lambda \geq f(\mu), f'(\mu)$ and for every $n' \geq n$,

$$f^{n'}_\mu p_{f(\mu)\lambda} = f'^{n'}_\mu p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation among S*-morphisms of inverse systems in $\mathcal{T}$. The equivalence class $[(f, f^n_\mu)]$ of an S*-morphism $(f, f^n_\mu) : \mathbf{X} \to \mathbf{Y}$ is denoted by $\mathbf{f}^*$. Let pro*-$\mathcal{T}$ be the quotient category corresponding to the equivalence relation $\sim$. In this category, objects are all inverse systems $\mathbf{X}$ in $\mathcal{T}$ and morphisms are all equivalence classes $\mathbf{f}^* = [(f, f_\mu^n)]$ of S*-morphisms (f, f_μ^n). The composition in pro*-$\mathcal{T}$ is well defined by putting

$$\mathbf{g}^*\mathbf{f}^* = \mathbf{h}^* = [(h, h_\nu^n)],$$

where $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) = (fg, g_\nu^n f_{g(\nu)}^n)$. For every inverse system $\mathbf{X}$ in $\mathcal{T}$, the identity morphism in pro*-$\mathcal{T}$ is $\mathbf{1}_\mathbf{X}^* = [(1_\Lambda, 1_{X_\Lambda}^n)]$.

A functor $\underline{\mathcal{J}} = \underline{\mathcal{J}}_\mathcal{T} : pro - \mathcal{T} \to pro^* - \mathcal{T}$ is defined. If $\mathbf{X}$ is an inverse system in $\mathcal{T}$, then $\underline{\mathcal{J}}(\mathbf{X}) = \mathbf{X}$ and if $\mathbf{f} \in pro - \mathcal{T}(\mathbf{X}, \mathbf{Y})$ is represented by (f, f_μ), then $\underline{\mathcal{J}}(\mathbf{f}) = \mathbf{f}^* = [(f, f_\mu^n)] \in pro^* - \mathcal{T}(\mathbf{X}, \mathbf{Y})$ is represented by the S*-morphism (f, f_μ^n), where $f_\mu^n = f_\mu$ for all $\mu \in M$ and $n \in \mathbb{N}$. Since the functor $\underline{\mathcal{J}}$ is faithful, we may consider the category pro-$\mathcal{T}$ as a subcategory of pro*-$\mathcal{T}$.

Let $\mathcal{P}$ be a subcategory of $\mathcal{T}$. For an object X in $\mathcal{T}$, a *$\mathcal{P}$-expansion* of X is a morphism $\mathbf{p} : X \to \mathbf{X}$ in pro-$\mathcal{T}$, where $\mathbf{X}$ belongs to pro-$\mathcal{P}$ with the following two properties:

(E1) For every object P of $\mathcal{P}$ and every map $h : X \to P$ in $\mathcal{T}$, there exist a $\lambda \in \Lambda$ and a map $f : X_\lambda \to P$ in $\mathcal{P}$ such that $fp_\lambda = h$;
(E2) If $f_0, f_1 : X_\lambda \to P$ in $\mathcal{P}$ satisfy $f_0 p_\lambda = f_1 p_\lambda$, then there exists a $\lambda' \geq \lambda$ such that $f_0 p_{\lambda\lambda'} = f_1 p_{\lambda\lambda'}$.

The subcategory $\mathcal{P}$ is said to be *pro-reflective* (*dense*) subcategory of $\mathcal{T}$ provided that every object X in $\mathcal{T}$ admits a $\mathcal{P}$-expansion $\mathbf{p} : X \to \mathbf{X}$.

Every two $\mathcal{P}$-expansions of an object are isomorphic as the objects of pro-$\mathcal{P}$. Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{p}' : X \to \mathbf{X}'$ be two $\mathcal{P}$-expansions of an object X in $\mathcal{T}$, and let $\mathbf{q} : Y \to \mathbf{Y}$ and $\mathbf{q}' : Y \to \mathbf{Y}'$ be two $\mathcal{P}$-expansions of an object Y in $\mathcal{T}$. Then there exist two natural (unique) isomorphisms $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ and $\mathbf{j} : \mathbf{Y} \to \mathbf{Y}'$ in pro-$\mathcal{P}$ with respect to $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{q}$, $\mathbf{q}'$, respectively. Consequently $\underline{\mathcal{J}}(\mathbf{i}) : \mathbf{X} \to \mathbf{X}'$ and $\underline{\mathcal{J}}(\mathbf{j}) : \mathbf{Y} \to \mathbf{Y}'$ are isomorphisms in pro*-$\mathcal{P}$. A morphism $\mathbf{f}^* : \mathbf{X} \to \mathbf{Y}$ is said to be *pro*-$\mathcal{P}$ equivalent* to a morphism $\mathbf{f}'^* : \mathbf{X}' \to \mathbf{Y}'$, denoted by $\mathbf{f}^* \sim \mathbf{f}'^*$, if the following diagram commutes in pro*-$\mathcal{P}$:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\underline{\mathcal{J}}(\mathbf{i})} & \mathbf{X}' \\ \Big\downarrow \mathbf{f}^* & & \mathbf{f}'^* \Big\downarrow \\ \mathbf{Y} & \xrightarrow{\underline{\mathcal{J}}(\mathbf{j})} & \mathbf{Y}'. \end{array}$$

The relation $\sim$ is an equivalence relation on each set $pro^* - \mathcal{P}(\mathbf{X}, \mathbf{Y})$, such that if $\mathbf{f}^* \sim \mathbf{f}'^*$ and $\mathbf{g}^* \sim \mathbf{g}'^*$, then $\mathbf{g}^*\mathbf{f}^* \sim \mathbf{g}'^*\mathbf{f}'^*$ whenever it is defined. The equivalence class of morphism $\mathbf{f}^*$ is denoted by $< \mathbf{f}^* >$.

Let $\mathcal{P}$ be a pro-reflective subcategory of $\mathcal{T}$. Now, the *(abstract) coarse shape category* $\mathrm{Sh}^*_{(\mathcal{T},\mathcal{P})}$ for the pair $(\mathcal{T}, \mathcal{P})$ is defined as follows: The objects of $\mathrm{Sh}^*_{(\mathcal{T},\mathcal{P})}$ are all objects of $\mathcal{T}$. A morphism $F^* : X \to Y$ which is called a coarse shape morphism, is the pro*-$\mathcal{P}$ equivalence class $< \mathbf{f}^* >$ of a mapping $\mathbf{f}^* : \mathbf{X} \to \mathbf{Y}$ in pro*-$\mathcal{P}$, with respect to any pair of $\mathcal{P}$-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$. The *composition* of $F^* =< \mathbf{f}^* >: X \to Y$ and $G^* =< \mathbf{g}^* >: Y \to Z$ is defined by $G^*F^* =< \mathbf{g}^*\mathbf{f}^* >: X \to Z$. The *identity coarse shape morphism* on an object X, $1_X^* : X \to X$, is the pro*-$\mathcal{P}$ equivalence class $< \mathbf{1_X}^* >$ of the identity morphism $\mathbf{1_X}^*$ in pro*-$\mathcal{P}$.

Since the homotopy category of polyhedra HPol is pro-reflective in the homotopy category HTop [7], the coarse shape category $\mathrm{Sh}^*_{(HTop,HPol)} = \mathrm{Sh}^*$ is well defined. Also, from [7], the pointed homotopy category of polyhedra HPol_0 is pro-reflective in the pointed homotopy category HTop_0. Hence the pointed coarse shape category Sh_0^* can be defined by $\mathrm{Sh}^*_{(\mathcal{T},\mathcal{P})}$, where $\mathcal{T} = \mathrm{HTop}_0$ and $\mathcal{P} = \mathrm{HPol}_0$.

The faithful functor $\mathcal{J} = \mathcal{J}_{(\mathcal{T},\mathcal{P})} : Sh_{(\mathcal{T},\mathcal{P})} \to Sh^*_{(\mathcal{T},\mathcal{P})}$ is defined as follows: If X is an object in $\mathcal{T}$, then $\mathcal{J}(X) = X$ and if $F : X \to Y$ is a shape morphism given by $\langle \mathbf{f} \rangle$ in which $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism in pro-$\mathcal{P}$, for $\mathcal{P}$-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ of X and Y, respectively, then $\mathcal{J}(F) = F^*$ that is a coarse shape morphism given by $\langle \mathbf{f}^* \rangle$, where $\mathbf{f}^* = \underline{\mathcal{J}}(\mathbf{f})$.

Remark 2.1. Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ be $\mathcal{P}$-expansions of X and Y, respectively. For every morphism $f : X \to Y$ in $\mathcal{T}$, there is a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{P}$ such that the following diagram commutes in pro-$\mathcal{P}$:

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow[\mathbf{p}]{} & X \\ \downarrow{\mathbf{f}} & & f\downarrow \\ \mathbf{Y} & \xleftarrow[\mathbf{q}]{} & Y. \end{array}$$

If we take other $\mathcal{P}$-expansions $\mathbf{p}' : X \to \mathbf{X}'$ and $\mathbf{q}' : Y \to \mathbf{Y}'$, we obtain another morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in pro-$\mathcal{P}$ such that $\mathbf{f}'\mathbf{p}' = \mathbf{q}'f$, and so we have $\mathbf{f} \sim \mathbf{f}'$ and hence $\underline{\mathcal{J}}(\mathbf{f}) \sim \underline{\mathcal{J}}(\mathbf{f}')$ in pro*-$\mathcal{P}$. Therefore, every morphism $f \in \mathcal{T}(X, Y)$ yields an pro*-$\mathcal{P}$ equivalence class $< \underline{\mathcal{J}}(\mathbf{f}) >$, i.e., a coarse shape morphism $F^* : X \to Y$, denoted by $\mathcal{S}^*(f)$. If we put $\mathcal{S}^*(X) = X$ for every object X of $\mathcal{T}$, then we obtain a functor $\mathcal{S}^* : \mathcal{T} \to Sh^*_{(\mathcal{T},\mathcal{P})}$, which is called the *coarse shape functor*.

Let X and Y be objects in $\mathcal{T}$. Corresponding to any shape morphism $F : X \to Y$, one can consider a coarse shape morphism $F^* : X \to Y$ as follows: Let $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : Y \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be $\mathcal{P}$-expansions of X and Y, respectively and F is given by $\langle \mathbf{f} \rangle$, where $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is represented by (f, f_μ). Thus, the morphism $\mathbf{f}^* : \mathbf{X} \to \mathbf{Y}$ in pro*-$\mathcal{P}$ which is represented by (f, f^n_μ) and $f^n_\mu = f_\mu$, for all $\mu \in M$ and $n \in \mathbb{N}$, gives a coarse shape morphism $F^* = \langle \mathbf{f}^* \rangle : X \to Y$.

Recall from [7], an inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{T}$ is said to be movable if every $\lambda \in \Lambda$ admits an $m(\lambda) \geq \lambda$ (called a movability index of λ) such that for any $\lambda'' \geq \lambda$ there is a morphism $r^\lambda : X_{m(\lambda)} \to X_{\lambda''}$ of $\mathcal{T}$ which satisfies

$$p_{\lambda\lambda''} \circ r^\lambda = p_{\lambda m(\lambda)}.$$

An inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{T}$ is uniformly movable if every $\lambda \in \Lambda$ admits an $m(\lambda) \geq \lambda$ (called a uniformly movability index of λ) such that there is a morphism $\mathbf{r}(\lambda) : \mathbf{X}_{m(\lambda)} \to \mathbf{X}$ in pro-$\mathcal{T}$ satisfying

$$\mathbf{p}_\lambda \circ \mathbf{r}(\lambda) = p_{\lambda m(\lambda)},$$

where $\mathbf{p}_\lambda : \mathbf{X} \to X_\lambda$ is the morphism of pro-$\mathcal{T}$ given by 1_{X_λ}.

An object $X \in \mathcal{T}$ is called movable (uniformly movable) if it has a movable (uniformly movable) $\mathcal{P}$-expansion.

From [11], for every inverse sequence $(\mathbf{X}, *) = ((X_n, *), p_{nn+1})$ in Top$_0$, one can associate a movable inverse sequence $(\mathbf{X}^*, *) = ((X^*_n, *), p^*_{nn+1})$ in Top$_0$ by the star construction. If X_n's are compact connected polyhedra, then X^*_n's are so, and hence $(X^*, *) = \lim\limits_{\leftarrow}(\mathbf{X}^*, *)$ is a movable continuum (see [7]).

Mashayekhy and the authors [6], defined a topology on the set of coarse shape morphisms as follows: Let X and Y be topological spaces, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be an inverse system in pro-HPol and $\mathbf{q} : Y \to \mathbf{Y}$ be an HPol-expansion of Y. For every $\mu \in M$ and $F^* \in Sh^*(X, Y)$ put $V^{F^*}_\mu = \{G^* \in Sh^*(X, Y) |\ \mathcal{S}^*(q_\mu) \circ F^* = \mathcal{S}^*(q_\mu) \circ G^*\}$. They proved that the family $\{V^{F^*}_\mu |\ F^* \in Sh^*(X, Y)$ and $\mu \in M\}$ is a basis for a topology $T_\mathbf{q}$ on $Sh^*(X, Y)$. Moreover, this topology depends only on X and Y.

Also, for topological spaces X, Y and Z and a coarse shape morphism $F^* : X \to Y$, consider $\hat{F^*} : Sh^*(Y, Z) \longrightarrow Sh^*(X, Z)$ and $\tilde{F^*} : Sh^*(Z, X) \longrightarrow Sh^*(Z, Y)$ with $\hat{F^*}(H^*) = H^* \circ F^*$ and $\tilde{F^*}(G^*) = F^* \circ G^*$. They proved that $\hat{F^*}$ and $\tilde{F^*}$ are continuous.

Now, let (X, x) and (Y, y) be pointed topological spaces, $(\mathbf{Y}, \mathbf{y}) = ((Y_\mu, y_\mu), q_{\mu\mu'}, M)$ be an inverse system in pro-HPol$_0$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y})$ be an HPol$_0$-expansion of (Y, y). For every $\mu \in M$ and $F^* \in Sh_0^*((X, x), (Y, y))$, put

$$V_\mu^{F^*} = \{G^* \in Sh_0^*((X, x), (Y, y)) | \ \mathcal{S}^*(q_\mu) \circ F^* = \mathcal{S}^*(q_\mu) \circ G^*\},$$

where $\mathcal{S}^* : \mathrm{HTop}_0 \to Sh_0^*$ is the coarse shape functor defined in Remark 2.1. Similar to [6], one can see that the family $\{V_\mu^{F^*} | \ F^* \in Sh_0^*((X, x), (Y, y))$ and $\mu \in M\}$ is a basis for a topology $T_{\mathbf{q}}$ on $Sh_0^*((X, x), (Y, y))$. Moreover, if (X, x), (Y, y) and (Z, z) are pointed topological spaces and $F^* : (X, x) \to (Y, y)$ is a coarse shape morphism, then the maps $\hat{F^*} : Sh_0^*((Y, y), (Z, z)) \longrightarrow Sh_0^*((X, x), (Z, z))$ and $\tilde{F^*} : Sh_0^*((Z, z), (X, x)) \longrightarrow Sh_0^*((Z, z), (Y, y))$ with $\hat{F^*}(H^*) = H^* \circ F^*$ and $\tilde{F^*}(G^*) = F^* \circ G^*$ are continuous.

3. Main results

First, we recall the notion *weak coarse shape equivalence* from [2]:

Definition 3.1. [2] Let $m \in \mathbb{N}$. A morphism $\mathbf{f}^* : (\mathbf{X}, \mathbf{x}) \to (\mathbf{Y}, \mathbf{y})$ is said to be an m-equivalence of pro*-HTop$_0$ if the induced morphism

$$\pi_k^*(\mathbf{f}^*) : \pi_k(\mathbf{X}, \mathbf{x}) \to \pi_k(\mathbf{Y}, \mathbf{y})$$

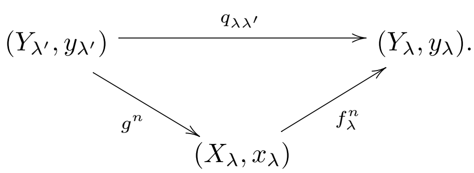
is an isomorphism of pro*-Set for $k = 0$, an isomorphism of pro*-Group for each $k = 1, 2, \ldots, m-1$ and an epimorphism of pro*-Group for $k = m$. A pointed coarse shape morphism $F^* : (X, x) \to (Y, y)$ is said to be a coarse (shape) m-equivalence if there exists a representative $\mathbf{f}^* : (\mathbf{X}, \mathbf{x}) \to (\mathbf{Y}, \mathbf{y})$ which is an m-equivalence in pro*-HPol$_0$.

Definition 3.2. [2] A weak coarse shape equivalence is a coarse shape morphism $F^* : (X, x) \to (Y, y)$ which is coarse (shape) m-equivalence, for all $m \in \mathbb{N}$, i.e., it induces isomorphism between all the homotopy pro*-groups.

In the sense of Dydak [4], for pointed topological spaces (X, x) and (Y, y) and shape morphism $F : (X, x) \to (Y, y)$ with HPol$_0$-expansions $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ of (X, x) and (Y, y), respectively and level representative morphism $(1_\lambda, f_\lambda)$ of F, shape morphism F is a weak shape domination if and only if for any $\lambda \in \Lambda$ there exist $\lambda' \geq \lambda$ and a pointed map $g : (Y_{\lambda'}, y_{\lambda'}) \to (X_\lambda, x_\lambda)$ such that $f_\lambda \circ g \simeq_0 q_{\lambda\lambda'}$ (by $f \simeq_0 g$, we mean f is homotopic to g relative to the base point).

In the following, by a similar way, we define the notion of paradomination.

Definition 3.3. Let $F^* : (X, x) \to (Y, y)$ be a coarse shape morphism between pointed topological spaces (X, x) and (Y, y), $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ be HPol$_0$-expansions of (X, x) and (Y, y), respectively and $(1_\lambda, f_\lambda^n)$ be a level morphism representative of F^*. We say F^* is a paradomination, if for every $\lambda \in \Lambda$ there exist $\lambda' \geq \lambda$ and $n' \in \mathbb{N}$ such that for any $n \geq n'$ there exists a pointed map $g^n : (Y_{\lambda'}, y_{\lambda'}) \to (X_\lambda, x_\lambda)$ such that the following diagram commutes in HPol$_0$



Proposition 3.4. *Let (X,x) and (Y,y) be pointed continua. If $F^*:(X,x)\to(Y,y)$ is a paradomination and (X,x) is uniformly movable, then $\tilde{F}^*(Sh_0^*((Z,z),(X,x)))$ is a dense subspace of $Sh_0^*((Z,z),(Y,y))$, for any pointed continuum (Z,z).*

Proof. Let $\beta^*\in Sh_0^*((Z,z),(Y,y))$. Consider HPol$_0$-expansions $\mathbf{p}:(X,x)\to(\mathbf{X},\mathbf{x})=((X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda)$ and $\mathbf{q}:(Y,y)\to(\mathbf{Y},\mathbf{y})=((Y_\lambda,y_\lambda),q_{\lambda\lambda'},\Lambda)$ of (X,x) and (Y,y), respectively, and level representative $(1_\lambda,f_\lambda^n)$ of F^*.

Let $\lambda\in\Lambda$. (X,x) is uniformly movable, so there exist a $\lambda'\geq\lambda$ and a morphism $\mathbf{r}(\lambda):(X_{\lambda'},x_{\lambda'})\to(\mathbf{X},\mathbf{x})$ in pro-HPol$_0$ such that $\mathbf{p}_\lambda\circ\mathbf{r}(\lambda)=p_{\lambda\lambda'}$, where $\mathbf{p}_\lambda:(\mathbf{X},\mathbf{x})\to(X_\lambda,x_\lambda)$ is the morphism of pro-HPol$_0$ given by 1_{X_λ}. Note that $\mathbf{r}(\lambda)$ determines the morphisms $\mathbf{r}(\lambda)^\mu:(X_{\lambda'},x_{\lambda'})\to(X_\mu,x_\mu)$, $\mu\in\Lambda$ such that

$$p_{\mu\mu'}\circ\mathbf{r}(\lambda)^{\mu'}\simeq_0\mathbf{r}(\lambda)^\mu \text{ (if } \mu'\geq\mu), \quad \text{and} \quad \mathbf{r}(\lambda)^\lambda\simeq_0 p_{\lambda\lambda'}. \tag{1}$$

Then $r=\langle[(\mathbf{r}(\lambda)^\mu)]\rangle$ is a shape morphism and induces a coarse shape morphism $r^*:(X_{\lambda'},x_{\lambda'})\to(X,x)$ given by $\langle[(\mathbf{r}(\lambda)^{\mu n})]\rangle$, where $\mathbf{r}(\lambda)^{\mu n}=\mathbf{r}(\lambda)^\mu$, for all $\mu\in\Lambda$ and every $n\in\mathbb{N}$.

F^* is a coarse shape morphism, so for $\lambda'\geq\lambda$, there exists $n_1\in\mathbb{N}$ such that for every $n\geq n_1$

$$f_\lambda^n p_{\lambda\lambda'}\simeq_0 q_{\lambda\lambda'}f_{\lambda'}^n \tag{2}$$

and since F^* is a paradomination, there are $\lambda''\geq\lambda'$ and $n_2\in\mathbb{N}$ such that for all $n\geq n_2$ there exists a pointed map $g^n:(Y_{\lambda''},y_{\lambda''})\to(X_{\lambda'},x_{\lambda'})$ such that the following diagram commutes in HPol$_0$:

$$\begin{array}{ccc} (Y_{\lambda''},y_{\lambda''}) & \xrightarrow{\quad q_{\lambda'\lambda''}\quad} & (Y_{\lambda'},y_{\lambda'}) \\ & {\scriptstyle g^n}\searrow \qquad \nearrow{\scriptstyle f_{\lambda'}^n} & \\ & (X_{\lambda'},x_{\lambda'}) & \end{array} \;. \tag{3}$$

For every $n<n_2$, consider g^n is the constant map at the point $x_{\lambda'}$ of $X_{\lambda'}$ and hence we have a coarse shape morphism $g^*:(Y_{\lambda''},y_{\lambda''})\to(X_{\lambda'},x_{\lambda'})$ is given by $\langle[(g^n)]\rangle$. Define $\alpha^*=r^*\circ g^*\circ\mathcal{S}^*(q_{\lambda''})\circ\beta^*$ which is a coarse shape morphism from (Z,z) to (X,x). We show that $\tilde{F}^*(\alpha^*)\in V_\lambda^{\beta^*}$.

Suppose $\mathbf{s}:(Z,z)\to(\mathbf{Z},\mathbf{z})=((Z_\nu,z_\nu),s_{\nu\nu'},N)$ is an HPol$_0$-expansion of (Z,z) and $\beta^*=\langle[(\beta_\lambda^n,\eta)]\rangle$. Hence for $\lambda''\geq\lambda$ there exist $\nu\geq\eta(\lambda),\eta(\lambda'')$ and $n_3\in\mathbb{N}$ such that for all $n\geq n_3$

$$q_{\lambda\lambda''}\circ\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0\beta_\lambda^n s_{\eta(\lambda)\nu}. \tag{4}$$

We know $\mathcal{S}^*(q_\lambda)\circ F^*\circ\alpha^*$ and $\mathcal{S}^*(q_\lambda)\circ\beta^*$ are coarse shape morphisms from (Z,z) to (Y_λ,y_λ) are given by $\langle[(f_\lambda^n\mathbf{r}(\lambda)^{\lambda^n}g^n\beta_{\lambda''}^n,\eta_0)]\rangle$ and $\langle[(\beta_\lambda^n,\eta_1)]\rangle$, respectively, in which $\eta_0,\eta_1:\{\lambda\}\to N$ with $\eta_0(\lambda)=\eta(\lambda'')$ and $\eta_1(\lambda)=\eta(\lambda)$. Put $n_0=\max\{n_1,n_2,n_3\}$. For every $n\geq n_0$, by (1), $f_\lambda^n\mathbf{r}(\lambda)^{\lambda^n}g^n\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0 f_\lambda^n p_{\lambda\lambda'}g^n\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}$ and by (2), $f_\lambda^n p_{\lambda\lambda'}g^n\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0 q_{\lambda\lambda'}f_{\lambda'}^n g^n\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}$. Also, by (3) and (4), one obtains $q_{\lambda\lambda'}f_{\lambda'}^n g^n\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0 q_{\lambda\lambda'}q_{\lambda'\lambda''}\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0 q_{\lambda\lambda''}\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0\beta_\lambda^n s_{\eta(\lambda)\nu}$. Therefore, one can conclude that $f_\lambda^n\mathbf{r}(\lambda)^{\lambda^n}g^n\beta_{\lambda''}^n s_{\eta(\lambda'')\nu}\simeq_0\beta_\lambda^n s_{\eta(\lambda)\nu}$, for every $n\geq n_0$ and hence $\mathcal{S}^*(q_\lambda)\circ F^*\circ\alpha^*=\mathcal{S}^*(q_\lambda)\circ\beta^*$. □

As it is mentioned in [9], out of pointed compact connected polyhedra, there is a countable set $\{(P_n,p_n):n\in\mathbb{N}\}$ containing one of each pointed homotopy type that forms the inverse sequence $((P_n,p_n),q_{nn+1})$, where $q_n:(P_{n+1},p_{n+1})\to(P_n,p_n)$ is the constant pointed map. Applying the star-construction of Overton–Segal [11] to the inverse sequence $((P_n,p_n),q_{nn+1})$, one obtains the pointed movable connected space (W,w) which shape dominates every pointed finite polyhedron (see [9]).

Proposition 3.5. *Let (X,x) and (Y,y) be pointed continua and $F^* : (X,x) \to (Y,y)$ be a coarse shape morphism. If $\tilde{F}^*(Sh_0^*((W,w),(X,x)))$ is a dense subspace of $Sh_0^*((W,w),(Y,y))$ and (Y,y) is uniformly movable, then F^* is a paradomination.*

Proof. Let $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x}) = ((X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda)$ and $\mathbf{q} : (Y,y) \to (\mathbf{Y},\mathbf{y}) = ((Y_\lambda,y_\lambda),q_{\lambda\lambda'},\Lambda)$ be HPol$_0$-expansions of (X,x) and (Y,y), respectively and $(1_\lambda, f_\lambda^n)$ be a level representative of F^*. Given $\lambda \in \Lambda$. Since (Y,y) is uniformly movable, there exist a $\lambda' \geq \lambda$ and a morphism $\mathbf{r}(\lambda) : (Y_{\lambda'},y_{\lambda'}) \to (\mathbf{Y},\mathbf{y})$ in pro-HPol$_0$ such that $\mathbf{q}_\lambda \circ \mathbf{r}(\lambda) = q_{\lambda\lambda'}$, where $\mathbf{q}_\lambda : (\mathbf{Y},\mathbf{y}) \to (Y_\lambda,y_\lambda)$ is the morphism of pro-HPol$_0$ given by 1_{Y_λ}. Hence there are the morphisms $\mathbf{r}(\lambda)^\mu : (Y_{\lambda'},y_{\lambda'}) \to (Y_\mu,y_\mu)$, $\mu \in \Lambda$ such that

$$q_{\mu\mu'} \circ \mathbf{r}(\lambda)^{\mu'} \simeq_0 \mathbf{r}(\lambda)^\mu \text{ (if } \mu' \geq \mu), \quad \text{and} \quad \mathbf{r}(\lambda)^\lambda \simeq_0 q_{\lambda\lambda'}$$

and so $r = \langle[(r_\mu = \mathbf{r}(\lambda)^\mu)]\rangle$, $\mu \in \Lambda$, is a shape morphism from $(Y_{\lambda'},y_{\lambda'}) \to (Y,y)$ such that

$$r_\lambda = \mathbf{r}(\lambda)^\lambda \simeq_0 q_{\lambda\lambda'}. \tag{5}$$

From [9, Proposition 5], (W,w) shape dominates every pointed finite polyhedron, so there are shape morphisms $i' : (Y_{\lambda'},y_{\lambda'}) \to (W,w)$ and $r' : (W,w) \to (Y_{\lambda'},y_{\lambda'})$ such that $\mathcal{S}(1_{Y_{\lambda'}}) = r' \circ i'$. Consider the coarse shape morphisms r^*, i'^* and r'^* corresponding to the shape morphisms r, i' and r', respectively and put $\beta^* = r^* \circ r'^* \in Sh_0^*((W,w),(Y,y))$. By the hypothesis, there exists a coarse shape morphism $\alpha^* \in Sh_0^*((W,w),(X,x))$ given by $\langle[(\alpha_\lambda^n,\alpha)]\rangle$ such that $\tilde{F}^*(\alpha^*) = F^* \circ \alpha^* \in V_\lambda^{\beta^*}$.

Let $\mathbf{s} : (W,w) \to (\mathbf{W},\mathbf{w}) = ((W_\nu,w_\nu),s_{\nu\nu'},N)$ be an HPol$_0$-expansion of (W,w) and r' and i' given by $\langle[(r'_{\lambda'},\varphi)]\rangle$ and $\langle[(i'_\nu,\psi)]\rangle$. Put $\nu' = \varphi(\lambda')$. We know $r' \circ i' = \mathcal{S}(1_{Y_{\lambda'}})$, so

$$r'_{\lambda'} \circ i'_{\nu'} \simeq_0 1_{Y_{\lambda'}}. \tag{6}$$

Also, we know $\mathcal{S}^*(q_\lambda) \circ F^* \circ \alpha^* = \mathcal{S}^*(q_\lambda) \circ \beta^*$ and β^* is given by $(r_\mu \circ r'_{\lambda'})$, $\mu \in \Lambda$. Hence there exist $\nu \geq \alpha(\lambda), \nu'$ and $n' \in \mathbb{N}$ such that for every $n \geq n'$

$$f_\lambda^n \circ \alpha_\lambda^n \circ s_{\alpha(\lambda)\nu} \simeq_0 r_\lambda \circ r'_{\lambda'} \circ s_{\nu'\nu}. \tag{7}$$

Now for every $n \geq n'$, put $g^n = \alpha_\lambda^n \circ i'_{\alpha(\lambda)} : (Y_{\lambda'},y_{\lambda'}) \to (X_\lambda,x_\lambda)$. From (7), $f_\lambda^n \circ \alpha_\lambda^n \circ s_{\alpha(\lambda)\nu} i'_\nu \simeq_0 r_\lambda \circ r'_{\lambda'} \circ s_{\nu'\nu} i'_\nu$ and by (5) and (6), it follows that $f_\lambda^n \circ g^n = f_\lambda^n \circ \alpha_\lambda^n \circ i'_{\alpha(\lambda)} \simeq_0 r_\lambda \circ r'_{\lambda'} \circ i'_{\nu'} \simeq_0 r_\lambda \simeq_0 q_{\lambda\lambda'}$. □

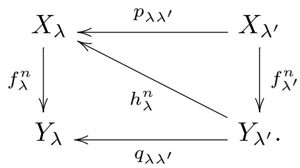
In the following, by Proposition 3.4 and Proposition 3.5, we characterize paradominations of uniformly movable pointed continua.

Corollary 3.6. *Let (X,x) and (Y,y) be uniformly movable continua and $F^* : (X,x) \to (Y,y)$ be a coarse shape morphism. Then the following statements are equivalent:*

a) F^* is a paradomination.
b) $\tilde{F}^*(Sh_0^*((Z,z),(X,x)))$ is a dense subspace of $Sh_0^*((Z,z),(Y,y))$, for every pointed continuum (Z,z).
c) $\tilde{F}^*(Sh_0^*((W,w),(X,x)))$ is a dense subspace of $Sh_0^*((W,w),(Y,y))$.

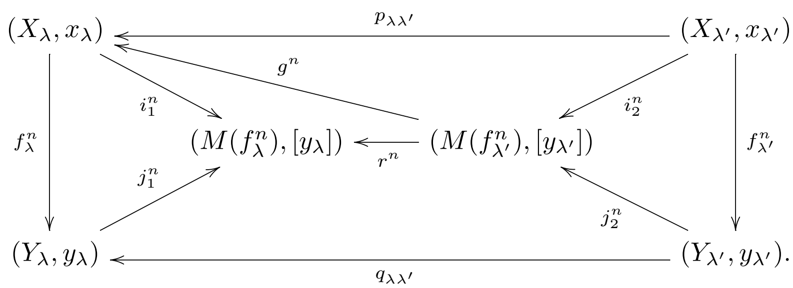
Bilan in [1], proved the following lemma which is a similar result to the well known Morita lemma [8], and characterizes isomorphisms in the category pro*-$\mathcal{T}$.

Lemma 3.7. *[1] Let $\mathbf{X} = (X_\lambda,p_{\lambda\lambda'},\Lambda)$ and $\mathbf{Y} = (Y_\lambda,q_{\lambda\lambda'},\Lambda)$ be inverse systems over the same index set and $\mathbf{f}^* : \mathbf{X} \to \mathbf{Y}$ be a morphism in pro*-$\mathcal{T}$ which admits a level representative $(1_\Lambda, f_\lambda^n)$. Then $\mathbf{f}^*$ is an* *isomorphism if and only if for every $\lambda \in \Lambda$ there exist $\lambda' \geq \lambda$ and $n' \in \mathbb{N}$ such that, for every $n \geq n'$ there exists a morphism $h^n_\lambda : Y_{\lambda'} \to X_\lambda$ in $\mathcal{T}$, such that the following diagram commutes in $\mathcal{T}$:*



Now, using techniques similar to those employed by Geoghegan in [5], we give another characterizations of isomorphisms in pro*-HPol$_0$.

Lemma 3.8. *Assume $((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ are inverse systems in pro*-HPol$_0$. A level map $(f^n_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, y_\lambda))$ in pro*-HPol$_0$ is an isomorphism if and only if for every $\lambda \in \Lambda$, there exist $\lambda' \geq \lambda$ and $n' \in \mathbb{N}$ such that for every $n \geq n'$, there exist morphisms r^n and g^n making the following diagram commute in HPol$_0$*



(8)

The space $M(f)$ is the mapping cylinder of $f : (X, x_0) \to (Y, y_0)$ with base point $[x_0, 1] = [f(x_0)] = [y_0]$ and $i : (X, x_0) \to (M(f), [y_0])$ and $j : (Y, y_0) \to (M(f), [y_0])$ are maps given by $i(x) = [x, 1]$ and $j(y) = [y]$, for all $x \in X$ and $y \in Y$.

Proof. First, suppose for every $\lambda \in \Lambda$, there are $\lambda' \geq \lambda$ and $n' \in \mathbb{N}$ such that for every $n \geq n'$, there exist morphisms r^n and g^n that commute the above diagram. For every $n \geq n'$, put $h^n_\lambda = g^n \circ j^n_2 : (Y_{\lambda'}, y_{\lambda'}) \to (X_\lambda, x_\lambda)$. We have

$$h^n_\lambda \circ f^n_{\lambda'} = g^n \circ j^n_2 \circ f^n_{\lambda'} \simeq_0 g^n \circ i^n_2 \simeq_0 p_{\lambda\lambda'},$$

and

$$f^n_\lambda \circ h^n_\lambda = f^n_\lambda \circ g^n \circ j^n_2 \simeq_0 q_{\lambda\lambda'}.$$

Then by the previous lemma, the level representative (f^n_λ) gives an isomorphism in pro*-HPol$_0$.

Conversely, since (f^n_λ) is an isomorphism in pro*-HPol$_0$, by the previous lemma, for every $\lambda \in \Lambda$, there exist $\lambda' \geq \lambda$ and $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ there exists a morphism $h^n_\lambda : (Y_{\lambda'}, y_{\lambda'}) \to (X_\lambda, x_\lambda)$ that $h^n_\lambda \circ f^n_{\lambda'} \simeq_0 p_{\lambda\lambda'}$ and $f^n_\lambda \circ h^n_\lambda \simeq_0 q_{\lambda\lambda'}$. Since F^* is a coarse shape morphism, there is $n_2 \in \mathbb{N}$ such that $f^n_\lambda \circ p_{\lambda\lambda'} \simeq_0 q_{\lambda\lambda'} \circ f^n_{\lambda'}$, for every $n \geq n_2$.

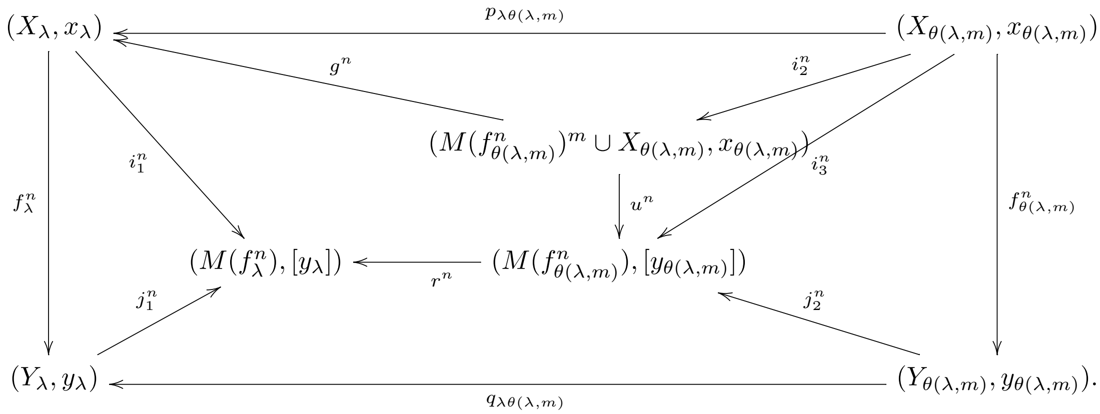
Put $n' = \max\{n_1, n_2\}$. For every $n \geq n'$, consider the map $H^n : X_{\lambda'} \times I \to Y_\lambda$ such that $H^n(-, 0) = f^n_\lambda \circ p_{\lambda\lambda'}$ and $H^n(-, 1) = q_{\lambda\lambda'} \circ f^n_{\lambda'}$. Define $r^n : (M(f^n_{\lambda'}), [y_{\lambda'}]) \to (M(f^n_\lambda), [y_\lambda])$ by $r^n([x, t]) = [p_{\lambda\lambda'}(x), 2t]$, if $0 \leq t \leq \dfrac{1}{2}$ and $r^n([x, t]) = [H^n(x, 2t - 1)]$, if $\dfrac{1}{2} \leq t \leq 1$ and $r^n([y]) = [q_{\lambda\lambda'}(y)]$, for every $x \in X_{\lambda'}$ and $y \in Y_{\lambda'}$, and define $g^n = h^n_\lambda \circ \pi^n : (M(f^n_{\lambda'}), [y_{\lambda'}]) \to (X_\lambda, x_\lambda)$ in which $\pi^n : (M(f^n_{\lambda'}), [y_{\lambda'}]) \to (Y_{\lambda'}, y_{\lambda'})$ is projection. It is obvious that the diagram (8) commutes, for every $n \geq n'$. $\square$

Consider the following diagram which commutes up to homotopy, for all but finitely many n

$$\begin{array}{ccc} (X_\lambda, x_\lambda) & \xleftarrow{p_{\lambda\lambda'}} & (X_{\lambda'}, x_{\lambda'}) \\ \Big\downarrow f^n_\lambda & & \Big\downarrow f^n_{\lambda'} \\ (Y_\lambda, y_\lambda) & \xleftarrow[q_{\lambda\lambda'}]{} & (Y_{\lambda'}, y_{\lambda'}). \end{array} \tag{9}$$

The maps $r^n : (M(f^n_{\lambda'}), [y_{\lambda'}]) \to (M(f^n_\lambda), [y_\lambda])$, which always exist by Lemma 3.8, are said the maps associated with the bonds.

Lemma 3.9. *Let $((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ be inverse systems and $(f^n_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, y_\lambda))$ be a level morphism in pro*-HPol$_0$. Suppose the level map $(f^n_{\lambda *} : \pi_k(X_\lambda, x_\lambda) \to \pi_k(Y_\lambda, y_\lambda))$ is an isomorphism in pro*-Group, for all $k \leq m$. Then for every $\lambda \in \Lambda$, there exist $\theta(\lambda, m) \geq \lambda$ and $N \in \mathbb{N}$ such that for every $n \geq N$, there exist maps r^n and g^n making the following diagram commute in HPol$_0$*



(10)

Proof. The level map $(f^n_{\lambda *} : \pi_k(X_\lambda, x_\lambda) \to \pi_k(Y_\lambda, y_\lambda))$ is an isomorphism in pro*-Group, so by Lemma 3.7, for every $\lambda \in \Lambda$ there exist $\beta \geq \lambda$ and $\gamma \geq \beta$ and $n_0 \in \mathbb{N}$ that for every $n \geq n_0$ there exist homomorphisms $a^n : \pi_k(Y_\beta, y_\beta) \to \pi_k(X_\lambda, x_\lambda)$ and $b^n : \pi_k(Y_\gamma, y_\gamma) \to \pi_k(X_\beta, x_\beta)$, where $f^n_{\lambda *} \circ a^n = q_{\alpha\beta *}$ and $b^n \circ f^n_{\gamma *} = p_{\beta\gamma *}$.

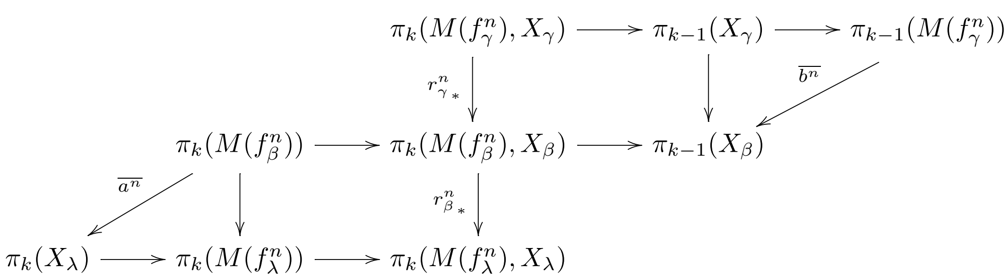
Also, there is $n_1 \in \mathbb{N}$ so that for every $n \geq n_1$, there are maps

$$(M(f^n_\gamma), [y_\gamma]) \xrightarrow{r^n_\gamma} (M(f^n_\beta), [y_\beta]) \xrightarrow{r^n_\beta} (M(f^n_\lambda), [y_\lambda])$$

associated with the bonds.

Consider X_λ as a subspace of $M(f^n_\lambda)$, with the map $l : X_\lambda \to M(f^n_\lambda)$, where $l(x) = [x, 0]$, $x \in X_\lambda$, $\lambda \in \Lambda$, $n \in \mathbb{N}$. We abbreviate the pointed triple $(M(f^n_\lambda), X_\lambda, x_\lambda)$ to $(M(f^n_\lambda), X_\lambda)$. For the map r^n : $(M(f^n_{\lambda'}), [y_{\lambda'}]) \to (M(f^n_\lambda), [y_\lambda])$ associated with the bonds, it is obvious that $r^n(X_{\lambda'}) \subseteq X_\lambda$, and so we have the induced homomorphism $r^n_* : \pi_k(M(f^n_{\lambda'}), X_{\lambda'}) \to \pi_k(M(f^n_\lambda), X_\lambda)$.

Put $n' = \max\{n_0, n_1\}$ and for every $n \geq n'$ consider the following commutative diagram



in which horizontal rows are exact. It is obvious that $r^n_{\beta *} \circ r^n_{\gamma *} : \pi_k(M(f^n_\gamma), X_\gamma) \to \pi_k(M(f^n_\lambda), X_\lambda)$ is zero. Now take $\gamma = \gamma_{\lambda,k}$, then for every $\lambda \in \Lambda$ and $k \leq m$ there is $n' \in \mathbb{N}$ such that for every $n \geq n'$ there exists $r^n_\lambda : (M(f^n_{\gamma_{\lambda,k}}), [y_{\gamma_{\lambda,k}}]) \to (M(f^n_\lambda), [y_\lambda])$ associated with the bonds such that the induced homomorphism $r^n_{\lambda *} : \pi_k(M(f^n_{\gamma_{\lambda,k}}), X_{\gamma_{\lambda,k}}) \to \pi_k(M(f^n_\lambda), X_\lambda)$ is zero.

Consider the sequence $\lambda_0 = \lambda, \lambda_1 = \gamma_{\lambda_0,m}, \ldots, \lambda_i = \gamma_{\lambda_{i-1},m-(i-1)}, \ldots, \lambda_m = \gamma_{\lambda_{m-1},1}$. For every $1 \leq i \leq m$, there exists $n_i \in \mathbb{N}$ such that for every $n \geq n_i$ there exists map $r^n_i : (M(f^n_{\lambda_i}), [y_{\lambda_i}]) \to (M(f^n_{\lambda_{i-1}}), [y_{\lambda_{i-1}}])$ which induces the zero homomorphism $r^n_{i *} : \pi_{m-(i-1)}(M(f^n_{\lambda_i}), X_{\lambda_i}) \to \pi_{m-(i-1)}(M(f^n_{\lambda_{i-1}}), X_{\lambda_{i-1}})$. Put $N = \max\{n_i\}$, then for every $n \geq N$, the map $s^n = r^n_1 r^n_2 \ldots r^n_m : (M(f^n_{\lambda_m}), [y_{\lambda_m}]) \to (M(f^n_\lambda), [y_\lambda])$ induces the zero homomorphism $s^n_* : \pi_m(M(f^n_{\lambda_m}), X_{\lambda_m}) \to \pi_m(M(f^n_\lambda), X_\lambda)$. We can find a cellular map $r^n : (M(f^n_{\lambda_m}), [y_{\lambda_m}]) \to (M(f^n_\lambda), [y_\lambda])$ homotopic to s^n rel X_{λ_m} such that $r^n(M(f^n_{\lambda_m})^m) \subset X_\lambda$. Now take $\theta(\lambda, m) = \lambda_m \geq \lambda$ and $N = \max\{n_i\}$, then for every $n \geq N$, we have maps $r^n : (M(f^n_{\theta(\lambda,m)}), [y_{\theta(\lambda,m)}]) \to (M(f^n_\lambda), [y_\lambda])$ and $g^n = r^n|_{M(f^n_{\theta(\lambda,m)})^m \cup X_{\theta(\lambda,m)}} : (M(f^n_{\theta(\lambda,m)})^m \cup X_{\theta(\lambda,m)}, x_{\theta(\lambda,m)}) \to (X_\lambda, x_\lambda)$ such that the diagram (10) commutes. □

Let $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ be an HPol$_0$-expansion of a pointed topological space (X, x). Consider the inverse system

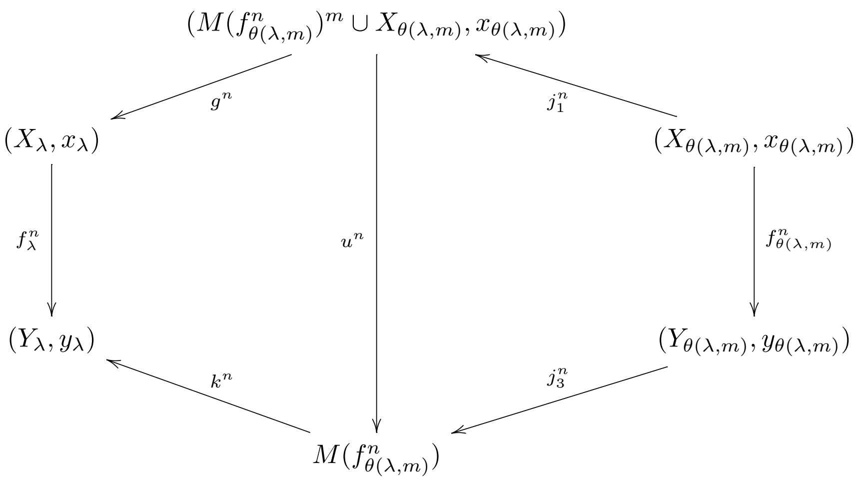
$$\mathbf{Sh}^*_0((Z, z), (X, x)) = (Sh^*_0((Z, z), (X_\lambda, x_\lambda)), p_{\lambda\lambda' *}, \Lambda)$$

in pro-Top$_0$, for every pointed topological space (Z, z). Now, similar to the Theorem 1 of [10], we prove the following useful result.

Theorem 3.10. *Let $F^* : (X, x) \to (Y, y)$ be a weak coarse shape equivalence. Then the induced morphism $\overline{F^*} : \mathbf{Sh}^*_0((P, p), (X, x)) \to \mathbf{Sh}^*_0((P, p), (Y, y))$ is an isomorphism in pro*-Top$_0$, for every compact connected pointed polyhedron (P, p).*

Proof. Let $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ be HPol$_0$-expansions of (X, x) and (Y, y), respectively and $(1_\lambda, f^n_\lambda)$ be a level representative of F^*.

Let (P, p) be a compact connected pointed polyhedron, so $\dim P = m < \infty$. Given $\lambda \in \Lambda$. By hypothesis, $(f^n_{\lambda *} : \pi_k(X_\lambda, x_\lambda) \to \pi_k(Y_\lambda, y_\lambda))$ is an isomorphism in pro*-Group, for all $k \leq m$, then by Lemma 3.9, there exist $\theta(\lambda, m) \geq \lambda$ and $N \in \mathbb{N}$ such that for every $n \geq N$ there are maps r^n and g^n for which diagram (10) commutes. Consider the following commutative diagram



in which $k^n = \pi^n \circ r^n$ and $\pi^n : (M(f^n_\lambda), [y_\lambda]) \to (Y_\lambda, y_\lambda)$ is projection. By the approximation theorem, we conclude that $u^n_* : Sh^*_0((P, p), (M(f^n_{\theta(\lambda,m)})^m \cup X_{\theta(\lambda,m)}, x_{\theta(\lambda,m)})) \to Sh^*_0((P, p), (M(f^n_{\theta(\lambda,m)}), [y_{\theta(\lambda,m)}]))$ is a bijection, so for every $n \geq N$ we have the map $h_\lambda^n = g_*^n \circ {u_*^n}^{-1} \circ {j_3^n}_* = Sh_0^*((P,p),(Y_{\theta(\lambda,m)}, y_{\theta(\lambda,m)})) \to Sh_0^*((P,p),(X_\lambda, x_\lambda))$ such that the following diagram commutes

$$\begin{array}{ccc} Sh_0^*((P,p),(X_\lambda,x_\lambda)) & \xleftarrow{\;{p_{\lambda\theta(\lambda,m)}}_*\;} & Sh_0^*((P,p),(X_{\theta(\lambda,m)},x_{\theta(\lambda,m)})) \\ {\scriptstyle {f_\lambda^n}_*}\downarrow & \nwarrow^{h_\lambda^n} & \downarrow{\scriptstyle {f_{\theta(\lambda,m)}^n}_*} \\ Sh_0^*((P,p),(Y_\lambda,y_\lambda)) & \xleftarrow{\;{q_{\lambda\theta(\lambda,m)}}_*\;} & Sh_0^*((P,p),(Y_{\theta(\lambda,m)},y_{\theta(\lambda,m)})). \end{array}$$

Now by Lemma 3.7, the result holds. □

Theorem 3.11. *Let (X,x) and (Y,y) be pointed continua and $F^* : (X,x) \to (Y,y)$ be a coarse shape morphism. If F^* is a weak coarse shape equivalence and (Y,y) is movable, then F^* is a paradomination.*

Proof. Let $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x}) = ((X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda)$ and $\mathbf{q} : (Y,y) \to (\mathbf{Y},\mathbf{y}) = ((Y_\lambda,y_\lambda),q_{\lambda\lambda'},\Lambda)$ be HPol$_0$-expansions of (X,x) and (Y,y), respectively and $(1_\lambda, f_\lambda^n)$ be a level representative of F^*.

Given $\lambda \in \Lambda$. (Y,y) is movable, so there is movability index $\lambda' \geq \lambda$ such that for every $\lambda'' \geq \lambda$, there exists $r_\lambda : (Y_{\lambda'},y_{\lambda'}) \to (Y_{\lambda''},y_{\lambda''})$ with $q_{\lambda\lambda''} \circ r_\lambda \simeq_0 q_{\lambda\lambda'}$.

By Theorem 3.10, $\overline{F^*} : \mathbf{Sh}_0^*((Y_{\lambda'},y_{\lambda'}),(X,x)) \to \mathbf{Sh}_0^*((Y_{\lambda'},y_{\lambda'}),(Y,y))$ is an isomorphism in pro*-Top$_0$. Hence by Lemma 3.7, there exist $\lambda'' \geq \lambda$ and $n' \in \mathbb{N}$ so that for every $n \geq n'$, there exists $h_\lambda^n : Sh_0^*((Y_{\lambda'},y_{\lambda'}),(Y_{\lambda''},y_{\lambda''})) \to Sh_0^*((Y_{\lambda'},y_{\lambda'}),(X_\lambda,x_\lambda))$ that commutes the following diagram

$$\begin{array}{ccc} Sh_0^*((Y_{\lambda'},y_{\lambda'}),(X_\lambda,x_\lambda)) & \xleftarrow{\;{p_{\lambda\lambda''}}_*\;} & Sh_0^*((Y_{\lambda'},y_{\lambda'}),(X_{\lambda''},x_{\lambda''})) \\ {\scriptstyle {f_\lambda^n}_*}\downarrow & \nwarrow^{h_\lambda^n} & \downarrow{\scriptstyle {f_{\lambda''}^n}_*} \\ Sh_0^*((Y_{\lambda'},y_{\lambda'}),(Y_\lambda,y_\lambda)) & \xleftarrow{\;{q_{\lambda\lambda''}}_*\;} & Sh_0^*((Y_{\lambda'},y_{\lambda'}),(Y_{\lambda''},y_{\lambda''})). \end{array}$$

Let $r^* : (Y_{\lambda'},y_{\lambda'}) \to (Y_{\lambda''},y_{\lambda''})$ be the coarse shape morphism corresponding to the map $r_\lambda : (Y_{\lambda'},y_{\lambda'}) \to (Y_{\lambda''},y_{\lambda''})$. Consider the coarse shape morphism $h_\lambda^n(r^*)$ which is given by $(a^m : (Y_{\lambda'},y_{\lambda'}) \to (X_\lambda,x_\lambda))$, $m \in \mathbb{N}$. We know ${f_\lambda^n}_*(h_\lambda^n(r^*)) = {q_{\lambda\lambda''}}_*(r^*)$, in which ${f_\lambda^n}_*(h_\lambda^n(r^*))$ and ${q_{\lambda\lambda''}}_*(r^*)$ are coarse shape morphisms from $(Y_{\lambda'},y_{\lambda'})$ to (Y_λ,y_λ) given by $(f_\lambda^n \circ a^m)$ and $(q_{\lambda\lambda''} \circ r_\lambda)$, respectively. Then there exists $N_n \in \mathbb{N}$ such that for every $m \geq N_n$, $f_\lambda^n \circ a^m \simeq_0 q_{\lambda\lambda''} \circ r_\lambda$.

Now, if we consider $\lambda' \geq \lambda$ and $n' \in \mathbb{N}$, then there exists the map $g^n = a^{N_n} : (Y_{\lambda'},y_{\lambda'}) \to (X_\lambda,x_\lambda)$ with $f_\lambda^n \circ g^n = f_\lambda^n \circ a^{N_n} \simeq_0 q_{\lambda\lambda''} \circ r_\lambda \simeq_0 q_{\lambda\lambda'}$, for every $n \geq n'$. □

Theorem 3.12. *Let (X,x) and (Y,y) be pointed continua and $F^* : (X,x) \to (Y,y)$ be a coarse shape morphism. If F^* is a weak coarse shape equivalence and (Y,y) is movable, then F^* is an epimorphism in the category Sh_0^*.*

Proof. Let $\mathbf{p} : (X,x) \to (\mathbf{X},\mathbf{x}) = ((X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda)$ and $\mathbf{q} : (Y,y) \to (\mathbf{Y},\mathbf{y}) = ((Y_\lambda,y_\lambda),q_{\lambda\lambda'},\Lambda)$ be HPol$_0$-expansions of (X,x) and (Y,y), respectively and $(1_\lambda, f_\lambda^n)$ be a level representative of F^*.

Consider the coarse shape morphisms $G^*, H^* : (Y,y) \to (Z,z)$ such that $G^* \circ F^* = H^* \circ F^*$, in which (Z,z) is a pointed topological space with HPol$_0$-expansion $\mathbf{r} : (Z,z) \to (\mathbf{Z},\mathbf{z}) = ((Z_\nu,z_\nu),r_{\nu\nu'},N)$ and G^* and H^* are given by $\langle[(g_\nu^n,\varphi)]\rangle$ and $\langle[(h_\nu^n,\psi)]\rangle$, respectively. We show that $G^* = H^*$.

Given $\nu \in N$. Since $G^* \circ F^* = H^* \circ F^*$, there exist $\lambda \geq \varphi(\nu), \psi(\nu)$ and $n_1 \in \mathbb{N}$ such that $g_\nu^n \circ f_{\varphi(\nu)}^n \circ p_{\varphi(\nu)\lambda} \simeq_0 h_\nu^n \circ f_{\psi(\nu)}^n \circ p_{\psi(\nu)\lambda}$, for every $n \geq n_1$.

Also, since F^* is a coarse shape morphism, there is $n_2 \in \mathbb{N}$ so that $f_{\varphi(\nu)}^n \circ p_{\varphi(\nu)\lambda} \simeq_0 q_{\varphi(\nu)\lambda} \circ f_\lambda^n$ and $f_{\psi(\nu)}^n \circ p_{\psi(\nu)\lambda} \simeq_0 q_{\psi(\nu)\lambda} \circ f_\lambda^n$, for every $n \geq n_2$. Hence for every $n \geq n_1, n_2$,

$$g_\nu^n \circ q_{\varphi(\nu)\lambda} \circ f_\lambda^n \simeq_0 h_\nu^n \circ q_{\psi(\nu)\lambda} \circ f_\lambda^n. \tag{11}$$

On the other hand, by Theorem 3.11 F^* is a paradomination, so there exist $\lambda' \geq \lambda$ and $n_3 \in \mathbb{N}$ such that for every $n \geq n_3$ there exists a map $g^n : (Y_{\lambda'}, y_{\lambda'}) \to (X_\lambda, x_\lambda)$ with

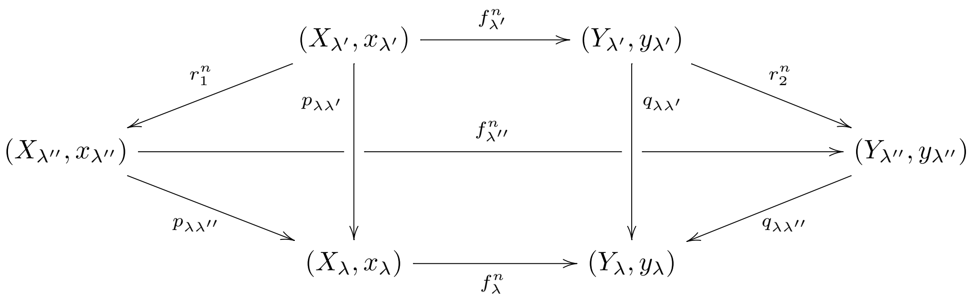
$$f_\lambda^n \circ g^n \simeq_0 q_{\lambda\lambda'}. \tag{12}$$

Now, if we consider $\lambda' \geq \varphi(\nu), \psi(\nu)$ and $n' = \max\{n_1, n_2, n_3\}$, then by (11) and (12),

$$\begin{aligned} g_\nu^n \circ q_{\varphi(\nu)\lambda'} &\simeq_0 g_\nu^n \circ q_{\varphi(\nu)\lambda} \circ q_{\lambda\lambda'} \\ &\simeq_0 g_\nu^n \circ q_{\varphi(\nu)\lambda} \circ f_\lambda^n \circ g^n \\ &\simeq_0 h_\nu^n \circ q_{\psi(\nu)\lambda} \circ f_\lambda^n \circ g^n \\ &\simeq_0 h_\nu^n \circ q_{\psi(\nu)\lambda} \circ q_{\lambda\lambda'} \\ &\simeq_0 h_\nu^n \circ q_{\psi(\nu)\lambda'}, \end{aligned}$$

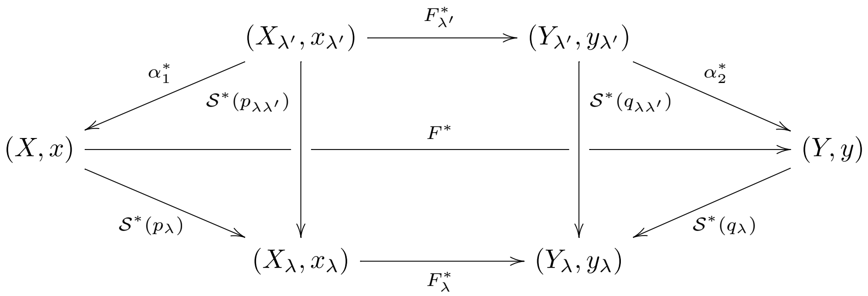
for every $n \geq n'$. It follows that $G^* = H^*$. □

Definition 3.13. Let (X, x) and (Y, y) be pointed topological spaces and $F^* : (X, x) \to (Y, y)$ be a coarse shape morphism. We say (X, x) and (Y, y) are simultaneously movable according to F^* if there are HPol$_0$-expansions $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ of (X, x) and (Y, y), respectively and a level representative $(1_\lambda, f_\lambda^n)$ of F^* such that for every $\lambda \in \Lambda$, there exist $\lambda' \geq \lambda$ and $n' \in \mathbb{N}$ so that for every $\lambda'' \geq \lambda$ and $n \geq n'$ there exist maps $r_1^n : (X_{\lambda'}, x_{\lambda'}) \to (X_{\lambda''}, x_{\lambda''})$ and $r_2^n : (Y_{\lambda'}, y_{\lambda'}) \to (Y_{\lambda''}, y_{\lambda''})$ with the following commutative diagram



(13)

Also, we say (X, x) and (Y, y) are simultaneously uniformly movable according to F^* if for every $\lambda \in \Lambda$, there exist $\lambda' \geq \lambda$ and coarse shape morphisms $\alpha_1^* : (X_{\lambda'}, x_{\lambda'}) \to (X, x)$ and $\alpha_2^* : (Y_{\lambda'}, y_{\lambda'}) \to (Y, y)$ such that the following diagram commutes



in which F_λ^* and $F_{\lambda'}^*$ are coarse shape morphisms given by $\langle[(f_\lambda^n)[\rangle$ and $\langle[(f_{\lambda'}^n)]\rangle$, respectively.

Theorem 3.14. *Let (X, x) and (Y, y) be pointed continua and $F^* : (X, x) \to (Y, y)$ be a weak coarse shape equivalence. If (X, x) and (Y, y) are simultaneously movable according to F^*, then F^* is a coarse shape equivalence.*

Proof. Let $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), q_{\lambda\lambda'}, \Lambda)$ be HPol$_0$-expansions of (X, x) and (Y, y), respectively and $(1_\Lambda, f^n_\lambda)$ be a level representative of F^*.

First, we show that for every $\lambda \in \Lambda$, there exist $\lambda' \geq \lambda$ and $M_\lambda \in \mathbb{N}$ such that the triple $(\lambda, \lambda', M_\lambda)$ satisfies the following condition:

$(**)$ For any $n \geq M_\lambda$ and any compact connected pointed polyhedron (P, p),

(i) Every map $h : (P, p) \to (Y_{\lambda'}, y_{\lambda'})$ admits a map $k^n : (P, p) \to (X_\lambda, x_\lambda)$ so that

$$f^n_\lambda \circ k^n \simeq_0 q_{\lambda\lambda'} \circ h.$$

(ii) For any two maps $k_1, k_2 : (P, p) \to (X_{\lambda'}, x_{\lambda'})$ with $f^n_{\lambda'} \circ k_1 \simeq_0 f^n_{\lambda'} \circ k_2$, we have

$$p_{\lambda\lambda'} \circ k_1 \simeq_0 p_{\lambda\lambda'} \circ k_2.$$

Given $\lambda \in \Lambda$. (X, x) and (Y, y) are simultaneously movable according to F^*, so there exist $\lambda' \geq \lambda$ and $n_0 \in \mathbb{N}$ which satisfy in Definition 3.13.

Let (P, p) be a compact connected pointed polyhedron. By Theorem 3.10, the morphism $\overline{F^*} : \mathbf{Sh}^*_0((P, p), (X, x)) \to \mathbf{Sh}^*_0((P, p), (Y, y))$ is an isomorphism in pro*-Top$_0$, so there exists a morphism $G : \mathbf{Sh}^*_0((P, p), (Y, y)) \to \mathbf{Sh}^*_0((P, p), (X, x))$ given by (g^n_λ, g) which is the inverse of $\overline{F^*}$. Since $G \circ \overline{F^*} = id$, so by the definition there exist $\lambda_1 \geq \lambda, g(\lambda)$ and $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ the following diagram commutes:

$$\begin{array}{ccc}
Sh^*_0((P,p),(X_{g(\lambda)}, x_{g(\lambda)})) & \xrightarrow{\;f^n_{g(\lambda)*}\;} & Sh^*_0((P,p),(Y_{g(\lambda)}, y_{g(\lambda)})) \\
\uparrow {\scriptstyle p_{g(\lambda)\lambda_1 *}} & & \downarrow {\scriptstyle g^n_\lambda} \\
Sh^*_0((P,p),(X_{\lambda_1}, x_{\lambda_1})) & \xrightarrow[\;p_{\lambda\lambda_1 *}\;]{} & Sh^*_0((P,p),(X_\lambda, x_\lambda)),
\end{array}$$

and since $\overline{F^*} \circ G = id$, so there exist $\lambda_2 \geq \lambda, g(\lambda)$ and $n_2 \in \mathbb{N}$ such that for every $n \geq n_2$ the following diagram commutes:

$$\begin{array}{ccc}
Sh^*_0((P,p),(Y_{g(\lambda)}, y_{g(\lambda)})) & \xrightarrow{\;g^n_\lambda\;} & Sh^*_0((P,p),(X_\lambda, x_\lambda)) \\
\uparrow {\scriptstyle q_{g(\lambda)\lambda_2 *}} & & \downarrow {\scriptstyle f^n_{\lambda *}} \\
Sh^*_0((P,p),(Y_{\lambda_2}, y_{\lambda_2})) & \xrightarrow[\;q_{\lambda\lambda_2 *}\;]{} & Sh^*_0((P,p),(Y_\lambda, y_\lambda)).
\end{array}$$

Hence for a $\mu \geq \lambda_1, \lambda_2$, the following diagrams are commutative, for every $n \geq \max\{n_1, n_2\}$

$$\begin{array}{ccc}
Sh_0^*((P,p),(X_{g(\lambda)},x_{g(\lambda)})) & \xrightarrow{\;f^n_{g(\lambda)_*}\;} & Sh_0^*((P,p),(Y_{g(\lambda)},y_{g(\lambda)})) \\
\Big\uparrow {\scriptstyle p_{g(\lambda)\mu_*}} & & \Big\downarrow {\scriptstyle g^n_\lambda} \\
Sh_0^*((P,p),(X_\mu,x_\mu)) & \xrightarrow[\;p_{\lambda\mu_*}\;]{} & Sh_0^*((P,p),(X_\lambda,x_\lambda)),
\end{array}$$

$$\begin{array}{ccc}
Sh_0^*((P,p),(Y_{g(\lambda)},y_{g(\lambda)})) & \xrightarrow{\;g^n_\lambda\;} & Sh_0^*((P,p),(X_\lambda,x_\lambda)) \\
\Big\uparrow {\scriptstyle q_{g(\lambda)\mu_*}} & & \Big\downarrow {\scriptstyle f^n_{\lambda\,*}} \\
Sh_0^*((P,p),(Y_\mu,y_\mu)) & \xrightarrow[\;q_{\lambda\mu_*}\;]{} & Sh_0^*((P,p)(Y_\lambda,y_\lambda)).
\end{array}$$

Since $\mu \geq \lambda$, by Definition 3.13, for any $n \geq n_0$, there exist the maps $r_1^n : (X_{\lambda'}, x_{\lambda'}) \to (X_\mu, x_\mu)$ and $r_2^n : (Y_{\lambda'}, y_{\lambda'}) \to (Y_\mu, y_\mu)$ satisfying (13).

Also, since F^* is a coarse shape morphism, for $\mu \geq g(\lambda)$, there exists $n_3 \in \mathbb{N}$ such that for every $n \geq n_3$,

$$f^n_{g(\lambda)} \circ p_{g(\lambda)\mu} \simeq_0 q_{g(\lambda)\mu} \circ f^n_\mu. \tag{14}$$

Put $M_\lambda = \max\{n_0, n_1, n_2, n_3\}$ and let $n \geq M_\lambda$.

To prove (i), consider a map $h : (P,p) \to (Y_{\lambda'}, y_{\lambda'})$. Let α^* be the coarse shape morphism from (P,p) to (Y_μ, y_μ) given by $\langle[(a^m)]\rangle$, where $a^m = r_2^n \circ h : (P,p) \to (Y_\mu, y_\mu)$, for every $m \in \mathbb{N}$. Then $\beta^* = (g^n_\lambda \circ q_{g(\lambda)\mu_*})(\alpha^*)$ is a coarse shape morphism from (P,p) to (X_λ, x_λ) given by $\langle[(b^m)]\rangle$. We know $f^n_{\lambda\,*} \circ g^n_\lambda \circ q_{g(\lambda)\mu_*}(\alpha^*) = q_{\lambda\mu_*}(\alpha^*)$, i.e., two coarse shape morphisms $f^n_{\lambda\,*}(\beta^*)$ and $q_{\lambda\mu_*}(\alpha^*)$ given by $\langle[(c^m = f^n_\lambda \circ b^m)]\rangle$ and $\langle[(d^m = q_{\lambda\mu} \circ r_2^n \circ h)]\rangle$, respectively, are equal. So by the definition, there exists $M_n \in \mathbb{N}$ that for every $m \geq M_n$,

$$f^n_\lambda \circ b^m \simeq_0 q_{\lambda\mu} \circ r_2^n \circ h.$$

Take $k^n = b^{M_n} : (P,p) \to (X_\lambda, x_\lambda)$. Hence, by (13)

$$f^n_\lambda \circ k^n = f^n_\lambda \circ b^{M_n} \simeq_0 q_{\lambda\mu} \circ r_2^n \circ h \simeq_0 q_{\lambda\lambda'} \circ h.$$

To prove (ii), suppose $k_1, k_2 : (P,p) \to (X_{\lambda'}, x_{\lambda'})$ are maps with $f^n_{\lambda'} \circ k_1 \simeq_0 f^n_{\lambda'} \circ k_2$ and so $r_2^n \circ f^n_{\lambda'} \circ k_1 \simeq_0 r_2^n \circ f^n_{\lambda'} \circ k_2$. By (13), $r_2^n \circ f^n_{\lambda'} \simeq_0 f^n_\mu \circ r_1^n$, so $f^n_\mu \circ r_1^n \circ k_1 \simeq_0 f^n_\mu \circ r_1^n \circ k_2$ and then $q_{g(\lambda)\mu} \circ f^n_\mu \circ r_1^n \circ k_1 \simeq_0 q_{g(\lambda)\mu} \circ f^n_\mu \circ r_1^n \circ k_2$. From (14), $f^n_{g(\lambda)} \circ p_{g(\lambda)\mu} \circ r_1^n \circ k_1 \simeq_0 f^n_{g(\lambda)} \circ p_{g(\lambda)\mu} \circ r_1^n \circ k_2$, hence $f^n_{g(\lambda)_*} \circ p_{g(\lambda)\mu_*}(l_1^*) = f^n_{g(\lambda)_*} \circ p_{g(\lambda)\mu_*}(l_2^*)$, in which l_1^* and l_2^* are coarse shape morphisms in $Sh_0^*((P,p),(X_\mu,x_\mu))$ given by $\langle[(l_1^m = r_1^n \circ k_1)]\rangle$ and $\langle[(l_2^m = r_1^n \circ k_2)]\rangle$, respectively. Then $g^n_\lambda \circ f^n_{g(\lambda)_*} \circ p_{g(\lambda)\mu_*}(l_1^*) = g^n_\lambda \circ f^n_{g(\lambda)_*} \circ p_{g(\lambda)\mu_*}(l_2^*)$ and so $p_{\lambda\mu_*}(l_1^*) = p_{\lambda\mu_*}(l_2^*)$ as coarse shape morphisms which are given by $\langle[(p_{\lambda\mu} \circ l_1^m = p_{\lambda\mu} \circ r_1^n \circ k_1)]\rangle$ and $\langle[(p_{\lambda\mu} \circ l_2^m = p_{\lambda\mu} \circ r_1^n \circ k_2)]\rangle$, respectively. Therefore,

$$p_{\lambda\mu} \circ r_1^n \circ k_1 \simeq_0 p_{\lambda\mu} \circ r_1^n \circ k_2$$

and by (13),

$$p_{\lambda\lambda'} \circ k_1 \simeq_0 p_{\lambda\lambda'} \circ k_2.$$

Finally, to prove that F^* is a coarse shape equivalence, by Lemma 3.7, it is sufficient to show that the morphism $\mathbf{f}^* : (\mathbf{X}, \mathbf{x}) \to (\mathbf{Y}, \mathbf{y})$ given by the level map (f_λ^n) is an isomorphism in pro*-HPol$_0$.

Given $\lambda \in \Lambda$. By the above argument, there exist triples $(\lambda, \lambda', M_\lambda)$ and $(\lambda', \lambda'', M_{\lambda'})$ satisfying the condition $(**)$.

Since F^* is a coarse shape morphism, there exists $n_0 \in \mathbb{N}$ that for every $n \geq n_0$,

$$f_\lambda^n \circ p_{\lambda\lambda'} \simeq_0 q_{\lambda\lambda'} \circ f_{\lambda'}^n,$$

and there exists $n_1 \in \mathbb{N}$ that for every $n \geq n_1$,

$$f_{\lambda'}^n \circ p_{\lambda'\lambda''} \simeq_0 q_{\lambda'\lambda''} \circ f_{\lambda''}^n.$$

Now consider $\lambda'' \geq \lambda$ and put $N = \max\{n_0, n_1, M_\lambda, M_{\lambda'}\}$. Let $n \geq N$. If $P = Y_{\lambda''}$ and $h = id : (Y_{\lambda''}, y_{\lambda''}) \to (Y_{\lambda''}, y_{\lambda''})$, then by (i) there exists a map $k^n : (Y_{\lambda''}, y_{\lambda''}) \to (X_{\lambda'}, x_{\lambda'})$ such that

$$f_{\lambda'}^n \circ k^n \simeq_0 q_{\lambda'\lambda''}. \tag{15}$$

Also, put $P = X_{\lambda''}$ and $k_1 = k^n \circ f_{\lambda''}^n$ and $k_2 = p_{\lambda'\lambda''}$ which are maps from $X_{\lambda''} \to X_{\lambda'}$. Hence

$$f_{\lambda'}^n \circ k_1 = f_{\lambda'}^n \circ k^n \circ f_{\lambda''}^n \simeq_0 q_{\lambda'\lambda''} \circ f_{\lambda''}^n \simeq_0 f_{\lambda'}^n \circ p_{\lambda'\lambda''} \simeq_0 f_{\lambda'}^n \circ k_2.$$

So by (ii),

$$p_{\lambda\lambda'} \circ k_1 \simeq_0 p_{\lambda\lambda'} \circ k_2.$$

Take $h_\lambda^n = p_{\lambda\lambda'} \circ k^n : (Y_{\lambda''}, y_{\lambda''}) \to (X_\lambda, x_\lambda)$. We have

$$f_\lambda^n \circ h_\lambda^n = f_\lambda^n \circ p_{\lambda\lambda'} \circ k^n \simeq_0 q_{\lambda\lambda'} \circ f_{\lambda'}^n \circ k^n \simeq_0 q_{\lambda\lambda'} \circ q_{\lambda'\lambda''} \simeq_0 q_{\lambda\lambda''},$$

and

$$h_\lambda^n \circ f_{\lambda''}^n = p_{\lambda\lambda'} \circ k^n \circ f_{\lambda''}^n = p_{\lambda\lambda'} \circ k_1 \simeq_0 p_{\lambda\lambda'} \circ k_2 = p_{\lambda\lambda'} \circ p_{\lambda'\lambda''} \simeq_0 p_{\lambda\lambda''}. \qquad \square$$

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References

[1] N.K. Bilan, N. Uglesic, The coarse shape, Glas. Mat. 42 (2007) 145–187.
[2] N.K. Bilan, N. Uglesic, The Whitehead type theorems in coarse shape theory, Homol. Homotopy Appl. 15 (2013) 103–125.
[3] E. Cuchillo-Ibanez, M.A. Morón, F.R. Ruiz del Portal, J.M.R. Sanjurjo, A topology for the sets of shape morphisms, Topol. Appl. 94 (1999) 51–60.
[4] J. Dydak, The Whitehead and the Smale theorems in shape theory, Diss. Math. 156 (1979) 1–55.
[5] R. Geoghegan, Elementary proofs of stability theorems in pro-homotopy and shape, Gen. Topol. Appl. 8 (1978) 265–281.
[6] F. Ghanei, H. Mirebrahimi, B. Mashayekhy, T. Nasri, Topological coarse shape homotopy groups, Topol. Appl. 36 (2010) 255–266.
[7] S. Mardešić, J. Segal, Shape Theory, North-Holland, Amsterdam, 1982.
[8] K. Morita, The Hurewicz and the Whitehead theorems in shape theory, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A 12 (1974) 246–258.
[9] M.A. Morón, F.R. Ruiz del Portal, On weak shape equivalences, Topol. Appl. 92 (1999) 225–236.
[10] M.A. Morón, F.R. Ruiz del Portal, Ultrametrics and infinite dimensional Whitehead theorems in shape theory, Manuscr. Math. 89 (1996) 325–333.
[11] R.H. Overton, J. Segal, A new construction of movable compacta, Glas. Mat. 6 (1971) 361–363.