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When is a Local Homeomorphism a Semicovering Map?

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Abstract In this paper, by reviewing the concept of semicovering maps, we present some conditions under which a local homeomorphism becomes a semicovering map. We also obtain some conditions under which a local homeomorphism is a covering map.

Keywords Local homeomorphism · Fundamental group · Covering map · Semicovering map

Mathematics Subject Classification (2010) 57M10 · 57M12 · 57M05

1 Introduction

It is well-known that every covering map is a local homeomorphism. The converse seems to be an interesting question that when a local homeomorphism is a covering map (see [4, 6, 8]). Recently, Brazas [2, Definition 3.1] generalized the concept of covering map by the phrase *A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies*. Note that a map $p : Y \longrightarrow X$ has *continuous lifting of paths* if $\mathcal{P}_p : (\mathcal{P}Y)_y \longrightarrow (\mathcal{P}X)_{p(y)}$ defined by $\mathcal{P}_p(\alpha) = p \circ \alpha$ is a homeomorphism for all $y \in Y$,where $(\mathcal{P}Y)_y = \{\alpha : [0, 1] \longrightarrow Y | \alpha(0) = y\}$. Also, a map $p : Y \longrightarrow X$ *has*

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continuous lifting of homotopies if $\Phi_p : (\Phi Y)_y \longrightarrow (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. It is easy to see that any covering map is semicovering.

The quasitopological fundamental group $\pi_1^{qtop}(X, x)$ is the quotient space of the loop space $\Omega(X, x)$ equipped with the compact-open topology with respect to the function $\Omega(X, x) \longrightarrow \pi_1(X, x)$ identifying path components (see [1]). Torabi et al. [11, Theorem 3.7] showed that for a connected, locally path connected space X, there is a one to one correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of subgroups of its fundamental group $\pi_1(X, x)$ with open core in $\pi_1^{qtop}(X, x)$. Using this classification, it can be concluded that for a locally path connected space X, a semicovering map $p : \tilde{X} \longrightarrow X$ is a covering map if and only if the core of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ is an open subgroup of $\pi_1^{qtop}(X, x_0)$. By using this fact, we give some conditions under which a semicovering map becomes a covering map which extend some results of [4].

In Section 2, among reviewing the concept of local homeomorphism, path lifting property and unique path lifting property, we mention a result on uniqueness of lifting for local homeomorphism and a simplified definition [7, Corollary 2.1] for a semicovering map. Note that there is a misstep in the simplification of semicovering which we give another proof to remedy this defect. In Section 3, we intend to find some conditions under which a local homeomorphism is a semicovering map. Among other things, we prove that if $p : \tilde{X} \longrightarrow X$ is a local homeomorphism, $\tilde{X}$ is Hausdorff and sequential compact, then p is a semicovering map. Also, a closed local homeomorphism from a Hausdorff space is a semicovering map. Moreover, a proper local homeomorphism from a Hausdorff space onto a Hausdorff space is semicovering.

Finally in Section 4, we generalize some results of [4]. In fact, we obtain some conditions under which a semicovering map is a covering map. More precisely, we prove that every finite sheeted semicovering map from a Hausdorff space is a covering map. Also, a proper local homeomorphism from a Hausdorff space onto a Hausdorff space is a finite sheeted covering map.

2 Notations and Preliminaries

In this paper, all maps $f : X \longrightarrow Y$ between topological spaces X and Y are continuous. We recall that a continuous map $p : \tilde{X} \longrightarrow X$, is called a *local homeomorphism* if for every point $\tilde{x} \in \tilde{X}$ there exists an open neighborhood $\tilde{W}$ of $\tilde{x}$ such that $p(\tilde{W}) \subset X$ is open and the restriction map $p|_{\tilde{W}} : \tilde{W} \longrightarrow p(\tilde{W})$ is a homeomorphism. In this paper, we assume that $\tilde{X}$ is path connected and p is surjective.

Definition 2.1 Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map and $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of f.

The map p has *path lifting property* if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f. Also, the map p has *unique path lifting property* if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f (see [9]).

The following lemma is stated in [5, Lemma 5.5] for $Y = I$. One can state it for an arbitrary map $f : X \longrightarrow Y$ for a connected space Y.

Lemma 2.2 *Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism, Y a connected space, $\tilde{X}$ a Hausdorff space and let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map. Given $\tilde{x} \in p^{-1}(x_0)$ there is at most one lifting $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f.*

The following interesting result seemingly can be concluded from [7, Definition 7, Lemma 2.1, Proposition 2.2].

A map $p : \tilde{X} \longrightarrow X$ is a semicovering map if and only if it is a local homeomorphism with unique path lifting and path lifting properties (see [7, Corollary 2.1]*).* $(\star)$

Unfortunately, there exists a misstep in the proof of [7, Lemma 2.1]. More precisely, in the proof of [7, Lemma 2.1], it is not guaranteed that $h_t(K_n^j) \cap U) \neq \phi$, i.e., $h_t|_{K_n^j}$ might be a lift of γ that is different from $(p|_U)^{-1} \circ \gamma$. After some attempts to find a proof for the misstep, we found out that the method in the proof of [7, Lemma 2.1] does not work. Now, we give a proof for [7, Lemma 2.1] with a different method as follows.

Lemma 2.3 *(Local Homeomorphism Homotopy Theorem) Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism with unique path lifting and path lifting properties. Consider the diagram of continuous maps*

$$\begin{array}{ccc} I & \xrightarrow{\ \tilde{f}\ } & (\tilde{X}, \tilde{x}_0) \\ {\scriptstyle j}\downarrow & \overset{\tilde{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ I \times I & \xrightarrow{\ F\ } & (X, x_0), \end{array}$$

where $j(t) = (t, 0)$ for all $t \in I$. Then there exists a unique continuous map $\tilde{F} : I \times I \longrightarrow \tilde{X}$ making the diagram commute.

Proof Put $\tilde{F}(t, 0) = \tilde{f}(t)$ for all $t \in I$, and let $W_{\tilde{f}(t)}$ be an open neighborhood of $\tilde{f}(t)$ in $\tilde{X}$ such that $p|_{W_{\tilde{f}(t)}} : W_{\tilde{f}(t)} \longrightarrow p(W_{\tilde{f}(t)})$ is a homeomorphism. Then, $\{\tilde{f}^{-1}(W_{\tilde{f}(t)})|t \in I\}$ is an open cover for I. Since I is compact, there exists $n \in \mathbb{N}$ such that for every $0 \le i \le n-1$ the interval $[\frac{i}{n}, \frac{i+1}{n}]$ is contained in $\tilde{f}^{-1}(W_{\tilde{f}(t_i)})$ for some $t_i \in I$. For every $0 \le i \le n-1$, $F^{-1}(p(W_{\tilde{f}(t_i)}))$ is open in $I \times I$ which contains $(\frac{i}{n}, 0)$. Hence there exists $s_i \in I$ such that $[\frac{i}{n}, \frac{i+1}{n}] \times [0, s_i]$ is contained in $F^{-1}(p(W_{\tilde{f}(t_i)}))$ and so $F([\frac{i}{n}, \frac{i+1}{n}] \times [0, s_i]) \subseteq p(W_{\tilde{f}(t_i)})$. Since $p|_{W_{\tilde{f}(t_i)}} : W_{\tilde{f}(t_i)} \longrightarrow p(W_{\tilde{f}(t_i)})$ is a homeomorphism, we can define $\tilde{F}$ on $k_i = [\frac{i}{n}, \frac{i+1}{n}] \times [0, s_i]$ by $p^{-1}|_{W_{\tilde{f}(t_i)}} \circ F|_{k_i}$. Let $s = \min\{s_i | 0 \le i \le n-1\}$, then by gluing lemma, we can define $\tilde{F}$ on $I \times [0, s]$ since $\{\frac{i+1}{n}\} \times [0, s] \in k_i \cap k_{i+1}$. Put $A = \{r \in I|$ there exists $\tilde{F}_r : I \times [0, r] \longrightarrow \tilde{X}$ such that $F(x, y) = p \circ \tilde{F}_r(x, y)$for every$(x, y) \in I \times [0, r]$ and $\tilde{F}(t, 0) = \tilde{f}(t)\}$. Note that A is nonempty since $0 \in A$. We show that there exists $M \in I$ such that $M = \max A$. For this, consider an increasing sequence $\{a_n\}_{n \in \mathbb{N}}$ in A such that $a_n \longrightarrow a$. We show that $a \in A$. Let $s < a$, then there exists $n_s \in \mathbb{N}$ such that $s \le a_{n_s}$. We define $H : I \times [0, a] \longrightarrow \tilde{X}$ by

$$H(t, s) = \begin{cases} \tilde{F}_{a_{n_s}}(t, s) & s < a \\ \lambda_a(t) & s = a, \end{cases}$$

where $\lambda_a(t)$ is the lifting of the path $F(I \times \{a\})$ starting at $\gamma(a)$ and γ is the lifting of the path $F(0, t)$ for $t \in I$ starting at $\tilde{f}(0)$. Note that the existence of $\lambda_a(t)$ and γ are due to path lifting property of p. The map $H|_{I \times [0,a)}$ is well defined and continuous since if $r_1, r_2 \in A$ and

$r_1 < r_2$, then $\tilde{F}_{r_2}|_{I\times[0,r_1)} = \tilde{F}_{r_1}$ and p is a local homeomorphism with unique path lifting property. Therefore, $H|_{\{0\}\times[0,a)}$ is a lifting of $F|_{\{0\}\times[0,a)}$ starting at $\tilde{f}(0)$ which implies that $H(0,t) = \gamma(t)$ for every $0 \leq t < a$. Put $B = \{t \in I | H|_{\{I\times[0,a)\}\cup\{[0,t]\times\{a\}\}}\text{is continuous}\}$. We show that, there exists $0 < \epsilon < 1$ such that $H|_{I\times[0,a)\cup\{[0,\epsilon)\times\{a\}\}}$ is continuous. For this, consider $U = U_{\gamma(a)}$ an open neighborhood of $\gamma(a)$ such that $p|_{U_{\gamma(a)}} : U_{\gamma(a)} \longrightarrow p(U_{\gamma(a)})$ is a homeomorphism. There exists $0 < \epsilon < 1$ such that $F([0,\epsilon] \times [a-\epsilon, a]) \subseteq p(U_{\gamma(a)})$ so we have a lifting of $F|_{[0,\epsilon]\times[a-\epsilon,a]}$ in $U_{\gamma(a)}$ by $p^{-1}|_U \circ F|_{[0,\epsilon]\times[a-\epsilon,a]}$. Note that $p^{-1}|_U \circ F|_{[0,\epsilon]\times[a-\epsilon,a]}(0,t)$ and $\gamma(t)$ are two liftings of $F|_{\{0\}\times[a-\epsilon,a]}(0,t)$ such that $p^{-1}|_U \circ F(0,a) = \lambda_a(0) = \gamma(a)$. By unique path lifting property, we have $p^{-1}|_U \circ F|_{[0,\epsilon]\times[a-\epsilon,a]}(0,t) = \gamma(t)$ for $t \in [a-\epsilon, a]$ so $p^{-1}|_U \circ F|_{[0,\epsilon]\times[a-\epsilon,a]}(0, a-\frac{\epsilon}{2}) = \gamma(a-\frac{\epsilon}{2})$. Since $H(0, a-\frac{\epsilon}{2}) = \gamma(a-\frac{\epsilon}{2})$, by unique path lifting property we have $p^{-1}|_U \circ F|_{[0,\epsilon]\times[a-\epsilon,a)} = H|_{[0,\epsilon]\times[a-\epsilon,a)}$. Note that $p^{-1}|_U \circ F|_{[0,\epsilon]\times[a-\epsilon,a]}(t,a)$ and $\lambda_a(t)$ are two liftings of $F|_{[0,\epsilon]\times\{a\}}(t,a)$ such that $p^{-1}|_U \circ F(0,a) = \gamma(a) = \lambda_a(0)$. By unique path lifting property, we have $p^{-1} \circ F(t,a) = \lambda_a(t) = H(t,a)$ for $t \in [0,\epsilon]$. Hence $H|_{I\times[0,a)\cup\{[0,\epsilon)\times\{a\}\}}$ is continuous which implies that B is nonempty. We show that B has a maximum element and $\max B = 1$. For this, consider an increasing sequence $\{b_n\}_{n\in\mathbb{N}}$ in B such that $b_n \longrightarrow b$. We know that H is continuous on $\{I \times [0,a)\} \cup \{[0,b) \times \{a\}\}$. By a similar argument for the continuity of $H|_{I\times[0,a)\cup\{[0,\epsilon)\times\{a\}\}}$, we can prove that H is continuous on $\{I \times [0,a)\} \cup \{[0,b] \times \{a\}\}$ and $\max B = 1$. Thus $B = I$. Therefore $a \in A$, which implies that A has a maximum. Finally, by a similar idea for constructing $\tilde{F}$ on $I \times [0,s]$, we can show that $M = 1$. Hence we have a lifting for F by p making the above diagram commute. Uniqueness of $\tilde{F}$ is obtained by Lemma 2.2 since $I \times I$ is connected. □

Also, there exists a misstep in the proof of [7, Proposition 2.2]. More precisely, in the proof of [7, Proposition 2.2], the equality

$$\mathcal{P}_p(U) = \bigcap_{j=1}^{n} \langle K_n^j, p(U_j)\rangle \cap (\mathcal{P}X)_x$$

does not hold in general. Now, we give a proof for the result (⋆).

Theorem 2.4 *A map $p : \tilde{X} \longrightarrow X$ is a semicovering map if and only if it is a local homeomorphism with unique path lifting and path lifting properties.*

Proof If $p : \tilde{X} \longrightarrow X$ is a semicovering map then continuous lifting of paths guarantees unique path lifting and path lifting properties. Hence, p is a local homeomorphism with unique path lifting and path lifting properties. For the converse, let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism with unique path lifting and path lifting properties, $x \in X$ and $\tilde{x} \in p^{-1}(x)$. The map $\mathcal{P}_p : (\mathcal{P}Y)_y \longrightarrow (\mathcal{P}X)_{p(y)}$ is bijective since p is a local homeomorphism with unique path lifting and path lifting properties. Also by Lemma 2.3, we can conclude that the map $\Phi_p : (\Phi Y)_y \longrightarrow (\Phi X)_{p(y)}$ is bijective. It is known that $\mathcal{P}_p$ and Φ_p are continuous so it is enough to show that $\mathcal{P}_p$ and Φ_p are open. A basic open set in $(\mathcal{P}\tilde{X})_{\tilde{x}}$ is of the form

$$\tilde{U} = \bigcap_{j=1}^{n} \langle K_n^j, \tilde{U}_j\rangle \cap (\mathcal{P}\tilde{X})_{\tilde{x}},$$

where $K_n^j = [\frac{j-1}{n}, \frac{j}{n}]$ and $\tilde{U}_j$ is an open neighborhood in $\tilde{X}$ such that $p|_{\tilde{U}_j} : \tilde{U}_j \longrightarrow p(\tilde{U}_j)$ is a homeomorphism for $1 \leq j \leq n$. Suppose

$$U = \bigcap_{j=1}^{n} \langle K_n^j, p(\tilde{U}_j)\rangle \cap \bigcap_{j=1}^{n-1} \langle K_n^j \cap K_n^{j+1}, p(\tilde{U}_j \cap \tilde{U}_{j+1})\rangle \cap (\mathcal{P}X)_x.$$

Note that $\mathcal{P}_p(\tilde{U}) = U$. Since p is an open map, $p(\tilde{U}_j)$ is open in X for $1 \leq j \leq n$. Since U is open in $\tilde{X}$, $\mathcal{P}_p$ is open and therefore $\mathcal{P}_p$ is a homeomorphism. It is obvious that $\mathcal{P}_p(\tilde{U}) \subseteq U$. Let $\alpha \in U$, since p has unique path lifting and path lifting properties, we can find a lift $\tilde{\alpha} \in (\mathcal{P}\tilde{X})_{\tilde{x}}$. Put $t \in K_n^j$. Since $p|_{\tilde{U}_j}$ and $p|_{\tilde{U}_j \cap \tilde{U}_{j+1}}$ are homeomorphisms and $\alpha(K_n^j \cap K_n^{j+1}) \subseteq p(\tilde{U}_j \cap \tilde{U}_{j+1})$, we have $p|_{\tilde{U}_j}(\tilde{\alpha}(t)) = \alpha(t)$ therefore $\tilde{\alpha}(t) \in \tilde{U}_j$. Thus $\tilde{\alpha} \in \tilde{U}$ and hence $\mathcal{P}_p(\tilde{U}) = U$.

To show that Φ_p is an open map, suppose $\tilde{U} \in (\Phi\tilde{X})_{\tilde{x}}$ is a basic open set of the form

$$\tilde{U} = \bigcap_{0<i,j\leq n} \langle K_n^{i,j}, \tilde{U}_{i,j}\rangle \cap (\Phi\tilde{X})_{\tilde{x}},$$

where $K_n^{i,j} = K_n^i \times K_n^j$ and $\tilde{U}_{i,j}$ is an open neighborhood in $\tilde{X}$ such that $p|_{\tilde{U}_{i,j}} : \tilde{U}_{i,j} \longrightarrow p(\tilde{U}_{i,j})$ is a homeomorphism for $0 < i, j \leq n$. Suppose

$$\begin{aligned} U = & \bigcap_{0<i,j\leq n} \langle K_n^{i,j}, p(\tilde{U}_{i,j})\rangle \cap \bigcap_{0<i\leq n, 0<j\leq n-1} \langle K_n^{i,j} \cap K_n^{i,j+1}, p(\tilde{U}_{i,j} \cap \tilde{U}_{i,j+1})\rangle \\ & \cap \bigcap_{0<i,j\leq n-1} \langle K_n^{i,j} \cap K_n^{i,j+1} \cap K_n^{i+1,j} \cap K_n^{i+1,j+1}, p(\tilde{U}_{i,j} \cap \tilde{U}_{i,j+1} \cap \tilde{U}_{i+1,j} \cap \tilde{U}_{i+1,j+1})\rangle \\ & \cap \bigcap_{0<i\leq n-1, 0<j\leq n} \langle K_n^{i,j} \cap K_n^{i+1,j}, p(\tilde{U}_{i,j} \cap \tilde{U}_{i+1,j})\rangle \cap (\Phi X)_x. \end{aligned}$$

Note that $\Phi_p(\tilde{U}) = U$. It is obvious that $\Phi_p(\tilde{U}) \subseteq U$. Let $f \in U$. Since p has unique path lifting and path lifting properties, by Lemma 2.3, we can find a lift $\tilde{f} \in (\Phi\tilde{X})_{\tilde{x}}$. Put $t \in K_n^{i,j}$. Since $p|_{\tilde{U}_{i,j}}$, $p|_{\tilde{U}_{i,j} \cap \tilde{U}_{i,j+1}}$, $p|_{\tilde{U}_{i,j} \cap \tilde{U}_{i+1,j}}$ and $p|_{\tilde{U}_{i,j} \cap \tilde{U}_{i,j+1} \cap \tilde{U}_{i+1,j} \cap \tilde{U}_{i+1,j+1}}$ are homeomorphisms, we have $p|_{\tilde{U}_{i,j}}(\tilde{f}(t)) = f(t)$ and so $\tilde{f}(t) \in \tilde{U}_{i,j}$. Thus $\tilde{f} \in \tilde{U}$ which implies $\Phi_p(\tilde{U}) = U$. Hence Φ_p is an open map. □

Note that there exists a local homeomorphism without unique path lifting and path lifting properties and so it is not a semicovering map .

Example 2.5 Let $\tilde{X} = ((0, 1) \times \{0\}) \bigcup (\{\frac{1}{2}\} \times [\frac{1}{2}, \frac{3}{4}))$ with a topology by open basis

$$\left\{\left(\left(a, \frac{1}{2}\right) \times \{0\}\right) \cup \left(\left\{\frac{1}{2}\right\} \times \left[\frac{1}{2}, b\right)\right) | a \in \left(0, \frac{1}{2}\right), \ b \in \left(\frac{1}{2}, \frac{3}{4}\right)\right\} \bigcup$$
$$\{(a, b) \times \{0\} | a, b \in (0, 1), \ a < b\} \bigcup \left\{\left\{\frac{1}{2}\right\} \times (a, b) | a, b \in \left(\frac{1}{2}, \frac{3}{4}\right), \ a < b\right\}$$

and let $X = (0, 1)$. Define $p : \tilde{X} \longrightarrow X$ by

$$p(s,t) = \begin{cases} s & t = 0 \\ t & t \neq 0. \end{cases}$$

It is routine to check that p is an onto local homeomorphism which does not have unique path lifting and path lifting properties.

3 When Is a Local Homeomorphism a Semicovering Map?

In this section, we obtained some conditions under which a local homeomorphism is a semicovering map. First, we intend to show that if $p : \tilde{X} \longrightarrow X$ is a local homeomorphism, $\tilde{X}$ is Hausdorff and sequential compact, then p is a semicovering map. In order to do this, we are going to study a local homeomorphism with a path which has no lifting.

Lemma 3.1 *Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism, f be an arbitrary path in X and $\tilde{x}_0 \in p^{-1}(f(0))$ such that there is no lifting of f starting at $\tilde{x}_0$. If $A_f = \{t \in I \mid f|_{[0,t]} has\ a\ lifting\ \hat{f}_t on [0,t] with \hat{f}_t(0) = \tilde{x}_0\}$, then A_f is open and connected. Moreover, there exists $\alpha \in I$ such that $A_f = [0, \alpha)$.*

Proof Let β be an arbitrary element of A_f. Since p is a local homeomorphism, there exists an open neighborhood W at $\hat{f}_\beta(\beta)$ such that $p|_W : W \longrightarrow p(W)$ is a homeomorphism. Since $\hat{f}_\beta(\beta) \in W$, there exists an $\epsilon \in I$ such that $f[\beta, \beta+\epsilon]$ is a subset of $p(W)$. We can define a map $\hat{f}_{\beta+\epsilon}$ as follows:

$$\hat{f}_{\beta+\epsilon}(t) = \begin{cases} \hat{f}_\beta(t) & t \in [0,\beta] \\ p|_W^{-1}(f(t)) & t \in [\beta, \beta+\epsilon]. \end{cases}$$

Hence, $(0, \beta+\epsilon)$ is a subset of A_f and so A_f is open.

Suppose $t, s \in A$. Without loss of generality, we can suppose that $t \geq s$. By the definition of A_f, there exists $\hat{f}_t$ and so $[0, t]$ is a subset of A_f. Also, every point between s and t belongs to A_f hence A_f is connected. Since A_f is open connected and $0 \in A_f$, there exists $\alpha \in I$ such that $A_f = [0, \alpha)$. □

Now, we prove the existence and uniqueness of a concept of a defective lifting.

Lemma 3.2 *Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism with unique path lifting property, f be an arbitrary path in X and $\tilde{x}_0 \in p^{-1}(f(0))$, such that there is no lifting of f starting at $\tilde{x}_0$. Then, using the notation of the previous lemma, there exists a unique continuous map $\tilde{f}_\alpha : A_f = [0,\alpha) \longrightarrow \tilde{X}$ such that $p \circ \tilde{f}_\alpha = f|_{[0,\alpha)}$.*

Proof First, we defined $\tilde{f}_\alpha : A_f = [0,\alpha) \longrightarrow \tilde{X}$ by $\tilde{f}_\alpha(s) = \hat{f}_s(s)$. The map $\tilde{f}_\alpha$ is well defined since if $s_1 = s_2$, then by unique path lifting property of p we have $\hat{f}_{s_1} = \hat{f}_{s_2}$ and so $\hat{f}_{s_1}(s_1) = \hat{f}_{s_2}(s_2)$ hence $\tilde{f}_\alpha(s_1) = \tilde{f}_\alpha(s_2)$. The map $\tilde{f}_\alpha$ is continuous since for any element s of A_f, $\hat{f}_{\frac{\alpha+s}{2}}$ is continuous at s and $\hat{f}_{\frac{\alpha+s}{2}} = \hat{f}_s$ on $[0, s]$. Thus, there exists $\epsilon > 0$ such that $\tilde{f}_\alpha|_{(s-\epsilon,s+\epsilon)} = \hat{f}_{\frac{\alpha+s}{2}}|_{(s-\epsilon,s+\epsilon)}$. Hence, $\tilde{f}_\alpha$ is continuous at s. For uniqueness, if there exists $\hat{f}_\alpha : [0,\alpha) \longrightarrow \tilde{X}$ such that $p \circ \hat{f}_\alpha = f|_{[0,\alpha)}$, then by unique path lifting property of $\tilde{X}$ we must have $\tilde{f}_\alpha = \hat{f}_\alpha$. □

Definition 3.3 By Lemmas 3.1 and 3.2, we called $\tilde{f}_\alpha$ the *incomplete lifting* of f by p starting at $\tilde{x}_0$.

Theorem 3.4 *If $\tilde{X}$ is Hausdorff and sequential compact and $p : \tilde{X} \longrightarrow X$ is a local homeomorphism, then p is a semicovering map.*

Proof Let $f : I \longrightarrow X$ be a path which has no lifting starting at $\tilde{x}_0 \in p^{-1}(f(0))$. Using the notion of Lemma 3.1, let $\tilde{f} : A_f = [0, \alpha) \longrightarrow \tilde{X}$ be the incomplete lifting of f at $\tilde{x}_0$. Suppose $\{t_n\}_0^\infty$ is a sequence in A_f which tends to t_0 and $t_n \leq t_0$. Since $\tilde{X}$ is sequential compact, there exists a convergent subsequence of $\tilde{f}(t_n)$, $\{\tilde{f}(t_{n_k})\}_0^\infty$ say, such that $\tilde{f}(t_{n_k})$ tends to l. We define

$$g(t) = \begin{cases} \tilde{f}(t) & 0 \leq t < t_0 \\ l = \lim_{k\to\infty} \tilde{f}(t_{n_k}) & t = t_0. \end{cases}$$

We have $p(l) = p(\lim_{k\to\infty} \tilde{f}(t_{n_k})) = \lim_{k\to\infty} p(\tilde{f}(t_{n_k})) = \lim_{k\to\infty} f(t_{n_k}) = f(t_0)$ and so $p \circ g = f$. We show that g is continuous on $[0, t_0]$, for this we show that g is continuous at t_0. Since p is a local homeomorphism, there exists a neighborhood W at l such that $p|_W : W \longrightarrow p(W)$ is a homeomorphism. Hence, there is $a \in I$ such that $f([a, t_0]) \subseteq p(W)$. Let V be a neighborhood at l and $W' = V \cap W$, then $p(W') \subseteq p(W)$ is an open set. Put $U = f^{-1}(p(W')) \cap (a, t_0]$ which is open in $[0, t_0]$ at t_0. It is enough to show that $g(U) \subseteq W'$. Since $f(U) \subseteq p(W')$ and p is a homeomorphism on W', $(p|_W)^{-1}(f(U)) \subseteq (p|_W)^{-1}(p(W'))$ and so $(p|_W)^{-1} \circ f = \tilde{f}$ on $[a, t_0)$ since $p(l) = f(t_0)$. Hence $(p|_W)^{-1} \circ f = g$ on $[a, t_0]$. Thus $g(U) \subseteq g(f^{-1}(p(W'))) = (p|_W)^{-1} \circ f(f^{-1}(p(W'))) \subseteq (p|_W)^{-1} \circ p(W') \subseteq W' \subseteq V$, so g is continuous. Hence $t_0 \in A_f$, which is a contradiction. Thus the map p has path lifting property. Using Lemma 2.2 p has unique path lifting property. Hence, Theorem 2.4 implies that p is a semicovering map. □

Note that every compact metric space is sequential compact. In the following, we present two semicovering maps on compact metric spaces.

Example 3.5 We show that $p : S^1 \times S^1 \longrightarrow S^1 \times S^1$ defined by $(x, y) \longrightarrow (x^n y^m, x^s y^t)$ is a semicovering map, where $m, n, s, t \in \mathbb{N}$ such that $\frac{n}{s} \neq \frac{m}{t}$. Let $\exp(\theta) = e^{2\pi i\theta}$, then we can consider p as $p(\exp(\alpha), \exp(\beta)) = (\exp(n\alpha + m\beta), \exp(s\alpha + t\beta))$. As a notation put $\exp(\gamma, \eta) = \{\exp(\theta) \in S^1 | \gamma \leq \theta \leq \eta\}$. Suppose $l = Max\{n, m, s, t\}$ and $U = (\exp(\alpha - \frac{\pi}{2l}, \alpha + \frac{\pi}{2l})) \times (\exp(\beta - \frac{\pi}{2l}, \beta + \frac{\pi}{2l}))$ is an open neighborhood of an element $(\exp(\alpha), \exp(\beta)) \in S^1 \times S^1$. It is clear that $p|_U : U \longrightarrow \exp(n(\alpha - \frac{\pi}{2l}) + m(\beta - \frac{\pi}{2l}), n(\alpha + \frac{\pi}{2l}) + m(\beta + \frac{\pi}{2l})) \times \exp(s(\alpha - \frac{\pi}{2l}) + t(\beta - \frac{\pi}{2l}), s(\alpha + \frac{\pi}{2l}) + t(\beta + \frac{\pi}{2l}))$ is a homeomorphism. Note that

$$\left(m\left(\alpha + \frac{\pi}{2l}\right) + n\left(\alpha + \frac{\pi}{2l}\right)\right) - \left(m\left(\alpha - \frac{\pi}{2l}\right) + n\left(\alpha - \frac{\pi}{2l}\right)\right) < \frac{m}{l}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{n}{l}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) < 2\pi$$

and

$$\left(s\left(\beta + \frac{\pi}{2l}\right) + t\left(\beta + \frac{\pi}{2l}\right)\right) - \left(s\left(\beta - \frac{\pi}{2l}\right) + t\left(\beta - \frac{\pi}{2l}\right)\right) < \frac{s}{l}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{t}{l}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) < 2\pi.$$

Therefore, if $p(\exp(\alpha_1), \exp(\beta_1)) = p((\exp(\alpha_2), \exp(\beta_2)))$, then

$$\begin{cases} n\alpha_1 + m\beta_1 &= n\alpha_2 + m\beta_2 \\ s\alpha_1 + t\beta_1 &= s\alpha_2 + t\beta_2 \end{cases} \quad \text{so} \quad \begin{cases} n(\alpha_1 - \alpha_2) &= m(\beta_2 - \beta_1) \\ s(\alpha_1 - \alpha_2) &= t(\beta_2 - \beta_1). \end{cases}$$

Since $\frac{n}{s} \neq \frac{m}{t}$, we have $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Thus, p is a local homeomorphism. Note that $S^1 \times S^1$ is a compact metric space and so it is sequential compact. Hence by Theorem 3.4, p is a semicovering map. It should be mentioned that every semicovering of a path-connected, locally path-connected, semilocally simply connected space is a covering. Hence, p is a covering map. Note that finding an evenly covered neighborhood by p for an arbitrary element of $S^1 \times S^1$ does not seem to be an easy computational task.

Example 3.6 In Fig. 1, the map p transfers every ci, j to ci directly for $i \in \mathbb{N}$ and $1 \leq j \leq 4$. Since the domain of p is compact metric, it is a sequential compact space and using Theorem 3.4 we can conclude that p is a semicovering map.

Clearly, the composition of two local homeomorphisms is a local homeomorphism hence by Theorem 3.4 we have the following corollary.

Corollary 3.7 *If $p_i : \tilde{X}_i \longrightarrow \tilde{X}_{i-1}$ for $i = 1, 2$ are local homeomorphisms and $\tilde{X}_2$ is Hausdorff and sequential compact, then $p_1 \circ p_2$ is a semicovering map.*

Chen and Wang [4, Theorem 1] showed that a closed local homeomorphism p from a Hausdorff space $\tilde{X}$ onto a connected space X is a covering map, when there exists at least one point $x_0 \in X$ such that $|p^{-1}(x_0)| = k$, for some finite number k. In the following theorem, we extend this result for semicovering map without finiteness condition on any fiber.

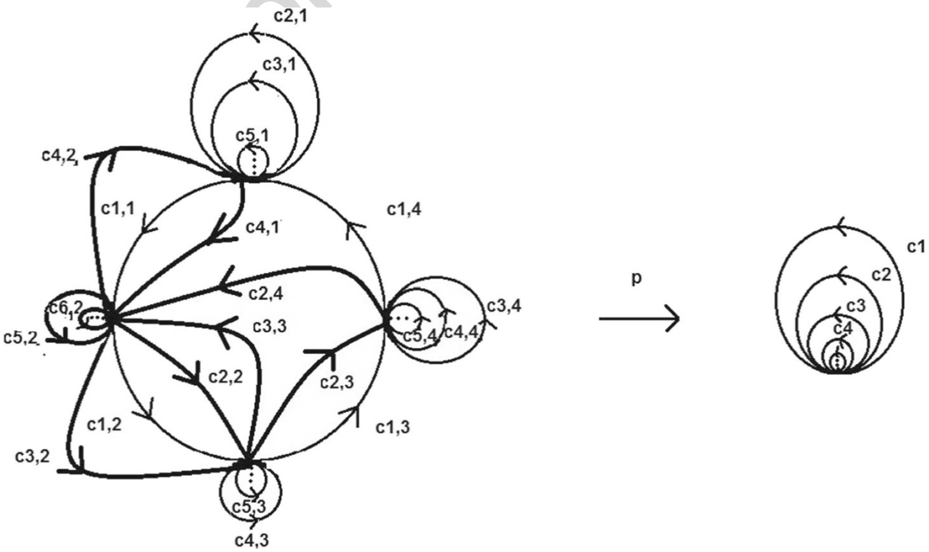

Fig. 1 A semicovering map of a sequential compact space

Theorem 3.8 *Let p be a closed local homeomorphism from a Hausdorff space $\tilde{X}$ onto a space X. Then p is a semicovering map.*

Proof Using Theorem 2.4, it is enough to show that p has unique path lifting and path lifting properties. By Lemma 2.2, p has the unique path lifting property. To prove the path lifting property for p, suppose there exists a path f in X such that it has no lifting starting at $\tilde{x}_0 \in p^{-1}(f(0))$. Let $g : A_f = [0, \alpha) \longrightarrow \tilde{X}$ be the incomplete lifting of f at $\tilde{x}_0$. Suppose $\{t_n\}_0^\infty$ is a sequence which tends to α. Put $B = \{t_n | n \in \mathbb{N}\}$, then $\overline{g(B)}$ is closed in $\tilde{X}$ and so $p(\overline{g(B)})$ is closed in X since p is a closed map. Since $\overline{f(B)} \subseteq p(\overline{g(B)})$, $f(\alpha) \in p(\overline{g(B)})$ and so there exists $\beta \in \overline{g(B)}$ such that $p(\beta) = f(\alpha)$. Since p is a local homeomorphism, there exists a neighborhood W_β at β such that $p|_{W_\beta} : W_\beta \longrightarrow p(W_\beta)$ is a homeomorphism. Since $\beta \in \overline{g(B)}$, there exists an $n_k \in \mathbb{N}$ such that $g(t_{n_r}) \in W_\beta$ for every $n_r \geq n_k$. Since $f(\alpha) \in p(W_\beta)$, there exists $k_1 \in \mathbb{N}$ such that $f([t_{n_{k_1}}, \alpha]) \subseteq p(W_\beta)$. Put $h = ((p|_{W_\beta})^{-1} \circ f)|_{[t_{n_{k_1}}, \alpha]}$, then $p \circ g = f = p \circ h$ on $[t_{n_{k_1}}, \alpha)$. Hence $p \circ h = p \circ g$ on $[t_{n_{k_1}}, \alpha)$. Since $p|_{W_\beta}$ is a homeomorphism, $g(t_{n_{k_1}}) = h(t_{n_{k_1}})$, thus $g = h$ on $[t_{n_{k_1}}, \alpha)$. Therefore, the map $\bar{g} : [0, \alpha] \longrightarrow \tilde{X}$ defined by

$$\bar{g}(t) = \begin{cases} g(t) & 0 \leq t < \alpha \\ h(\alpha) = \beta & t = \alpha \end{cases}$$

is continuous and $p \circ \bar{g} = f$ on $[0, \alpha]$. Hence $\alpha \in A_f$, which is a contradiction. □

Remark 3.9 Note that the local homeomorphism $p : S^1 \times S^1 \longrightarrow S^1 \times S^1$, introduced in Example 2.5, is a closed map since $S^1 \times S^1$ is Hausdorff and compact. Thus, using Theorem 3.8, we can obtain another proof to show that p is semicovering.

4 When is a Semicovering Map a Covering Map?

If $p : \tilde{X} \longrightarrow X$ is semicovering, then $\pi_1(X, x_0)$ acts on $Y = p^{-1}(x_0)$ by $\alpha\tilde{x}_0 = \tilde{\alpha}(1)$, where $\tilde{x}_0 \in Y$ and $\tilde{\alpha}$ is the lifting of α starting at $\tilde{x}_0$ (see [2]). Therefore, we can conclude that the stabilizer of $\tilde{x}_0$, $\pi_1(X, x_0)_{\tilde{x}_0}$, is equal to $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ for all $\tilde{x}_0 \in Y$ and so $|Y| = [\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$. Thus, if $x_0, x_1 \in X$, $Y_0 = p^{-1}(x_0)$, $Y_1 = p^{-1}(x_1)$ and $\tilde{X}$ is a path connected space, then $|Y_0| = |Y_1|$. Hence, one can define the concept of sheet for a semicovering map similar to covering maps.

The following example shows that there exists a local homeomorphism with finite fibers which is not a semicovering map.

Example 4.1 Let $\tilde{X} = (0, 2)$ and let $X = S^1$. Define $p : \tilde{X} \longrightarrow X$ by $p(t) = e^{2\pi i t}$. It is routine to check that p is an onto local homeomorphism whose fibers are finite but p is not a semicovering map since

$$|p^{-1}((0, 1))| = 2 \neq 1 = |p^{-1}((1, 0))|.$$

Theorem 4.2 *Suppose $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a semicovering map and $\tilde{X}$ is a Hausdorff space such that $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$ is finite, then p is a finite sheeted covering map.*

Proof Let x be an arbitrary element of X. Since p is a semicovering map and $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))] = m$, p is an m-sheeted semicovering map and so we have $\tilde{x} \in p^{-1}(x) =$

$\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_m\}$. Since $\tilde{X}$ is Hausdorff and p is a local homeomorphism, we can find an open neighborhood U of x and disjoint open neighborhoods $\tilde{U}_j$ such that for every j, $1 \leq j \leq m$, $\tilde{x}_j \in \tilde{U}_j$ and $p|_{\tilde{U}_j} : \tilde{U}_j \longrightarrow U$ is a homeomorphism. Since p is a semicovering map, we have $|p^{-1}(a)| = m$ for every $a \in X$ which implies that $p^{-1}(U) = \cup_{j=1}^{m} \tilde{U}_j$. Thus p is a finite sheeted covering map. □

The following result is an immediate consequence of the above theorem.

Corollary 4.3 *Every finite sheeted semicovering map from a Hausdorff space is a finite sheeted covering map.*

Note that Theorem 1 in [4] is an immediate consequence of our results Theorem 3.8 and Corollary 4.3.

Brazas presented an infinite sheeted semicovering map which is not a covering map (see [2, Example 3.8]). In a similar way, one can construct another example as follows.

Example 4.4 In Fig. 2, the map p_2 is a covering of $\mathbb{HE}$ with infinite fiber $\mathbb{Z}$. The map p_1 is a two-sheeted covering map, but the composition is not a covering map since it fails local triviality. Thus, the composition is a semicovering of $\mathbb{HE}$ which is not a covering map.

A map $f : X \longrightarrow Y$ is called *proper* if and only if $f^{-1}(H)$ is compact for any compact subset H of Y. It is well-known that a proper local homeomorphism from a Hausdorff space to a locally compact, Hausdorff space is a covering map. Chen and Wang [4, Corollary 2] showed that a proper local homeomorphism from a Hausdorff, first countable space onto a Hausdorff, connected space is a covering map. We extend this result without first countability as follows.

Theorem 4.5 *If p is a proper local homeomorphism from a Hausdorff space $\tilde{X}$ onto a Hausdorff space X, then p is a finite sheeted covering map.*

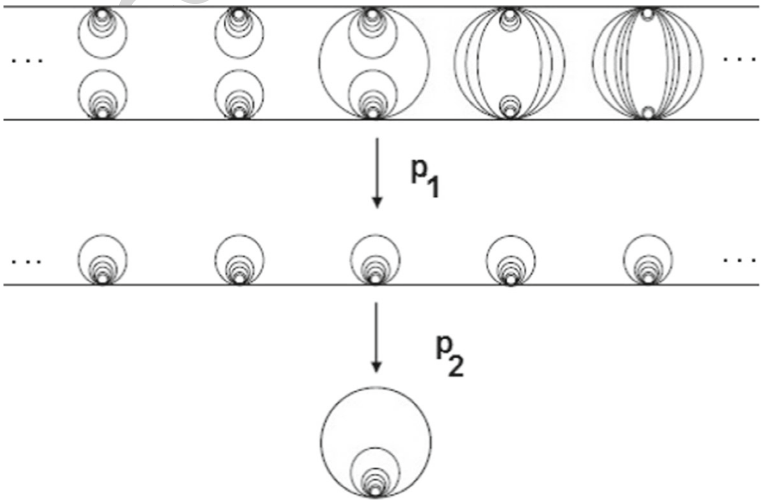

Fig. 2 A semicovering map which is not a covering map

Proof The map p is open since every local homeomorphism is an open map. Also, every open proper map is closed (see [10, Fact 2.3]). Hence, using Theorem 3.8, p is a semicovering map. Since singletons in X are compact and p is proper, every fiber of p is compact in $\tilde{X}$. Since fibers of a local homeomorphism are discrete, fibers of p are compact and so they are finite. Therefore, p is a finite sheeted semicovering map. Hence by Corollary 4.3, p is a finite sheeted covering map. □

The local homeomorphism $p : S^1 \times S^1 \longrightarrow S^1 \times S^1$, introduced in Example 3.5, is a proper map since $S^1 \times S^1$ is Hausdorff and compact. Thus by Theorem 4.5, we can prove that p is a finite sheeted covering map. Note that it is not easy to find an evenly covered neighborhood by p for an arbitrary element of $S^1 \times S^1$.

Corollary 4.6 *If $\tilde{X}$ is Hausdorff and sequential compact and $p : \tilde{X} \longrightarrow X$ is a local homeomorphism with a finite fiber, then p is a finite sheeted covering map.*

Proof By Theorem 3.4, p is a semicovering map. Hence, p is a finite sheeted semicovering map which is a finite sheeted covering map by Corollary 4.3. □

Note that the local homeomorphism $p : S^1 \times S^1 \longrightarrow S^1 \times S^1$, introduced in Example 3.5, has a finite fiber and $S^1 \times S^1$ is Hausdorff and sequential compact so by Corollary 4.6, p is a finite sheeted covering map.

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