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Ricci cubic gravity in d dimensions, gravitons and SAdS/Lifshitz black holes

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Abstract A special class of higher curvature theories of gravity, Ricci cubic gravity (RCG), in general d dimensional space-time has been investigated in this paper. We have used two different approaches, the linearized equations of motion and the auxiliary field formalism to study the massive and massless graviton propagating modes of the AdS background. Using the auxiliary field formalism, we have found the renormalized boundary stress tensor to compute the mass of the Schwarzschild–AdS and Lifshitz black holes in RCG theory.

1 Introduction

The Einstein–Hilbert action, as an effective gravitational theory, receives different higher curvature corrections. The origin of these corrections may come from quantum gravity or string theory [1–3]. Specifically these gravitational theories with higher curvature corrections in the presence of a cosmological parameter become more important in the context of AdS/CFT correspondence (see for example [4,5]).

There are many questions arising in these theories when one studies different black hole solutions. For example the existence of Schwarzschild–AdS (SAdS) or Lifshitz black holes is expected in these theories and consequently the computation of the mass or thermodynamical properties such as entropy will be a challenging problem.

The linear excitation of the gravitational field or graviton mode is another important object in these theories. It is a well-known property for these theories to have massive excitation modes in addition to the massless gravitons. The stability of vacuum solution requires tachyon-free conditions, which restrict the theory to specific regions of the parameter space.

Another common property in gravitational theories with higher curvature terms is the existence of the scalar and tensor ghost modes. In pure theories of gravity, although a scalar ghost mode can be eliminated by proper assumptions such as traceless condition of the linearized equations of motion, the tensor ghost modes may survive and destroy the unitarity of the dual CFTs. At first sight, the absence of tensor ghost modes can be achieved by going to the critical points, but at these points the massive modes degenerate into massless graviton mode and are replaced by ghost-like logarithmic modes. This theory may include a unitary subspace through the truncation of the logarithmic modes by imposing proper boundary conditions at the linear level. The unitarity problem of these theories has been discussed in [6–11].

Many different properties of higher curvature theories of gravity have been investigated in different space-time dimensions. For example in $d = 3$, gravitational theories known as massive gravities have been studied extensively [7,12–21]. Other higher curvature theories of gravity are also studied in five and six dimensions; for example see [22] and [23]. In general d dimensions one can follow recent work, for example [24–28].

In this paper we are interested in studying a special class of higher curvature theories of gravity, Ricci Cubic Gravity (RCG), in general d dimensional space-time in the context that we mentioned above. We will employ two different approaches, the linearized equations of motion and the auxiliary field formalism.

In the first approach we study the linear excitations around a d dimensional Anti-de Sitter (AdS_d) space-time and we find the stability conditions of this black hole. We will show that there are restrictions on the free parameters of the RCG when we are eliminating the scalar ghost modes. We also show that this model includes two massive graviton propagators and a massless one. We analyze various critical points of this theory where the massive modes are degenerate with the massless mode. We also compute the energy of the excitation modes and the Abbott–Deser [29] energy of different black hole solutions.

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In the second approach by a reformulation of RCG with the help of auxiliary fields we will find a Lagrangian of second order in derivatives of the fields. The linearization around the AdS$_d$ background up to the second order of the gravitational coupling generates the Fierz–Pauli massive action. We can read again the mass of the excitation modes by this approach.

The mass of SAdS and Lifshitz black holes can be computed in different ways, either by calculating the free energy and using the first law of thermodynamics or by computing the renormalized boundary energy-momentum tensor.

This paper is organized as follows: In Sect. 2, we begin with a six-derivative action constructed out of Ricci curvature tensor and its covariant derivatives. We study the graviton modes by linearizing the equations of motion around the AdS$_d$ vacuum. In order to construct a theory free of the scalar ghost modes, we should impose two constraints on the couplings of this theory. We show that RCG contains two massive graviton modes in addition to a massless one. We also discuss the stability of this vacuum solution. In last part of this section we calculate the conserved quantities of theory by the Abbott–Deser method [30].

In Sect. 3, we reformulate the RCG action with the help of two auxiliary fields and we linearize it around the AdS$_d$ background up to the second order of gravitational coupling. Then we rewrite this action as a linear combination of three Fierz-Pauli massive Lagrangians for spin-2 fields [7]. We also find the energy of the linear excitations to reconfirm the stability arguments in Sect. 2.

In Sect. 4, we will use the reformulated RCG action to compute the boundary energy-momentum tensor by using the technique which has been introduced in [31]. For this purpose, we will require a well-posed variational principle which is provided by some generalized Gibbons–Hawking terms.

We study a SAdS black hole solution of RCG in Sect. 5. We find the thermodynamical properties, such as temperature, free energy and entropy. We also compute the finite value of the mass of SAdS from the renormalized boundary energy-momentum tensor by adding a proper counter-term to the boundary terms. We show that this mass is compatible with the first law of thermodynamics for black holes.

As a more complicated case, the Lifshitz black hole is investigated in Sect. 6. We have tried to find a finite mass from the boundary stress tensor, consistent with the first law of thermodynamics. We observe that this is similar to the three dimensional case in [31]; there is an ambiguity for writing the counter-terms.

Section 7 contains the results of previous sections but in the special number of dimensions $d = 3$ to obtain the central charges of dual CFTs. In the last section we summarize and discuss our results. Almost all parts of the calculations in this paper have been done by the Mathematica package xAct [56].

2 Ricci cubic gravity in d dimensions

Let us start with the most general Ricci cubic gravity (RCG) in d dimensions by adding all possible independent contractions of the Ricci tensor and its covariant derivatives to the Einstein–Hilbert action in the presence of a cosmological parameter Λ_0. We restrict ourselves to terms with at most six derivatives, i.e.,

$$\begin{aligned}S = \frac{1}{\kappa^2}\int d^d x\sqrt{-g}\Big(&\sigma R - 2\Lambda_0 + a_1 R^{\mu\nu}R_{\mu\nu} + a_2 R^2\\&+ b_1\nabla_\mu R\nabla^\mu R + b_2\nabla_\mu R_{\alpha\beta}\nabla^\mu R^{\alpha\beta} + c_1 R^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma}\\&+ c_2 R R^{\alpha\beta}R_{\alpha\beta} + c_3 R^3\Big),\end{aligned} \tag{2.1}$$

where σ is a dimensionless parameter and κ is the gravitational coupling constant. These parameters together with the other couplings $a_1, a_2, b_1, b_2, c_1, c_2$ and c_3 make the parameter space of this theory. The six-derivative equations of motion for action (2.1) are given by

$$\sigma(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) + \Lambda_0 g_{\mu\nu} + \sum_{i=1}^{7} H^{(i)}_{\mu\nu} = 0, \tag{2.2}$$

where

$$\begin{aligned}H^{(1)}_{\mu\nu} = a_1\Big(&(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + \Box(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\\&+2\Big(R_{\mu\alpha\nu\beta} - \frac{1}{4}g_{\mu\nu}R_{\alpha\beta}\Big)R^{\alpha\beta}\Big),\end{aligned} \tag{2.3a}$$

$$H^{(2)}_{\mu\nu} = a_2\Big(2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + 2R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu})\Big), \tag{2.3b}$$

$$\begin{aligned}H^{(3)}_{\mu\nu} = b_1\Big(&2\nabla_\mu\nabla_\nu\Box R - 2R_{\mu\nu}\Box R + \nabla_\mu R\nabla_\nu R\\&- (2\Box^2 R + \frac{1}{2}\nabla^\gamma R\nabla_\gamma R)g_{\mu\nu}\Big),\end{aligned} \tag{2.3c}$$

$$\begin{aligned}H^{(4)}_{\mu\nu} = b_2\Big(&-\Box^2 R_{\mu\nu} + 2\nabla^\gamma\nabla_{(\mu}\Box R_{\nu)\gamma} - 2R^\gamma{}_{(\mu}\Box R_{\nu)\gamma}\\&- 2R^\alpha{}_{(\mu}\nabla^\beta\nabla_{\nu)}R_{\alpha\beta} + 2R^{\alpha\beta}\nabla_\alpha\nabla_{(\mu}R_{\nu)\beta}\\&+ \nabla^\gamma R\nabla_{(\mu}R_{\nu)\gamma} + \nabla_\mu R^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - 2\nabla_{(\mu}R^{\alpha\beta}\nabla_\alpha R_{\nu)\beta}\\&- (\nabla^\alpha\nabla^\beta\Box R_{\alpha\beta} + \frac{1}{2}\nabla^\gamma R^{\alpha\beta}\nabla_\gamma R_{\alpha\beta})g_{\mu\nu}\Big),\end{aligned} \tag{2.3d}$$

$$\begin{aligned}H^{(5)}_{\mu\nu} = c_1\Big(&3R_{\mu\alpha}R^{\alpha\beta}R_{\beta\nu} + \frac{3}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\alpha\gamma}R^\beta_\gamma) + \frac{3}{2}\Box R^\gamma{}_\mu R_{\gamma\nu}\\&- 6\nabla_\alpha\nabla_{(\mu}(R^\beta{}_{\nu)}R^\alpha{}_\beta) - \frac{1}{2}g_{\mu\nu}R^\alpha{}_\beta R^{\beta\gamma}R_{\gamma\alpha}\Big),\end{aligned} \tag{2.3e}$$

$$H^{(6)}_{\mu\nu} = c_2\Big((R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)R_{\alpha\beta}R^{\alpha\beta} + 2RR^{\gamma}{}_{\mu}R_{\gamma\nu} + g_{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\alpha\beta}R) + \Box(RR_{\mu\nu}) - 2\nabla_\gamma\nabla_{(\mu}(R^{\gamma}{}_{\nu)}R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)(R_{\alpha\beta}R^{\alpha\beta})\Big), \tag{2.3f}$$

$$H^{(7)}_{\mu\nu} = c_3\Big(3R_{\mu\nu}R^2 + 3(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R^2 - \frac{1}{2}g_{\mu\nu}R^3\Big). \tag{2.3g}$$

2.1 The linearized equations of motion

Let us consider a maximally symmetric space in d dimensions as a solution to the equations of motion (2.2). The Riemann, Ricci and scalar curvature tensors can be written as

$$R_{\alpha\mu\beta\nu} = \Lambda(g_{\alpha\beta}g_{\mu\nu} - g_{\mu\beta}g_{\alpha\nu}),\quad R_{\alpha\beta} = \Lambda(d-1)g_{\alpha\beta},\quad R = \Lambda d(d-1), \tag{2.4}$$

where Λ is the cosmological constant. By inserting the above tensors into the equations of motion (2.3a)–(2.3g) we will find that the cosmological parameter Λ_0 is related to the cosmological constant via

$$\Lambda_0 = \frac{1}{2}(d-1)\Lambda\Big((d-2)\sigma + (d-4)(d-1)\Lambda(a_1 + da_2) + (d-6)(d-1)^2\Lambda^2(c_1 + dc_2 + d^2c_3)\Big). \tag{2.5}$$

Now we suppose that the metric fluctuations $h_{\mu\nu}$ are around an AdS$_d$ background $\bar{g}_{\mu\nu}$ of which the radius has been fixed by Eq. (2.5) and the metric is given by $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$. If we insert this into the equation of motion (2.2) we will find the linearized equation of motion as follows[1]:

$$\mathcal{E}^L_{\mu\nu} = \bar{\sigma}\mathcal{G}_{\mu\nu}(h) + \sigma_1\mathcal{G}_{\mu\nu}(\mathcal{G}(h)) + \sigma_2\mathcal{G}_{\mu\nu}(\mathcal{G}(\mathcal{G}(h))) + \sigma_3(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + (d-1)\Lambda\bar{g}_{\mu\nu})R^{(1)} + \sigma_4(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + (d-1)\Lambda\bar{g}_{\mu\nu}\bar{\Box}R^{(1)}. \tag{2.6}$$

In the above equation the various constants are defined as follows:

$$\begin{aligned}
\bar{\sigma} &= \sigma + (d-1)\Lambda\Big(2a_1 + 2da_2 + 3(d-1) \times \Big(c_1 + d(c_2 + dc_3)\Big)\Lambda\Big),\\
\sigma_1 &= -2\Big(a_1 - \Big(2b_2 - (d-1)(3c_1 + dc_2)\Big)\Lambda\Big),\quad \sigma_2 = -4b_2,\\
\sigma_3 &= \frac{1}{2}\Big(4a_2 - (d-4)a_1 - \Big(b_2(d-3)(d-2)^2 + (d-1) \times \big(3dc_1 - 12(c_1 + dc_3) + c_2(d(d-4) - 8)\big)\Big)\Big),\\
\sigma_4 &= \frac{1}{2}\Big(-4b_1 - b_2(d(d-5) + 8)\Big).
\end{aligned} \tag{2.7}$$

[1] We have used the same approach and notation as [7].

In Eq. (2.6) we have used $\mathcal{G}_{\mu\nu}(h)$ as a linearized expression for the Einstein tensor, which we define by [32]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{2}(d-1)(d-2)\Lambda g_{\mu\nu}. \tag{2.8}$$

Therefore the linearized form of the Einstein tensor is given by

$$\begin{aligned}
\mathcal{G}_{\mu\nu}(h) &= R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\bar{g}_{\mu\nu} - (d-1)\Lambda h_{\mu\nu}\\
&= \bar{\nabla}_\alpha\bar{\nabla}_{(\mu}h^{\alpha}_{\nu)} - \frac{1}{2}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta} - \bar{\Box}h\right)\\
&\quad - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h + \frac{1}{2}(d-1)(\bar{g}_{\mu\nu}h - 2h_{\mu\nu})\Lambda,
\end{aligned} \tag{2.9}$$

where we have used the following linearized Ricci and scalar curvature tensors:

$$\begin{aligned}
R^{(1)}_{\mu\nu} &= \frac{1}{2}\left(\bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\nu\sigma} + \bar{\nabla}^\sigma\bar{\nabla}^\nu h_{\mu\sigma} - \bar{\Box}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\right),\\
R^{(1)} &= -\bar{\Box}h + \bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\sigma\mu} - (d-1)\Lambda h.
\end{aligned} \tag{2.10}$$

2.2 Massless and massive graviton modes

By multiplying Eq. (2.6) with $\bar{g}^{\mu\nu}$ one can find the trace of the linearized equation of motion in terms of covariant derivatives of the linearized scalar curvature tensor in (2.10),

$$z_1R^{(1)} + z_2\bar{\Box}R^{(1)} + z_3\bar{\Box}^2R^{(1)} = 0, \tag{2.11}$$

where

$$\begin{aligned}
z_1 &= -\frac{1}{2}(d-2)\bar{\sigma} - \frac{1}{4}(d-1)(d-2)^2\Lambda\sigma_1 - \frac{1}{8}(d-1)^2(d-2)^3\Lambda\sigma_2 + d(d-1)\Lambda\sigma_3,\\
z_2 &= -\frac{1}{4}(d-2)^2\sigma_1 - \frac{1}{4}(d-1)(d-2)^3\Lambda\sigma_2 + (d-1)\sigma_3 + d(d-1)\Lambda\sigma_4,\\
z_3 &= -\frac{1}{8}(d-2)^3\sigma_2 + (d-1)\sigma_4.
\end{aligned} \tag{2.12}$$

As indicated in [32], in order to avoid the propagating scalar degrees of freedom in AdS$_d$ background we will restrict ourselves to the parameters that satisfy the relation $z_3 = z_2 = 0$ or

$$a_2 = \frac{1}{4(d-1)}\Big(\Big(b_2(d-2)^2 - (d-1)\Big(3d(c_1 + 4c_3(d-1)) + c_2(d^2 + 8d - 8)\Big)\Big)\Lambda - da_1\Big),\quad b_1 = -\frac{d}{4(d-1)}b_2. \tag{2.13}$$

With these conditions, the D'Alembertian operator will be removed from Eq. (2.11) and therefore the trace of the linearized equation of motion reduces to a simpler form, $z_1 R^{(1)} = 0$. We also assume that $z_1 \neq 0$, therefore $R^{(1)}$ must vanish. As noted in [24] we may choose the gauge condition $\bar{\nabla}^\mu h_{\mu\nu} = \bar{\nabla}_\nu h$, which from (2.10) leads to $R^{(1)} = -(d-1)\Lambda h$, and therefore one can set $h = 0$. Consequently the gauge condition for $h_{\mu\nu}$ would be the transverse and traceless gauge $\bar{\nabla}^\mu h_{\mu\nu} = h = 0$. The linearized Ricci and Einstein tensors in this transverse traceless gauge become

$$R^{(1)}_{\mu\nu} = d\Lambda h_{\mu\nu} - \frac{1}{2}\bar{\Box} h_{\mu\nu}, \quad \mathcal{G}_{\mu\nu} = \Lambda h_{\mu\nu} - \frac{1}{2}\bar{\Box} h_{\mu\nu}, \tag{2.14}$$

and the linearized equation of motion (2.6) simplifies to

$$\begin{aligned}\mathcal{E}^L_{\mu\nu} = &-\frac{\sigma_2}{8}\bar{\Box}^3 h_{\mu\nu} + \frac{1}{4}(\sigma_1 + 3\Lambda\sigma_2)\bar{\Box}^2 h_{\mu\nu} \\ &- \frac{1}{2}(\bar{\sigma} + 2\Lambda\sigma_1 + 3\Lambda^2\sigma_2)\bar{\Box} h_{\mu\nu} \\ &+ \Lambda(\bar{\sigma} + \Lambda\sigma_1 + \Lambda^2\sigma_2) h_{\mu\nu}.\end{aligned} \tag{2.15}$$

As we see, this equation depends on three parameters $\bar{\sigma}$, σ_1 and σ_2 defined in Eq. (2.7). The linearized equation of motion (2.15) now can be rewritten as

$$-\frac{\sigma_2}{8}(\bar{\Box} - 2\Lambda)(\bar{\Box} - 2\Lambda - M_+^2)(\bar{\Box} - 2\Lambda - M_-^2) h_{\mu\nu} = 0, \tag{2.16}$$

so that the massless and massive modes satisfy the following Klein–Gordon equations in AdS_d background:

$$\begin{aligned}&(\bar{\Box} - A) h^0_{\mu\nu} = 0, \\ &(\bar{\Box} - 2\Lambda - M_+^2) h^{M_+}_{\mu\nu} = 0, \\ &(\bar{\Box} - 2\Lambda - M_-^2) h^{M_-}_{\mu\nu} = 0,\end{aligned} \tag{2.17}$$

where for the values of the masses

$$M_\pm^2 = \frac{\sigma_1 \pm \sqrt{\sigma_1^2 - 4\bar{\sigma}\sigma_2}}{\sigma_2}. \tag{2.18}$$

As we see, the parameter space defined by the parameters $\{\sigma, a_1, b_2, c_1, c_2, c_3\}$ now can be considered as a space with parameters $\{\bar{\sigma}, \sigma_1, \sigma_2\}$ when we study the mass of the graviton modes. In order to have a free tachyon condition we must restrict ourselves to $M_\pm^2 \geq 0$ together with $\sigma_1^2 \geq 4\bar{\sigma}\sigma_2$. We have summarized the analysis of these conditions in Table 1. This table shows that the only allowed regions (shown by asterisk) are those with all values of $\{\bar{\sigma}, \sigma_1, \sigma_2\}$ positive or all negative.

There are special subspaces in this three-parameter space:

- At $\sigma_1 = \bar{\sigma} = 0$ and for $\sigma_2 \neq 0$ in this parameter space, $M_\pm^2 = 0$. This corresponds to a tri-critical point where two massive modes degenerate into the massless one. At this point the massive gravitons are replaced by new solutions, called “log” and “log^2” ghost modes; for example see [7]. The linearized equation of motion at this point has a simple form of an equation of motion for a spin-2 version of the rank-three scalar field, i.e. $\mathcal{G}_{\mu\nu}(\mathcal{G}(\mathcal{G}(h))) = 0$.
- One can find another critical subspace in the parameter space as ($\sigma_2 \neq 0$, $\bar{\sigma} \neq 0$) at $\sigma_1^2 = 4\bar{\sigma}\sigma_2$, for which, in this case, two massive gravitons degenerate into each other, i.e. $M_+^2 = M_-^2 = \sigma_1/\sigma_2$.
- Moreover, we have another critical subspace which is defined by ($\sigma_2 \neq 0$, $\sigma_1 \neq 0$) at $\bar{\sigma} = 0$, where one of the massive modes degenerates into the massless mode, $M_- = 0$ and $M_+ = 2\sigma_1/\sigma_2$. In this critical line, the degenerate graviton is a logarithmic ghost mode.
- In the special situation when $\sigma_2 = 0$ the linearized equation of motion reduces to

$$\begin{aligned}\mathcal{E}^L_{\mu\nu} &= \frac{1}{4}\sigma_1\bar{\Box}^2 h_{\mu\nu} - \frac{1}{2}(\bar{\sigma} + 2\Lambda\sigma_1)\bar{\Box} h_{\mu\nu} \\ &\quad + \Lambda(\bar{\sigma} + \Lambda\sigma_1) h_{\mu\nu}, \\ &= \frac{\sigma_1}{4}(\bar{\Box} - 2\Lambda)(\bar{\Box} - 2\Lambda - \hat{M}^2) h_{\mu\nu} = 0,\end{aligned} \tag{2.19}$$

where we have just one massive mode with $\hat{M}^2 = 2\bar{\sigma}/\sigma_1$. The stability holds here when both the σ and the σ_1 parameters are positive or negative.

As we mentioned in introduction, there will be a unitary subspace if and only if the ghost-like logarithmic modes at the critical points are truncated by imposing certain boundary conditions [6–10]. But it should be noted that the unitary truncation method is valid only in free theories at the linear level [11].

Table 1 Tachyon-free conditions in parameter space

$\sigma_1^2 \geq 4\bar{\sigma}\sigma_2$	σ_1	σ_2	$\bar{\sigma}$
$(M_+^2 > 0, M_-^2 > 0)^*$	+	+	+
$(M_+^2 > 0, M_-^2 > 0)^*$	−	−	−
$(M_+^2 < 0, M_-^2 < 0)$	+	−	−
$(M_+^2 < 0, M_-^2 < 0)$	−	+	+
$(M_+^2 > 0, M_-^2 < 0)$	+	+	−
$(M_+^2 > 0, M_-^2 < 0)$	−	+	−
$(M_+^2 < 0, M_-^2 > 0)$	+	−	+
$(M_+^2 < 0, M_-^2 > 0)$	−	−	+

2.3 Conserved charges

In order to obtain the conserved charges corresponding to the symmetries of the theory, following [29,30,33], we may assume a Killing vector ξ_ν and use the linearized equation of motion to write $\xi_\nu \mathcal{E}_L^{\mu\nu}$ as a surface integral. We use this method to find the mass of asymptotically Schwarzschild–AdS black holes in Ricci cubic gravity.

In the Abbott–Deser method [29] the linearized equation of motion $\mathcal{E}^L_{\mu\nu}$ is considered as an effective energy-momentum tensor. This allows us to compute the conserved charges Q^μ as follows:

$$Q^\mu(\xi) = \int_\Sigma d^{d-1}x\sqrt{-\bar{g}}\xi_\nu\mathcal{E}^{\mu\nu}_L, \tag{2.20}$$

where Σ is a spatial $(d-1)$ dimensional hypersurface. For calculating the conserved charges, one can show that the integrand can be written as a divergence of a two-form i.e. $\xi_\nu\mathcal{E}^{\mu\nu}_L = \bar{\nabla}_\nu\mathcal{F}^{\mu\nu}$. Therefore the integral in (2.20) reduces to a surface integral at spatial infinity,

$$Q^\mu(\xi) = \int_{\partial\Sigma} dS_\alpha\mathcal{F}^{\mu\alpha}, \tag{2.21}$$

where $\partial\Sigma$ is the $(d-2)$ dimensional boundary of Σ. The conserved charge associated to the RCG can be found by this method from the linearized equation of motion (2.6) as

$$\begin{aligned} Q^\mu(\xi) = \frac{1}{4\Omega_{d-2}G_d}&\int_\Sigma d^{d-1}x\sqrt{-\bar{g}} \\ &\times\Big(\bar{\sigma}\xi_\nu\mathcal{G}^{\mu\nu}(h) + \sigma_1\xi_\nu\mathcal{G}^{\mu\nu}(\mathcal{G}(h)) \\ &+\sigma_2\xi_\nu\mathcal{G}^{\mu\nu}(\mathcal{G}(\mathcal{G}(h))) \\ &+\sigma_3\xi_\nu(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + (d-1)\Lambda\bar{g}^{\mu\nu})R^{(1)} \\ &+\sigma_4\xi_\nu(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + (d-1)\Lambda\bar{g}^{\mu\nu})\bar{\Box}R^{(1)}\Big), \end{aligned} \tag{2.22}$$

where we have found this result by generalizing the approach of [7] to the d dimensional space-time. The overall factor is chosen for future proposes in computing the mass of black hole solutions. Equation (2.22) is written into the form of $\bar{\nabla}_\nu\mathcal{F}^{\mu\nu}$ through the following relations:

$$\begin{aligned} \xi_\nu\mathcal{G}^{\mu\nu}(h) = \bar{\nabla}_\rho\Big(&\xi_\nu\bar{\nabla}^{[\mu}h^{\rho]\nu} + \xi^{[\mu}\bar{\nabla}^{\rho]}h + h^{\nu[\mu}\bar{\nabla}^{\rho]}\xi_\nu \\ &-\xi^{[\mu}\bar{\nabla}_\nu h^{\rho]\nu} + \frac{1}{2}h\bar{\nabla}^\mu\xi^\rho\Big), \end{aligned} \tag{2.23a}$$

$$\begin{aligned} \xi_\nu\mathcal{G}^{\mu\nu}(\mathcal{G}(h)) = \bar{\nabla}_\rho\Big(&\xi_\nu\bar{\nabla}^{[\mu}\mathcal{G}^{\rho]\nu}(h) + \xi^{[\mu}\bar{\nabla}^{\rho]}\mathcal{G}(h) \\ &+\mathcal{G}^{\nu[\mu}(h)\bar{\nabla}^{\rho]}\xi_\nu - \xi^{[\mu}\bar{\nabla}_\nu\mathcal{G}^{\rho]\nu}(h) \\ &+\frac{1}{2}\mathcal{G}(h)\bar{\nabla}^\mu\xi^\rho\Big), \end{aligned} \tag{2.23b}$$

$$\begin{aligned} \xi_\nu\mathcal{G}^{\mu\nu}(\mathcal{G}(\mathcal{G}(h))) = \bar{\nabla}_\rho\Big(&\xi_\nu\bar{\nabla}^{[\mu}\mathcal{G}^{\rho]\nu}(\mathcal{G}(h)) + \xi^{[\mu}\bar{\nabla}^{\rho]}\mathcal{G}(\mathcal{G}(h)) \\ &+\mathcal{G}^{\nu[\mu}(\mathcal{G}(h))\bar{\nabla}^{\rho]}\xi_\nu - \xi^{[\mu}\bar{\nabla}_\nu\mathcal{G}^{\rho]\nu}\mathcal{G}(h) \\ &+\frac{1}{2}\mathcal{G}(\mathcal{G}(h))\bar{\nabla}^\mu\xi^\rho\Big), \end{aligned} \tag{2.23c}$$

$$\begin{aligned} &\xi_\nu(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + (d-1)\Lambda\bar{g}^{\mu\nu})R^{(1)} \\ &= -\frac{4}{d-2}\bar{\nabla}_\rho\Big(\xi^{[\mu}\bar{\nabla}^{\rho]}\mathcal{G}(h) + \frac{1}{2}\mathcal{G}(h)\bar{\nabla}^\mu\xi^\rho\Big), \end{aligned} \tag{2.23d}$$

$$\begin{aligned} &\xi_\nu(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + (d-1)\Lambda\bar{g}^{\mu\nu})\bar{\Box}R^{(1)} \\ &= -\frac{4}{d-2}\bar{\nabla}_\rho\Big(\xi^{[\mu}\bar{\nabla}^{\rho]}\bar{\Box}\mathcal{G}(h) + \frac{1}{2}\bar{\Box}\mathcal{G}(h)\bar{\nabla}^\mu\xi^\rho\Big), \end{aligned} \tag{2.23e}$$

where $\mathcal{G}_{\mu\nu}(h)$ was introduced in Eq. (2.9). Finally the conserved quantities can be found from the following relation:

$$\begin{aligned} Q^\mu(\xi) = &\frac{1}{4\Omega_{d-2}G_d}\int_\Sigma d^{d-1}x\sqrt{-\bar{g}}\Big(\bar{\sigma}\xi_\nu\mathcal{G}^{\mu\nu}(h) \\ &+\sigma_1\xi_\nu\mathcal{G}^{\mu\nu}(\mathcal{G}(h)) + \sigma_2\xi_\nu\mathcal{G}^{\mu\nu}(\mathcal{G}(\mathcal{G}(h)))\Big) \\ &-\frac{1}{4\Omega_{d-2}G_d}\int_{\partial\Sigma} dS_\rho\sqrt{-\bar{g}}\Big(\frac{4\sigma_3}{d-2}\Big(\xi^{[\mu}\bar{\nabla}^{\rho]}\mathcal{G}(h) \\ &+\frac{1}{2}\mathcal{G}(h)\bar{\nabla}^\mu\xi^\rho\Big) + \frac{4\sigma_4}{d-2}(\xi^{[\mu}\bar{\nabla}^{\rho]}\bar{\Box}\mathcal{G}(h) \\ &+\frac{1}{2}\bar{\Box}\mathcal{G}(h)\bar{\nabla}^\mu\xi^\rho)\Big). \end{aligned} \tag{2.24}$$

We will use this relation to compute the mass of the black holes in AdS_d space-time.

3 Auxiliary field formalism

In this section we are going to rewrite the Ricci cubic action into the form of Fierz–Pauli massive action for spin-2 fields. Writing in this form we will be able again to calculate the mass of the graviton modes. We will do this by employing the auxiliary field formalism.

For this purpose we need to reformulate the six-derivative action (2.1) using the auxiliary fields which produce an action with just second order derivative terms.

To do this, we need to introduce two rank-two auxiliary fields $(f_{\mu\nu}, \lambda_{\mu\nu})$ [7,31,34]. Let us start from the following action[2]:

$$\begin{aligned} S = \frac{1}{\kappa^2}\int d^dx\sqrt{-g}\Big(&\sigma R - 2\Lambda_0 + \chi_1 f^{\alpha\beta}R_{\alpha\beta} + \chi_2 fR \\ &+\chi_3 f^{\alpha\beta}\lambda_{\alpha\beta} + \chi_4 f\lambda + \chi_5\lambda_{\alpha\beta}\lambda^{\alpha\beta} + \chi_6\lambda^2 \\ &+\chi_7\nabla^\mu\lambda^{\alpha\beta}\nabla_\mu\lambda_{\alpha\beta} + \chi_8\nabla^\mu\lambda\nabla_\mu\lambda + \chi_9\nabla_\alpha\lambda^{\alpha\beta}\nabla^\mu\lambda_{\mu\beta} \\ &+\chi_{10}\nabla^\mu\lambda\nabla^\nu\lambda_{\mu\nu} + \chi_{11}\nabla_\beta\lambda_{\alpha\mu}\nabla^\mu\lambda^{\alpha\beta} + \chi_{12}\lambda^3 \\ &+\chi_{13}\lambda\lambda_{\mu\nu}\lambda^{\mu\nu} + \chi_{14}\lambda_{\alpha\beta}\lambda^{\beta\mu}\lambda_\mu{}^\alpha\Big), \end{aligned} \tag{3.1}$$

[2] Note that our choice for those terms in 3.1 with covariant derivative of $\lambda_{\mu\nu}$, differs from the choice of [7]. We have considered all possible terms and we do not need to add extra boundary terms like those which appeared in [7].

where f and λ are traces of the auxiliary fields. We can find the unknown coefficients by computing the equations of motion for auxiliary fields in d dimensions as follows:

$$\lambda_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} - \frac{1}{2(d-1)}Rg_{\mu\nu}\right), \tag{3.2a}$$

$$\begin{aligned} f_{\mu\nu} = & \frac{1}{(d-2)(d-1)}\Big((d-1)\Big(2(\chi_5+\chi_{13}\lambda)\lambda_{\mu\nu} \\ & + 3\chi_{14}\lambda_\mu{}^\delta\lambda_{\nu\delta} + 2\chi_7\Box\lambda_{\mu\nu} + 2\chi_9\nabla_{(\mu}\nabla_\delta\lambda_{\nu)}{}^\delta \\ & + \chi_{10}\nabla_\mu\nabla_\nu\lambda + 2\chi_{11}\nabla_\delta\nabla_{(\mu}\lambda_{\nu)}{}^\delta\Big) \\ & - g_{\mu\nu}\Big((2(\chi_5+\chi_6) + (3\chi_{12}+2\chi_{13})\lambda)\lambda \\ & + (\chi_{13}+3\chi_{14})\lambda_{\alpha\beta}\lambda^{\alpha\beta} + \big(\chi_{10}+2(\chi_7+\chi_8)\big)\Box\lambda \\ & - (\chi_{10}+2(\chi_9+\chi_{11}))\nabla_\alpha\nabla_\beta\lambda^{\alpha\beta}\Big)\Big), \end{aligned} \tag{3.2b}$$

where we have fixed

$$\chi_1 = 1, \quad \chi_2 = -\frac{1}{2}, \quad \chi_3 = -(d-2), \quad \chi_4 = (d-2), \tag{3.3}$$

by using the freedom in scaling of the fields and demanding that the equation of motion from variation of $f_{\alpha\beta}$ gives a value of the auxiliary field $\lambda_{\mu\nu}$ equal to the Schouten tensor in d dimensions. By inserting the above results into the Lagrangian (3.1) and comparing with the Lagrangian in the original action (2.1) one finds the following values:

$$\begin{aligned} & \chi_5 = a_1(d-2)^2, \quad \chi_6 = 4a_2(d-1)^2 + a_1(3d-4), \\ & \chi_7 = b_2(d-2)^2, \\ & \chi_9 = -\chi_{10} - \chi_8 + b_2(3d-4) + 4b_1(d-1)^2, \\ & \chi_{11} = 0, \\ & \chi_{12} = c_1(4d-6) + 2(d-1)(4c_3(d-1)^2 + c_2(3d-4)), \\ & \chi_{13} = (3c_1 + 2c_2(d-1))(d-2)^2, \quad \chi_{14} = c_1(d-2)^3, \end{aligned} \tag{3.4}$$

where we see that all coefficients have been fixed except χ_8 and χ_{10}. In the next section we will show that we are able to fix these remaining coefficients too.

3.1 Graviton mass spectrum

Now we can expand the new action (3.1) around the AdS_d maximally space up to the second order of field perturbations. The perturbation of auxiliary fields around their background values can be defined through a linear combination of two fluctuating fields $k_{1\mu\nu}$ and $k_{2\mu\nu}$ together with the background metric perturbation $h_{\mu\nu}$, i.e.

$$\begin{aligned} \lambda_{\mu\nu} &= \frac{\Lambda}{2}(\bar{g}_{\mu\nu} + \kappa h_{\mu\nu}) + \kappa k_{1\mu\nu}, \\ f_{\mu\nu} &= \zeta\Lambda(\bar{g}_{\mu\nu} + \kappa h_{\mu\nu}) + \kappa k_{2\mu\nu}, \\ \zeta &= -\frac{2\Lambda(d-1)}{d-2}(2(a_1+da_2) \\ & \quad + 3(d-1)(c_1 + d(c_2 + dc_3))\Lambda), \end{aligned} \tag{3.5}$$

where the coefficients are chosen so that the background values satisfy Eqs. (3.2a) and (3.2b). By expanding (3.1) around these background fields up to the second order of perturbations and by substituting the following expressions for the Ricci tensor:

$$\begin{aligned} R_{\mu\nu} &= R^{(0)}_{\mu\nu} + \kappa R^{(1)}_{\mu\nu} + \kappa^2 R^{(2)}_{\mu\nu}, \\ R^{(0)}_{\mu\nu} &= \Lambda(d-1)\bar{g}_{\mu\nu}, \\ R^{(1)}_{\mu\nu} &= -\frac{1}{2}(\Box h_{\mu\nu} + \nabla_\nu\nabla_\mu h - \nabla^\alpha\nabla_\mu h_{\nu\alpha} - \nabla^\alpha\nabla_\nu h_{\mu\alpha}), \\ R^{(2)}_{\mu\nu} &= \frac{1}{4}\Big(\nabla_\mu h^{\alpha\beta}\nabla_\nu h_{\alpha\beta} + \nabla^\alpha h_{\mu\nu}(2\nabla_\beta h_\alpha{}^\beta - \nabla_\alpha h) \\ & \quad + 2(\nabla_\beta h_{\nu\alpha} - \nabla_\alpha h_{\nu\beta})\nabla^\beta h_\mu{}^\alpha \\ & \quad + 2(\nabla_\alpha h - 2\nabla_\beta h_\alpha{}^\beta)\nabla_{(\mu}h_{\nu)}{}^\alpha \\ & \quad + 2h^{\alpha\beta}\Big(\nabla_\beta\nabla_\alpha h_{\mu\nu} - 2\nabla_\beta\nabla_{(\mu}h_{\nu)\alpha} + \nabla_\mu\nabla_\nu h_{\alpha\beta}\Big)\Big), \end{aligned} \tag{3.6}$$

we will obtain the following Lagrangian:

$$\begin{aligned} \mathcal{L}^{(2)} = & -\frac{1}{2}\bar{\sigma}h^{\mu\nu}\mathcal{G}_{\mu\nu}(h) + \xi_1 k_1^{\mu\nu}\mathcal{G}_{\mu\nu}(k_1) + \xi_2 k_2^{\mu\nu}\mathcal{G}_{\mu\nu}(h) \\ & + \xi_3(k_1^{\mu\nu}k_{1\mu\nu} - k_1k_1) + \xi_4(k_1^{\mu\nu}k_{2\mu\nu} - k_1k_2), \end{aligned} \tag{3.7}$$

where $\mathcal{G}_{\mu\nu}(h)$ is the linearized Einstein tensor (2.9) or equivalently

$$\begin{aligned} \mathcal{G}_{\mu\nu}(h) = & -\frac{1}{2}\Big(\Box h_{\mu\nu} + \nabla_\nu\nabla_\mu h - 2\nabla_{(\mu}\nabla^\rho h_{\rho\nu)} \\ & - 2\Lambda h_{\mu\nu} - (d-3)\Lambda\eta_{\mu\nu}h\Big). \end{aligned} \tag{3.8}$$

To find $\mathcal{G}_{\mu\nu}(k_1)$ one needs to replace $h_{\mu\nu}$ with $k_{1\mu\nu}$ in the above equation. The coefficients in the Lagrangian (3.7) are given by

$$\begin{aligned} \bar{\sigma} = & \, \sigma - \frac{1}{2}(d-2)^2\Lambda(a_1 - d\Lambda b_2) - \frac{1}{2}(d-1) \\ & \times \Big(3(2-2d+d^2)c_1 + d(-2+2d+d^2)c_2 \\ & + 6d^2(d-1)c_3\Big)\Lambda^2, \end{aligned} \tag{3.9}$$

$$\begin{aligned} & \xi_1 = 2b_2(d-2)^2, \quad \xi_2 = 1, \quad \xi_4 = -d+2, \\ & \xi_3 = (d-2)^2(a_1 + (d-1)(3c_1 + dc_2)\Lambda - 2\Lambda b_2), \end{aligned}$$

where for computing these coefficients we have used the relation between the cosmological constant Λ and the cosmological parameter Λ_0 in equation (2.5). We have also used the constraints in (2.13). In order to write the Lagrangian in the specific form in Eq. (3.7), we need to fix the remaining unfixed coefficients as

$$\chi_8 = -b_2(d-2)^2, \quad \chi_{10} = 2b_2(d-2)^2. \tag{3.10}$$

We can go further and write the Lagrangian (3.7) as a diagonalized form by the following field redefinitions:

$$\begin{aligned} h_{\mu\nu} &= h'_{\mu\nu} + \frac{2b_2(d-2)M_\pm^2}{\bar\sigma} k'_{1\mu\nu} + \frac{1}{\bar\sigma} k'_{2\mu\nu}, \\ k_{1\mu\nu} &= k'_{1\mu\nu} - \frac{M_\pm^2}{2(d-2)\bar\sigma} k'_{2\mu\nu}, \\ k_{2\mu\nu} &= k'_{2\mu\nu} + 2b_2(d-2)M_\pm^2 k'_{1\mu\nu}, \end{aligned} \tag{3.11}$$

which immediately gives rise to

$$\begin{aligned} \mathcal{G}_{\mu\nu}(h) &= \mathcal{G}_{\mu\nu}(h') + \frac{2b_2(d-2)M_\pm^2}{\bar\sigma}\mathcal{G}_{\mu\nu}(k'_1) + \frac{1}{\bar\sigma}\mathcal{G}_{\mu\nu}(k'_2), \\ \mathcal{G}_{\mu\nu}(k_1) &= \mathcal{G}_{\mu\nu}(k'_1) - \frac{M_\pm^2}{2(d-2)\bar\sigma}\mathcal{G}_{\mu\nu}(k'_2). \end{aligned} \tag{3.12}$$

Thus, we find a linear combination of the massive Fierz–Pauli Lagrangians which contains a massless spin-2 field $h'_{\mu\nu}$ and two massive spin-2 fields $k'_{1\mu\nu}$ and $k'_{2\mu\nu}$ with $M_\pm^2$ mass squares, respectively,

$$\begin{aligned} \mathcal{L}^{(2)} = &-\frac{1}{2}\bar\sigma h'^{\mu\nu}\mathcal{G}_{\mu\nu}(h') + \frac{4(d-2)^2 b_2}{\bar\sigma}(\bar\sigma + b_2 M_\pm^4) \\ &\times \left(\frac{1}{2}k'^{\mu\nu}_1\mathcal{G}_{\mu\nu}(k'_1) - \frac{1}{4}M_\mp^2\left(k'^{\mu\nu}_1 k'_{1\mu\nu} - k'^2_1\right)\right) \\ &+ \frac{1}{\bar\sigma^2}(\bar\sigma + b_2 M_\pm^4)\Big(\frac{1}{2}k'^{\mu\nu}_2\mathcal{G}_{\mu\nu}(k'_2) \\ &- \frac{1}{4}M_\pm^2(k'^{\mu\nu}_2 k'_{2\mu\nu} - k'^2_2)\Big). \end{aligned} \tag{3.13}$$

The values of these masses obtained confirm exactly the values of mass we have found from Eq. (2.18) by linearizing the equation of motion. In order to have a ghost-free theory we need all kinetic terms to have the same sign. As we see from (3.13) for $\bar\sigma \neq 0$ this is impossible and we always have a rank-two ghost field. This is a general property for higher derivative gravity theories; this has been reported in various papers; for example see [34].

The holographic studies of critical gravities show that the dual gauge theories are log CFTs; for example see [32] and [34]. For RCG we have found the set of these critical points at the end of Sect. 2.2.

3.2 Energy of the linear excitations

Using the linearized form of the Lagrangian in (3.13) we are able to compute the energy of graviton modes by constructing the Hamiltonian. Let us redefine $h'_{\mu\nu}$, the massless mode, by $\psi^0_{\mu\nu}$ and massive modes, $k'_{1\mu\nu}$ and $k'_{2\mu\nu}$, by $\psi^\pm_{\mu\nu}$ as follows:

$$h'_{\mu\nu}=\psi^0_{\mu\nu}, \quad k'_{1\mu\nu} = \frac{\bar\sigma}{2b_2(d-2)M_\pm^2}\psi^+_{\mu\nu}, \quad k'_{2\mu\nu}=\bar\sigma\psi^-_{\mu\nu}, \tag{3.14}$$

and calculate the Hamiltonian by the Ostrogradsky formalism. We recall that the fields are fixed in the transverse and traceless gauge. The Hamiltonian is given by

$$\begin{aligned} H = \frac{1}{2\kappa^2}\int d^{d-1}x\sqrt{-\bar g}\Big[&-\bar\sigma \dot h'_{\mu\nu}\bar\nabla^0 h'^{\mu\nu} + \frac{4b_2(d-2)^2}{\bar\sigma m^4} \\ &\times\left(\bar\sigma + \frac{b_2 M_\pm^4}{m^4}\right)\dot k'_{1\mu\nu}\bar\nabla^0 k'^{\mu\nu}_1 \\ &+ \frac{1}{\bar\sigma^2}\left(\bar\sigma + \frac{b_2M_\pm^4}{m^4}\right)\dot k'_{2\mu\nu}\bar\nabla^0 k'^{\mu\nu}_2 - \mathcal{L}^{(2)}\Big]. \end{aligned} \tag{3.15}$$

Therefore the on-shell energies of the linearized modes are

$$E^0 = -\frac{\bar\sigma}{2\kappa^2}\int d^{d-1}x\sqrt{-\bar g}\dot\psi^0_{\mu\nu}\bar\nabla^0\psi^{0\mu\nu}, \tag{3.16a}$$

$$E^{M_\pm} = \frac{1}{2\kappa^2}\left(\bar\sigma + \frac{b_2M_\pm^4}{m^4}\right)\int d^{d-1}x\sqrt{-\bar g}\dot\psi^\pm_{\mu\nu}\bar\nabla^0\psi^{\pm\mu\nu}. \tag{3.16b}$$

To have ghost-free modes, the energy of the massless and massive gravitons should have the same sign in Eqs. (3.16a) and (3.16b). This is equivalent to requiring that all kinetic terms in the linear action (3.13) have the same sign. As we told before, in general we have ghost modes in this theory except at the critical points. The results here have been observed already for $d = 3$ in [7].

4 The boundary stress tensor

We showed that the generic Ricci cubic curvature theory in arbitrary d dimensions admits a reformulation by using two auxiliary fields in a two-derivative action. Variation of this action (3.1) produces the following boundary terms:

$$\begin{aligned} \delta\mathcal{S}^{(b)} = \frac{1}{\kappa^2}\int d^{d-1}x\sqrt{-\gamma}\Big(&\mathcal{B}^{1\,\beta}_{\alpha}\delta\lambda^\alpha{}_\beta \\ &+ \mathcal{B}^2_{\alpha\beta}\delta g^{\alpha\beta} + \mathcal{B}^3_{\alpha\beta\delta}\nabla^\alpha\delta g^{\beta\delta}\Big), \end{aligned}$$

$$\mathcal{B}_\alpha^{1\beta} = n^\mu\Big(2\chi_7\nabla_\mu\lambda_\alpha{}^\beta + g_{\alpha\mu}(\chi_{10}\nabla^\beta\lambda + 2\chi_9\nabla^\delta\lambda^\beta{}_\delta) + g_\alpha{}^\beta(\chi_{10}\nabla_\delta\lambda_\mu{}^\delta + 2\chi_8\nabla_\mu\lambda)\Big), \tag{4.1a}$$

$$\begin{aligned}\mathcal{B}^2_{\alpha\beta} = \frac{1}{2}n^\mu\Big(&\chi_1(\nabla_\mu f_{\alpha\beta} - 2\nabla_\beta f_{\mu\alpha} + g_{\alpha\beta}\nabla_\delta f_\mu{}^\delta)\\ &+2\chi_2(g_{\alpha\beta}\nabla_\mu f - g_{\alpha\mu}\nabla_\beta f)\\ &+4\chi_7(\lambda_\mu{}^\delta\nabla_\beta\lambda_{\alpha\delta} - \lambda_\alpha{}^\delta\nabla_\beta\lambda_{\mu\delta})\\ &+2\chi_9(\lambda_{\alpha\mu}\nabla_\delta\lambda_\beta{}^\delta - \lambda_{\alpha\beta}\nabla_\delta\lambda_\mu{}^\delta\\ &+(g_{\alpha\beta}\lambda_\mu{}^\delta - g_{\alpha\mu}\lambda_\beta{}^\delta)\nabla_\theta\lambda_\delta{}^\theta)\\ &+\chi_{10}(2\lambda_{\alpha\mu}\nabla_\beta\lambda + (g_{\alpha\beta}\lambda_\mu{}^\delta - 2g_{\alpha\mu}\lambda_\beta{}^\delta)\nabla_\delta\lambda\\ &-\lambda_{\alpha\beta}\nabla_\mu\lambda)\Big),\end{aligned} \tag{4.1b}$$

$$\mathcal{B}^3_{\alpha\beta\delta} = \frac{1}{2}n^\mu\Big(\chi_1(2f_{\alpha\beta}g_{\delta\mu} - f_{\beta\delta}g_{\alpha\mu} - f_{\alpha\mu}g_{\beta\delta}) + 2(\sigma+\chi_2 f)(g_{\alpha\delta}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\delta})\Big), \tag{4.1c}$$

where n^μ is a vector normal to the boundary and all coefficients are given in Eqs. (3.4) and (3.10). In the computation of the above expressions we have used $f^\mu{}_\nu$ and $\lambda^\mu{}_\nu$ as our fundamental fields; see [31]. In order to have a well-defined variational principle we need the generalized Gibbons–Hawking terms [35,36]. To do this we employ the method that is introduced in [31].

Let us choose the coordinates $x^\mu = (r, x^i)$ corresponding to a slicing of the d dimensional bulk, by the $(d-1)$ dimensional Lorentzian submanifolds for each value of the radial coordinate r. We can make an ADM-like split of the metric as

$$\mathrm{d}s^2 = N^2\mathrm{d}r^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}r)(\mathrm{d}x^j + N^j\mathrm{d}r), \tag{4.2}$$

where γ_{ij} defines the boundary metric while N and N^i denote the lapse and shift functions, respectively. Inserting (4.2) into (4.1c) one finds

$$\begin{aligned}\mathcal{B}^3_{\alpha\beta\delta}\nabla^\alpha\delta g^{\beta\delta} = \delta K_{ij}\Big(&\chi_1 f^{ij} + \gamma^{ij}(2\sigma + 2\chi_2 f^j{}_j + (\chi_1+2\chi_2)f^r{}_r)\Big) - \delta\gamma_{ij}\Big((\sigma + \chi_2 f^k{}_k\\ &+ (\chi_1+\chi_2)f^r{}_r)K^{ij} - \frac{1}{2}n_r\mathcal{D}_i f^{ir}\gamma^{jk}\Big),\end{aligned} \tag{4.3}$$

where $K_{ij} = -\frac{1}{2}\partial_r\gamma_{ij}$ is the extrinsic curvature tensor and $\mathcal{D}$ is the covariant derivative with respect to the boundary metric γ_{ij}. Here we have considered $N=1$ and $N_i=0$ for simplicity but we can always get the generalized results in the final answer, similar to the work of [31]. By using the above result we can read the Gibbons–Hawking terms,

$$\mathcal{S}^{\rm GH} = -\frac{1}{\kappa^2}\int d^{d-1}x\sqrt{-\gamma}\Big(\chi_1 f^{ij}K_{ij} + \big(2\sigma + (\chi_1+2\chi_2)f^r{}_r + 2\chi_2 f^i{}_i\big)K\Big), \tag{4.4}$$

where $K = K^i{}_i$. Now we are able to find the boundary energy-momentum tensor through a variation with respect to the boundary metric γ_{ij},

$$8\pi G_d T^{ij} = \frac{2}{\sqrt{-\gamma}}\frac{\delta\mathcal{S}^{\rm tot}}{\delta\gamma_{ij}}, \quad \delta\mathcal{S}^{\rm tot} = \delta\mathcal{S}^b + \delta\mathcal{S}^{\rm GH}, \tag{4.5}$$

by using the Gibbons–Hawking terms in Eq. (4.4) and the boundary terms (4.1b) and (4.1c),

$$\begin{aligned}4\pi G_d T^{ij} &= \sigma\Big(K^{ij} - K\gamma^{ij}\Big) + \mathcal{T}_1^{ij} + \mathcal{T}_2^{ij} + \mathcal{T}_7^{ij} + \mathcal{T}_9^{ij} + \mathcal{T}_{10}^{ij},\\ \mathcal{T}_1^{ij} &= \frac{1}{4}\chi_1\Big(4sK^{ij} + 2\mathcal{D}_r f^{ij} - 4\mathcal{D}^{(i}h^{j)} - 4f^{(i}{}_k K^{j)k}\\ &\quad + (2\mathcal{D}_r s - 4sK + 4\mathcal{D}_k h^k)\gamma^{ij}\Big),\\ \mathcal{T}_2^{ij} &= \chi_2\Big(sK^{ij} + (\mathcal{D}_r f - sK + \mathcal{D}_r s)\gamma^{ij}\\ &\quad + f(K^{ij} - K\gamma^{ij})\Big),\\ \mathcal{T}_7^{ij} &= 2\chi_7\Big(-(S^2K^{ij} + H_kH^k)K^{ij} - 3H^kH^{(i}K^{j)}{}_k\\ &\quad - K^{(i}{}_l\lambda^{j)k}\lambda^l{}_k - \lambda^{(ik}\mathcal{D}^{j)}H_k + H^k\mathcal{D}^{(i}\lambda_k{}^{j)}\\ &\quad - H^{(i}\mathcal{D}^{j)}S + S(2K^{(i}{}_k\lambda^{j)k} + \mathcal{D}^{(i}H^{j)})\Big),\\ \mathcal{T}_9^{ij} &= \chi_9\Big(\tfrac{1}{2}\mathcal{D}_r(H^iH^j) - 2H^kH^{(i}K^{j)}{}_k - H^iH^jK\\ &\quad + H^{(i}\mathcal{D}_k\lambda^{j)k} + \lambda^{ij}(SK - \mathcal{D}_rS - K_{kl}\lambda^{kl} - \mathcal{D}_kH^k)\\ &\quad + \gamma^{ij}\Big(H^k\mathcal{D}_l\lambda_k{}^l + H_k\mathcal{D}_rH^k - 2H^kH^lK_{kl}\\ &\quad - K(S^2 + H_kH^k) + S(\mathcal{D}_rS + K_{kl}\lambda^{kl} + \mathcal{D}_kH^k)\Big)\Big),\\ \mathcal{T}_{10}^{ij} &= \frac{1}{2}\chi_{10}\Big(-\mathcal{D}_r(S+\lambda)\lambda^{ij} + 2H^{(i}\mathcal{D}^{j)}S\\ &\quad + 2H^{(i}\mathcal{D}^{j)}\lambda + \gamma^{ij}\Big(S\mathcal{D}_r(S+\lambda)\\ &\quad + H^k\mathcal{D}_k(S+\lambda)\Big)\Big),\end{aligned} \tag{4.6}$$

where we have used $\lambda^i{}_r = H^i, \lambda^r{}_r = S, f^i{}_r = h^i, f^r{}_r = s, f^k{}_k = f$ and $\lambda^k{}_k = \lambda$ for simplicity in notation. We also use $\mathcal{D}_r$ as a covariant r-derivative and $\mathcal{D}_i$ as a covariant derivative with respect to the boundary metric,

$$\mathcal{D}_r\lambda^{ij} = \frac{1}{N}\Big(\partial_r\lambda^{ij} - N^k\partial_k\lambda^{ij} + 2\lambda^{k(j}\partial_k N^{i)}\Big),$$
$$\mathcal{D}_r\lambda = \frac{1}{N}(\partial_r\lambda - N^j\partial_j\lambda). \tag{4.7}$$

As has been shown in [31], in order to take into account the non-trivial lapse and shift functions, it is enough to replace all the above fields with the following combinations of auxiliary fields:

$$\hat{\lambda}^{ij} = \lambda^{ij} + 2H^{(i}N^{j)} + sN^iN^j,$$
$$\hat{f}^{ij} = f^{ij} + 2h^{(i}N^{j)} + sN^iN^j,$$
$$\hat{H}^i = N(H^i + SN^i),\ \ \hat{h}^i = N(h^i + sN^i),$$
$$\hat{S} = N^2S,\ \ \hat{s} = N^2s. \tag{4.8}$$

In the next section we will use the boundary stress tensor (4.6) to compute the conserved charges of the RCG for different background space-times. To do this, we decompose the boundary geometry in ADM-like form. Consider the boundary coordinates as $x^i = (t, x^a)$, where the x^a belong to the $d-2$ dimensional space-like hyper-surface σ, the metric on the boundary can be written as

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = -N_0^2\mathrm{d}t^2 + \hat{\gamma}_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t). \tag{4.9}$$

By using a time-like normal vector u^i, we can calculate the conserved charges associated to the Killing vector ξ^i:

$$\mathcal{Q}_\xi = \int_\Sigma d^{d-2}x\sqrt{\hat{\gamma}}u^iT_{ij}\xi^i. \tag{4.10}$$

For example we can find the mass as follows:

$$M = \int_\Sigma d^{d-2}x\sqrt{\hat{\gamma}}N_0T_{ij}u^iu^j. \tag{4.11}$$

We will use this relation to find the mass of different solutions of the RCG such as SAdS and Lifshitz black holes in d dimensions.

In addition to the boundary stress tensor we have two other boundary tensors which achieve by variation with respect to the auxiliary fields λ^i_j and f^i_j,

$$8\pi G_d\ {\tau_1}^i_j = \frac{2}{\sqrt{-\gamma}}\frac{\delta\mathcal{S}^{\mathrm{tot}}}{\delta\lambda^i_j},\quad 8\pi G_d\ {\tau_2}^i_j = \frac{2}{\sqrt{-\gamma}}\frac{\delta\mathcal{S}^{\mathrm{tot}}}{\delta f^i_j}. \tag{4.12}$$

By using the action (4.4) and the boundary terms in Eq. (4.1a) one can find these new boundary tensors,

$$8\pi G_d\ {\tau_1}^i_j = 2\chi_7\mathcal{D}_r\lambda^i_j + 2\chi_8\mathcal{D}_r(\lambda + S)\gamma^i_j$$
$$+\chi_{10}(\mathcal{D}_kH^k - KS + \mathcal{D}_rS)\gamma^i_j,$$
$$8\pi G_d\ {\tau_2}^i_j = -2\chi_2K\gamma^i_j - \chi_1K^i_j. \tag{4.13}$$

In holographic terms, it is well known that the energy-momentum boundary tensor T^{ij} is a holographic response function conjugate to the h_{ij} source. On the other hand the auxiliary field formalism in Sect. 3 and specifically equations (3.5) and (3.11) show that a mixed combination of fluctuations of the auxiliary fields describes the massive graviton modes. So it is probable that a mixture ${\tau_1}^i_j$ and ${\tau_2}^i_j$ plays the role of holographic response functions conjugate to the $k'_{1\mu\nu}$ and $k'_{2\mu\nu}$ (massive graviton modes).

To check this proposal holographically there are several suggestions. For example in [7] and for three dimensional tri-critical gravity the energy-momentum tensor has been expanded in terms of the leading and sub-leading terms in a Fefferman–Graham expansion of the metric and central charges have been computed. One may perform the same calculation for ${\tau_1}^i_j$ and ${\tau_2}^i_j$. We postpone the holographic study of RCG model for future work [57].

5 Schwarzschild–AdS black hole in RCG

In this section we study the SAdS black hole as a solution of the RCG. First we study the thermodynamics of this black hole and then we compute its mass by a renormalized boundary stress tensor.

Let us start with the following AdS$_d$ black hole which is a solution of the equations of motion (for simplicity we have considered a flat boundary space but it is possible to consider spherical or hyperboloid spaces)

$$\mathrm{d}s^2 = \frac{l^2}{f(r)r^2}\mathrm{d}r^2 + \frac{r^2}{l^2}\Big(-f(r)\mathrm{d}t^2 + \sum_{a=1}^{d-2}\delta_{ab}dx^adx^b\Big),$$
$$f(r) = 1 - \Big(\frac{r_0}{r}\Big)^{d-1}. \tag{5.1}$$

Here r_0 is the radius of the horizon and the cosmological parameter Λ_0 is related to value of l, the radius of AdS space-time, through the following relation:

$$\Lambda_0 = \frac{d-1}{2l^6}\Big((d-1)(-(d-6)(d-1)(c_1 + d(c_2 + c_3d))$$
$$+(d-4)(a_1 + a_2d)l^2) - (d-2)l^4\sigma\Big). \tag{5.2}$$

5.1 Black hole thermodynamics

To study the thermodynamics of this black hole we start from temperature and then compute the entropy. The value of the temperature can be read from the Euclidean version of the metric by using the following relation:

$$T = \frac{1}{2\pi}\sqrt{g^{rr}}\partial_r(\sqrt{g_{\tau\tau}})\Big|_{r=r_0} = \frac{(d-1)r_0}{4\pi l^2}. \tag{5.3}$$

To find the entropy we employ two techniques, the free energy and the Wald formula. To compute the free energy we must insert the metric into the Euclidean action,

$$I_E^{\mathrm{BH}}[T] = -\frac{1}{16\pi G_d}\int_0^{1/T} \mathrm{d}\tau \int_{r_0}^{R} \mathrm{d}r \int d^{d-2}x\sqrt{g_E}\mathcal{L} = -\frac{V_{d-2}}{2(d-1)G_d}\frac{r_0^{d-2}}{l^{d-2}}\left(1-\frac{R^{d-1}}{r_0^{d-1}}\right)\bar{\sigma}, \tag{5.4}$$

where $\bar{\sigma}$ is given in Eq. (2.7) and we have considered V_{d-2} as the regulator volume of $d-2$ dimensional flat space. In the above equation R is a regulator for the radial coordinate, which we will send it to infinity later. To remove the divergent part of the above expression we need to subtract the value of the action for the AdS background at temperature T',

$$I_E^{\mathrm{AdS}}[T'] = -\frac{1}{16\pi G_d}\int_0^{1/T'} \mathrm{d}\tau \int_0^{R} \mathrm{d}r \int d^{d-2}x\sqrt{g_E}\mathcal{L} = -\frac{V_{d-2}}{2(d-1)G_d}\frac{r^{d-1}}{r_0 l^{d-2}}\left(1-\frac{r_0^{d-1}}{R^{d-1}}\right)^{\frac{1}{2}}\bar{\sigma}, \tag{5.5}$$

where T' is defined in such a way that the time periodicity of the Euclidean AdS background will be equal to the black hole's one at the regulator surface $r=R$. In other words,

$$\frac{1}{T'} = \frac{1}{T}\sqrt{\frac{g_{tt}^{\mathrm{BH}}}{g_{tt}^{\mathrm{AdS}}}}\Bigg|_{r=R} = \frac{4\pi l^2}{r_0(d-1)}\left(1-\frac{r_0^{d-1}}{R^{d-1}}\right)^{\frac{1}{2}}. \tag{5.6}$$

Finally the free energy is given by

$$F[T] = T\left(I_E^{\mathrm{BH}}[T]-I_E^{\mathrm{AdS}}[T']\right)\Big|_{R\to\infty} = -\frac{V_{d-2}r_0^{d-1}}{8\pi G_d l^d}\bar{\sigma} \tag{5.7}$$

and the entropy by

$$S = -\frac{\partial F}{\partial T} = \frac{1}{2G_d}\left(\frac{4\pi l}{d-1}\right)^{d-2} T^{d-2}\bar{\sigma}. \tag{5.8}$$

It is worth to mention that the same result for the entropy can be found from the well-known Wald formula for the entropy in higher curvature theories of gravity [37]. Starting from

$$\begin{aligned} S^W &= 8\pi\int_H d^{d-2}x^H\sqrt{g^H}g^{\perp}_{\alpha\mu}g^{\perp}_{\beta\nu}\left(\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\mu\nu}}-\nabla_\gamma\frac{\partial\mathcal{L}}{\partial\nabla_\gamma R_{\alpha\beta\mu\nu}}\right),\\ &= \frac{1}{2G_d}\int_H d^{d-2}x^H\sqrt{g^H}\Big(\tfrac{1}{2}\Big(g^{\perp\delta}_{\alpha}g^{\perp\alpha\beta}\\ &\quad\times\left(-3c_1R_\beta{}^\nu R_{\delta\nu}-2R_{\beta\delta}(a_1+c_2R)\right)\\ &\quad-g^{\perp}_{\alpha\beta}g^{\perp\alpha\beta}\left(\sigma+c_2R^{\mu\nu}R_{\mu\nu}+R(2a_2+3c_3R)\right)\\ &\quad+g^{\perp\alpha}{}_{\alpha}\Big(g^{\perp\beta\delta}\left(3c_1R_\beta{}^\nu R_{\delta\nu}+2R_{\beta\delta}(a_1+c_2R)\right)\\ &\quad+g^{\perp\beta}{}_{\beta}\left(\sigma+c_2R^{\mu\nu}R_{\mu\nu}+R(2a_2+3c_3R)\right)\Big)\Big)\\ &\quad-2b_2g^{\perp\alpha}{}_{\alpha}g^{\perp\beta\delta}\nabla_\nu\nabla^\nu R_{\beta\delta}-2b_1g^{\perp\alpha}{}_{\alpha}g^{\perp\beta}{}_{\beta}\nabla_\nu\nabla^\nu R\Big), \end{aligned} \tag{5.9}$$

where $g^{\perp}_{\alpha\beta}$ denotes the metric projection onto the subspace orthogonal to the horizon; one can find the same entropy as (5.8) exactly.

5.2 Mass from renormalized boundary stress tensor

The auxiliary field components can be determined by their field equations (3.2a) and (3.2b) in terms of the SAdS black hole metric as follows:

$$\begin{aligned} \hat{\lambda}^{ij} &= -\frac{1}{2l^2}\gamma^{ij}, \quad \hat{\lambda}^{ir} = \hat{f}^{ir} = 0, \quad \hat{\lambda}^{rr} = -\frac{1}{2l^2}\frac{r^2}{l^2}f(r).\\ \hat{f}^{ij} &= -\frac{2(d-1)}{(d-2)l^4}\gamma^{ij}\Big(3(d-1)(c_1+d(c_2+c_3d))\\ &\quad-2(a_1+a_2d)l^2\Big),\\ \hat{f}^{rr} &= -\frac{2(d-1)}{(d-2)l^4}\Big(3(d-1)(c_1+d(c_2+c_3d))\\ &\quad-2(a_1+a_2d)l^2\Big)\frac{r^2}{l^2}f(r). \end{aligned} \tag{5.10}$$

Using these relations the value of the mass can be computed from Eq. (4.11). To find the mass we need T^{00} from (4.6) together with Eqs. (4.7) and (4.8). The value of the mass with this technique becomes

$$M = \frac{(d-2)V_{d-2}}{4\pi l G_d}\left(\frac{r_0}{l}\right)^{d-1}\left(1-\left(\frac{r}{r_0}\right)^{d-1}\right)\bar{\sigma}, \tag{5.11}$$

which diverges obviously when computed on boundary at $r\to\infty$. To remove this divergence we need to renormalize the energy-momentum tensor as follows:

$$T^{ren}_{ij} = T_{ij} - \frac{d-2}{8\pi G_d l}\bar{\sigma}\gamma_{ij}. \tag{5.12}$$

This can be done by adding a boundary counter-term to the Lagrangian just proportional to the volume of boundary. Consequently the mass is given by

$$M = \frac{(d-2)V_{d-2}}{8\pi l G_d}\left(\frac{r_0}{l}\right)^{d-1}\bar{\sigma}. \tag{5.13}$$

As a check of our results, it is easy to show that the value of mass in (5.13) and the values of entropy (5.8) and temperature (5.3) will satisfy the first law of thermodynamics for black holes i.e. $\mathrm{d}M = T\mathrm{d}S$, if one differentiates mass and entropy with respect to the location of the horizon at $r=r_0$. As another check one may compute the mass of the AdS black hole from the first approach by linearizing the equations of motion i.e. from Eq. (2.24). It can be shown that for asymptotically AdS solutions only the first term in (2.24) has a contribution to the AD mass. This computation reconfirms the value of the mass in Eq. (5.13).

6 Lifshitz vacuum and Lifshitz black hole in RCG

In addition to the SAdS black hole we discussed in the previous section, one can find other interesting solutions such as the Lifshitz vacuum and Lifshitz black hole [38–47]. It should be noted that the Lifshitz solutions with different scalings of space and time have interesting applications as gravity duals of non-relativistic quantum field theories. In this section we will find conditions to have such solutions in RCG. We also use the auxiliary field formalism to compute the mass of Lifshitz black holes.

Let us start from the Lifshitz vacuum as a solution for equations of motion. This background is characterized by a dynamical exponent z, which governs the anisotropy between spatial and temporal scalings i.e. $x \to \lambda x, r \to \lambda^{-1} r$ and $t \to \lambda^z t$,

$$\mathrm{d}s^2 = -\frac{r^{2z}}{l^{2z}}\mathrm{d}t^2 + \frac{l^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{l^2}\left(\sum_{a=1}^{d-2}\delta_{ab}dx^a dx^b\right). \tag{6.1}$$

Moreover, the cosmological parameter Λ_0 and the Lifshitz length l in the metric above can be fixed by the equations of motion and are given by

$$\begin{aligned}
\Lambda_0 = \frac{1}{l^6}\Big(& -\frac{1}{2}(d-2)(d-1)l^4\sigma + \tfrac{1}{2}l^2(z^2+d-2) \\
&\times\big(2+d^2-2(z-2)z-d(3+2z)\big)a_1 + \tfrac{1}{2}l^2\big((2 \\
&-3d+d^2)^2-4(d-2)^2z^2-8(d-2)z^3-4z^4\big)a_2 \\
&+(2-d)(z-1)^2 z\big(12-5z+d(3z-14 \\
&+4d)\big)b_2 + \tfrac{1}{2}\big((d-2)^3(6z-(d-1)) \\
&-6(d-3)(d-2)^2z^2+2(d-2)\big(5+(d-4)d\big)z^3 \\
&+3(d-2)(d-1)z^4+6(d-2)z^5+4z^6\big)c_1 \\
&-\tfrac{1}{2}\big(d^2+2(z-1)^2+d(2z-3)\big)^2\big(2+d^2-4(z \\
&-2)z-d(3+4z)\big)\big(c_3+(z^2+d-2)c_2\big)\Big),
\end{aligned} \tag{6.2a}$$

$$\begin{aligned}
l^2 = \frac{1}{\sigma}\Big(& a_1(z^2+d-2)+a_2\big(d^2+2(z-1)^2 \\
&+d(2z-3)\big) \pm \frac{1}{2}\sqrt{\delta}\Big), \\
\delta = 4\big(& a_1(z^2+d-2)+a_2\big(d^2+2(z-1)^2+d(2z-3)\big)\big)^2 \\
&-4\big((12-36d+39d^2-18d^3+3d^4 \\
&+(-48+96d-60d^2+12d^3)z+(72-84d+24d^2)z^2 \\
&+(-48+24d)z^3+12z^4)c_3+3\big(d^3 \\
&+2(z-1)^2(z^2-2)+d^2(z^2+2z-5) \\
&+d(8-8z-z^2)+2z^3\big)c_2+3(z^2+d-2)^2c_1+(2d^2z \\
&-4dz+6z^2-2d^2z^2-6z^3+4dz^3)b_2\big)\sigma.
\end{aligned} \tag{6.2b}$$

6.1 Lifshitz black hole

Motivated by SAdS solution at $z = 1$ one may choose the following ansatz to write the Lifshitz black hole:

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{r^{2z}}{l^{2z}}f(r)\mathrm{d}t^2 + \frac{l^2}{r^2 f(r)}\mathrm{d}r^2 + \frac{r^2}{l^2}\sum_{a=1}^{d-2}\delta_{ab}dx^a dx^b, \\
f(r) &= 1-\left(\frac{r_0}{r}\right)^{d-1}.
\end{aligned} \tag{6.3}$$

Similar to the Lifshitz background, the above metric is a solution for equations of motion in a special region on the parameter space of the theory. These special values are presented in Eqs. (B.1b)–(B.1h) in Appendix B.

The auxiliary fields in this background are given by

$$\begin{aligned}
\lambda^{ab} &= -\frac{\big((d-1)(d-2)+2z(1-z)+(3-3d+2z)(z-1)(\frac{r_0}{r})^{d-1}}{2(d-1)(d-2)}\frac{\delta^{ab}}{r^2}, \\
\lambda^{tt} &= \frac{\big(1+2(z-2)z+d(2z-1)\big)-(z-1)(1-d+2z)(\frac{r_0}{r})^{d-1}}{2(d-1)l^2\Big(1-(\frac{r_0}{r})^{d-1}\Big)}\frac{l^{2z}}{r^{2z}}, \\
N^2\lambda^{rr} &= \frac{-\big(d-1+2(z-1)z\big)+(3-3d+2z)(z-1)(\frac{r_0}{r})^{d-1}}{2(d-1)l^2}, \\
\lambda^{ar} &= \lambda^{at} = f^{ar} = f^{at} = 0, \\
f^{ab} &= \frac{1}{2(d-2)l^2}\frac{\delta^{ab}}{r^2}\left(f_{10}+f_{11}\frac{r_0^{d-1}}{r^{d-1}}+f_{12}\frac{r_0^{2d-2}}{r^{2d-2}}\right), \\
f^{tt} &= \frac{1}{4(d-2)l^4(1-(\frac{r_0}{r})^{d-1})}\frac{l^{2z}}{r^{2z}}\left(f_{20}+f_{21}\frac{r_0^{d-1}}{r^{d-1}}+f_{22}\frac{r_0^{2d-2}}{r^{2d-2}}\right), \\
N^2 f^{rr} &= -\frac{1}{4(d-2)l^4}\left(f_{30}+f_{31}\frac{r_0^{d-1}}{r^{d-1}}+f_{32}\frac{r_0^{2d-2}}{r^{2d-2}}\right)\frac{l^2}{r^2 f(r)},
\end{aligned} \tag{6.4}$$

where all the constant values f_{10} to f_{32} are given in Eqs. (A.1a)–(A.1i).

6.2 Thermodynamics of Lifshitz black hole

In the same approach as SAdS black holes we can read the value of the temperature from the Euclidean metric,

$$T = \frac{d-1}{4\pi l}\frac{r_0^z}{l^z}. \tag{6.5}$$

To compute the entropy we suppose that the Wald entropy formalism still holds here and its value can be found by putting the black hole solution (6.3) into the Wald formula which we have computed in Eq. (5.9)

$$
\begin{aligned}
S &= \frac{V_{d-2}}{8l^4 G_d}\frac{r_0^{d-2}}{l^{d-2}}\tilde{\sigma},\\
\tilde{\sigma} &= (d-1)\Big((d-1)(3c_1(1-3z)^2 + 6c_2(1+2(d-5)z\\
&\quad + 9z^2) + 12c_3(d-3+3z)^2\\
&\quad + 16(2b_1(1-d+2z) + b_2(3-2d+2z))(z-1))\Big)\\
&\quad + \frac{1}{l^2}\Big(a_1(1-3z) - 2a_2(d-3+3z)\Big) + 4l^4\sigma. \qquad (6.6)
\end{aligned}
$$

Using the first law of thermodynamics for black holes, $\mathrm{d}M = T\,\mathrm{d}S$, we find that the mass is given by

$$
M = \frac{V_{d-2}(d-1)(d-2)}{32\pi l^5 G_d(z+d-2)}\frac{r_0^{z+d-2}}{l^{z+d-2}}\tilde{\sigma}. \qquad (6.7)
$$

As we see the value of the mass leads to the SAdS black hole's mass (5.13), when we insert $z = 1$.

As a general result, we observe that the entropy (6.6) and mass (6.7) for SAdS or Lifshitz black holes in RCG, are both proportional to the critical parameter $\tilde{\sigma}$. Stability of these solutions restricts this critical parameter to positive values, i.e. $\tilde{\sigma} > 0$.

Several investigations in thermodynamical properties of the Lifshitz black holes have been done in other gravitational theories, for example see [48–50].

6.3 Mass from the renormalized boundary stress tensor

To find the mass of Lifshitz black hole from auxiliary field formalism we need to compute the integral in Eq. (4.11) and the value of T^{00} can be found by inserting the values of auxiliary field components in Eq. (6.4) into the relation (4.6). The non-renormalized black hole's mass in this way is given by

$$
\begin{aligned}
M &= -\frac{V_{d-2}}{32\pi G_d\, l^{3+d+z}}\left(1 - \frac{r_0^{d-1}}{r^{d-1}}\right) r^{d+z-2}\\
&\quad \times \left(M_1 + \frac{r_0^{d-1}}{r^{d-1}}M_2 + \frac{r_0^{2d-2}}{r^{2d-2}}M_3\right) \qquad (6.8)\\
&= -\frac{V_{d-2}}{32\pi G_d\, l^{3+d+z}} r^{d+z-2}\Big(M_1\\
&\quad + \frac{r_0^{d-1}}{r^{d-1}}(M_2 - M_1) + \frac{r_0^{2d-2}}{r^{2d-2}}(M_3 - M_2) - M_3\frac{r_0^{3d-3}}{r^{3d-3}}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= -2(d-2)f_{10} + f_{20} - f_{30} + 8(d-2)\\
&\quad \times \left(2b_2(d-2)(z-1)^2 z + l^4\sigma\right),\\
M_2 &= 2f_{11} + f_{21} - f_{31} + 4b_2(z-1)^2\\
&\quad \times \left(\left(4 + d(2d-5)\right)(d-1-2z) - 4(d-2)^2 z\right)\\
&\quad + 16b_1(d-1)^2(d-1-2z)(z-1)^2, \qquad (6.9)\\
M_3 &= 2df_{12} + f_{22} - f_{32} - 4b_2(z-1)^2\\
&\quad \times \left(4 + d(2d-5)\right)(d-1-2z)\\
&\quad - 16b_1(d-1)^2(d-1-2z)(z-1)^2,
\end{aligned}
$$

and this mass diverges as one goes to the boundaries at $r \to \infty$. To have a finite non-zero mass for Lifshitz black holes, the above relation suggests that the only possible massive black holes are those with $z = 1$, $z = d$ and $z = 2d - 1$. The case with $z = 1$ or SAdS black hole has been studied already in the previous section. We now try to find the renormalized mass for two other cases.

Similar to the SAdS black hole we need to find a renormalized boundary energy-momentum tensor here. As noted in [31], there is an ambiguity in choosing the boundary terms. For example we can choose the following scalar tensors to construct the counter-terms on the boundary:

$$
\begin{aligned}
S_{\text{ct}}^{(1)} &= \frac{1}{\kappa^2}\int d^{d-1}x\sqrt{-\gamma} \to M_{\text{ct}}^{(1)}\\
&= \frac{V_{d-2}}{4\pi G_d\, l^{d+z-2}}\left(1 - \frac{r_0^{d-1}}{r^{d-1}}\right) r^{d+z-2}\\
&\quad \times \left(1 + \frac{1}{2}\frac{r_0^{d-1}}{r^{d-1}} + \frac{3}{8}\frac{r_0^{2d-2}}{r^{2d-2}}\right),\\
S_{\text{ct}}^{(2)} &= \frac{1}{\kappa^2}\int d^{d-1}x\sqrt{-\gamma}\lambda^k{}_k \to M_{\text{ct}}^{(2)}\\
&= -\frac{M_{\text{ct}}^{(1)}}{2l^2}\left(d + 2z - 3 - 2(z-1)\frac{r_0^{d-1}}{r^{d-1}}\right), \qquad (6.10)
\end{aligned}
$$

and so forth. One may find different scalars in order to construct the renormalized action. For example if we restrict ourselves to at most cubic terms with at most two covariant derivatives we can choose the following scalars:

$$
\begin{aligned}
&\Big\{ f^k{}_k,\, f^r{}_r,\, \lambda^k{}_k,\, \lambda^r{}_r,\, f^i{}_i\lambda^j{}_j,\, f^k{}_k\lambda^r{}_r,\, f^r{}_r\lambda^j{}_j,\\
&f^r{}_r\lambda^r{}_r,\, f^i{}_j\lambda^j{}_i,\, \lambda^i{}_j\lambda^j{}_i,\, (\lambda^k{}_k)^2,\, (\lambda^r{}_r)^2,\\
&(\lambda^k{}_k)^3,\, (\lambda^r{}_r)^3,\, \lambda^i{}_i\lambda^j{}_k\lambda^k{}_j,\, \lambda^i{}_j\lambda^j{}_k\lambda^k{}_i,\, \lambda^i{}_i\lambda^r{}_r,\\
&(\lambda^i{}_i)^2\lambda^r{}_r,\, \lambda^i{}_i(\lambda^r{}_r)^2,\, \lambda^i{}_j\lambda^j{}_i\lambda^r{}_r \Big\}.
\end{aligned}
$$

Therefore to find a renormalized mass one encounters the ambiguity in choosing the correct combination of terms as indicated in [31]. To find a renormalized mass we will follow the same steps as [31] and fix the coefficients by using the value of the mass in (6.7) which is consistent with the first law of thermodynamics for black holes.

As an example let us start with the following combination of boundary counter-terms, which have been chosen for simplicity:

$$S_{cc}=\frac{1}{\kappa^2}\int d^{d-1}x\sqrt{-\gamma}(\alpha_1+\alpha_2\lambda^k{}_k+\alpha_3(\lambda^k{}_k)^2+\alpha_4\lambda_{ij}\lambda^{ij}+\alpha_5(\lambda^k{}_k)^3+\alpha_6\lambda_{ij}\lambda^{ij}\lambda^k{}_k+\alpha_7\lambda_{ij}\lambda^{jk}\lambda_k{}^i). \tag{6.11}$$

By adding these counter-terms and demanding a finite value for mass equal to the value in (6.7) for $z=d$ and $z=2d-1$ simultaneously, we can fix the unknown coefficients α_1 to α_7 which are given in Appendix C.

7 Ricci cubic gravity in three dimensions

In this section, as an application of our results, we are trying to study the RCG in three dimensions. We will compute the values of central charges corresponding to the dual CFT of the AdS_3 space-time. We also find the BTZ black hole mass and angular momentum from the renormalized energy-momentum tensor.

7.1 Central charges

The central charges of the dual CFT of AdS_3 space-time can be computed by applying the renormalized boundary stress tensor (4.6). We will review and use the method in [31,51].

To find the central charge we need the anomalous behavior of the energy-momentum tensor under the conformal transformation. In light-cone coordinates these transformations are

$$\delta x^+=-\xi^+(x^+),\quad \delta x^-=-\xi^-(x^-), \tag{7.1}$$

and consequently the energy-momentum tensor components transform as [51]

$$\delta T_{++}=\mathcal{L}_\xi T_{++}-\frac{c}{24\pi}\partial_+^3\xi^+,\quad \delta T_{--}=\mathcal{L}_\xi T_{--}-\frac{c}{24\pi}\partial_-^3\xi^-. \tag{7.2}$$

Each transformation contains two parts, a Lie derivative part, which comes from the boundary-preserving diffeomorphisms, and an anomalous part, which comes from the fact that the asymptotic symmetry group of AdS_3 is larger than the boundary-preserving diffeomorphisms.

To compute the central charges from the anomalous terms, let us start with the AdS_3 metric written in the light-cone coordinates,

$$\mathrm{d}s^2=\frac{l^2}{r^2}\mathrm{d}r^2-r^2\mathrm{d}x^+\mathrm{d}x^-, \tag{7.3}$$

together with the Brown and Henneaux boundary conditions [52] to define the asymptotic behavior of the metric,

$$g_{+-}=-\frac{r^2}{2}+\mathcal{O}(1),\quad g_{++}=\mathcal{O}(1),\quad g_{--}=\mathcal{O}(1),$$
$$g_{rr}=\frac{l^2}{r^2}+\mathcal{O}(\frac{1}{r^4}),\quad g_{+r}=\mathcal{O}(\frac{1}{r^3}),\quad g_{-r}=\mathcal{O}(\frac{1}{r^3}). \tag{7.4}$$

The diffeomorphisms which respect to the Brown and Henneaux boundary conditions are parametrized by the following vector fields[3]:

$$X^\pm=\xi^\pm(x^\pm)+\frac{l^2}{2r^2}\partial_\mp^2\xi^\mp(x^\mp),\quad X^r=-\frac{r}{2}\left(\partial_+\xi^++\partial_-\xi^-\right). \tag{7.5}$$

These diffeomorphisms (asymptotic symmetry group of AdS_3) do not belong to the class of boundary-preserving diffeomorphism [31] and therefore they will produce anomalous terms similar to those in (7.2).

To compute the transformation of the boundary energy-momentum tensor we need to compute the transformation of the boundary metric, the extrinsic curvature and the auxiliary fields components under (7.5). For example we have

$$\delta_X g_{\mu\nu}=\mathcal{L}_X g_{\mu\nu}=X^\rho\partial_\rho g_{\mu\nu}+\partial_\mu X^\rho g_{\rho\nu}+\partial_\nu X^\rho g_{\mu\rho},$$
$$\delta_X\lambda^{\mu\nu}=\mathcal{L}_X\lambda^{\mu\nu}=X^\rho\partial_\rho\lambda^{\mu\nu}-\lambda^{\mu\rho}\partial_\rho X^\nu-\lambda^{\rho\nu}\partial_\rho X^\mu. \tag{7.6}$$

Using these relations, all components of the metric remain invariant, except the two components g_{++} and g_{--}, which transform as

$$\delta_X g_{++}=-\frac{l^2}{2}\partial_+^3\xi^+,\quad \delta_X g_{--}=-\frac{l^2}{2}\partial_-^3\xi^-. \tag{7.7}$$

The extrinsic curvature K_{ij} and its trace are also invariant and the only possible non-trivial components for auxiliary fields (5.10) are (for more details of this computation see Appendix A in [31])

$$\delta_X\lambda^{++}=-\frac{1}{r^4}\partial_-^3\xi^-,\quad \delta_X\lambda^{--}=-\frac{1}{r^4}\partial_+^3\xi^+,$$
$$\delta_X f^{++}=-\frac{2}{r^4}\left(a_1-\frac{30}{l^2}c_1-\frac{78}{l^2}c_2-\frac{216}{l^2}c_3+\frac{3}{l^2}b_2\right)\partial_-^3\xi^-,$$
$$\delta_X f^{--}=-\frac{2}{r^4}\left(a_1-\frac{30}{l^2}c_1-\frac{78}{l^2}c_2-\frac{216}{l^2}c_3+\frac{3}{l^2}b_2\right)\partial_+^3\xi^+, \tag{7.8}$$

[3] We note that in the ADM decomposition, the vectorial diffeomorphism parameter X^μ can be decomposed as $X^\mu=(\xi^i,\lambda)$ where ξ^i and λ are arbitrary functions of the coordinate $x^\mu=(x^i,r)$ [31].

where we have used the scalar ghost-free conditions in (2.13) to write the above relations.

Now we can use (7.7) and (7.8) to compute $\delta_X T_{++}$ from the renormalized energy-momentum tensor which we found in (5.12)

$$8\pi G\delta_X T_{++} = -\frac{l}{2}\partial_+^3\xi^+\Big(\sigma + \frac{1}{2l^2}a_1 - \frac{15}{l^4}c_1 - \frac{39}{l^4}c_2 - \frac{108}{l^4}c_3 + \frac{3}{2l^4}b_2\Big) = -\frac{l}{2}\partial_+^3\xi^+\bar{\sigma}_{d=3}. \quad (7.9)$$

By comparing the above result with the second term in Eq. (7.2), one can read the central charge as

$$c = \frac{3l}{2G}\bar{\sigma}_{d=3}. \quad (7.10)$$

This value coincides with the value computed by another method in [16].

An example of the three dimensional RCG is Extended New Massive Gravity (ENMG), which is a theory free of scalar ghosts. It has been shown in [16] that it is possible to write the action of ENMG in terms of three dimensional Schouten tensor, $\lambda_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$ and Cotton tensor $C_{\mu\nu} = \epsilon^{\rho\kappa}{}_{\nu}\nabla_\rho(\lambda_{\mu\kappa})$ as

$$\mathcal{L} = \sigma R - 2\Lambda_0 + \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right) + \frac{1}{m^4}\left(2a\det(\lambda) - b\,C^{\mu\nu}C_{\mu\nu}\right), \quad (7.11)$$

where

$$\det(\lambda) = -\frac{1}{3}R^\nu{}_\mu R^\rho{}_\nu R^\mu{}_\rho + \frac{3}{8}RR_{\mu\nu}R^{\mu\nu} - \frac{17}{192}R^3,$$
$$C^{\mu\nu}C_{\mu\nu} = R_{\mu\nu}\Box R^{\mu\nu} - \frac{3}{8}R\Box R - 3R^\nu{}_\mu R^\rho{}_\nu R^\mu{}_\rho + \frac{5}{2}RR_{\mu\nu}R^{\mu\nu} - \frac{1}{2}R^3. \quad (7.12)$$

Simply one can match the two actions in (2.1) and in (7.11) as follows:

$$a_1 = \frac{1}{m^2}, \quad a_2 = -\frac{3}{8m^2}, \quad b_1 = -\frac{3b}{8m^4}, \quad b_2 = \frac{b}{m^4},$$
$$c_1 = \frac{9b-2a}{3m^4}, \quad c_2 = \frac{3a-10b}{4m^4}, \quad c_3 = \frac{48b-17a}{96m^4}. \quad (7.13)$$

By substitution of these values into the definition of $\bar{\sigma}_{d=3}$ in (2.7), the central charge (7.10) will be equal to the central charge reported in [16], i.e.

$$c_{ENMG} = \frac{3l}{2G}\bar{\sigma}_{d=3} = \frac{3l}{2G}\left(\sigma + \frac{1}{2m^2l^2} - \frac{a}{8m^4l^4}\right). \quad (7.14)$$

An alternative way is that the values of central charges can be computed from the conserved charges of (2.24). This method has been used in [7] and [20], which again confirms the above value for the central charges,

$$c_L = c_R = \frac{3l}{2G}\bar{\sigma}. \quad (7.15)$$

7.2 BTZ black hole

The BTZ black hole is a solution of pure gravity [53]. Here for RCG we have such a solution again. Starting from

$$\mathrm{d}s^2 = l^2\left[\frac{r^2}{(r^2-r_+^2)(r^2-r_-^2)}\mathrm{d}r^2 - \frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}\mathrm{d}t^2 + r^2(\mathrm{d}\phi - \frac{r_+r_-}{r^2}\mathrm{d}t)^2\right], \quad (7.16)$$

where $r_\pm$ are outer and inner event horizons, the ADM-like metric for rotating BTZ black hole is given by

$$\mathrm{d}s^2 = N^2\mathrm{d}r^2 - N^{-2}\mathrm{d}t^2 + r^2(\mathrm{d}\phi - N_\phi\mathrm{d}t)^2$$
$$N = \left(-8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}\right)^{-\frac{1}{2}}, \quad N_\phi = -\frac{4GJ}{r^2}$$
$$M = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad J = \frac{r_+r_-}{4Gl}. \quad (7.17)$$

We can use this metric to compute the conserved charges from (4.10) by using the stress tensor in (4.6) and after renormalization. Then the mass and angular momentum become

$$M_{\mathrm{BTZ}} = M\bar{\sigma}_{d=3}, \quad J_{\mathrm{BTZ}} = J\bar{\sigma}_{d=3}. \quad (7.18)$$

Moreover, the angular velocity at horizon is defined as

$$\Omega_H = \frac{1}{l}N_\phi(r_+) = \frac{1}{l}\frac{r_-}{r_+}. \quad (7.19)$$

To find the thermodynamical parameters of the BTZ black hole we observe that the Hawking temperature in ADM form of the metric can be obtained from the surface gravity κ as

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} = \frac{1}{2\pi l}\frac{\partial_r N}{\sqrt{g_{rr}}} = \frac{r_+}{2\pi l}\left(1 - \frac{r_-^2}{r_+^2}\right). \quad (7.20)$$

Now by using the Smarr relation $M = \frac{1}{2}T_{\mathrm{H}}S_{\mathrm{BH}} + \Omega_H J$ for BTZ black holes in three dimensions we can evaluate the entropy. The Bekenstein–Hawking entropy is given by

$$S_{\mathrm{BH}} = \frac{\pi r_+\bar{\sigma}_{d=3}}{2lG}. \quad (7.21)$$

As we see, all conserved charges such as mass, angular momentum and entropy are proportional to the central charge of the dual CFT, therefore the BTZ black hole is stable whenever this central charge is positive.

8 Summary and conclusion

In this paper we have studied the most general Ricci Cubic Gravity (RCG) in d dimensional space-time. Our Lagrangian in (2.1) is constructed out of the Ricci tensor up to cubic terms and its covariant derivatives such that the equations of motion only contain at most six partial derivatives. We have also considered a cosmological parameter. Since we are interested in studying this theory from the auxiliary field formalism point of view, we have restricted ourselves to Ricci tensors. As has been shown in [31], in three dimensions one needs to consider two rank-two auxiliary fields to construct a Lagrangian with at most two derivatives in its equations of motion. A similar situation holds in general d dimensions.

If we add terms including the Riemann tensor, then we will need to consider rank-four auxiliary field [31], where we have postponed study of these terms for future work. Our study of RCG is divided into two main parts.

• The linear excitations of gravitational fields around the maximally symmetric AdS_d space-time: We have used two different approaches to the study of gravitons in RCG. In the first approach in Sect. 2, we linearize the equations of motion, following the work of [7], to write these equations as (2.6). To get rid of scalar ghosts in this theory it would be enough to set the trace of linearized equations of motion to zero. Doing this, we will find two constraints among the nine free parameters (couplings and cosmological parameter) in this theory, (2.13).

The scalar ghost-free condition allows us to use the transverse traceless gauge so that the linearized equation of motion now can be decomposed as Eq. (2.16). This suggests the existence of three exciting modes in the AdS_d background, two massive gravitons in addition to a massless one. Although seven free parameters have remained in the parameter space of the theory, the mass of the massive modes, $(M_\pm)$, depends only on the three parameters $\{\bar{\sigma}, \sigma_1, \sigma_2\}$ in (2.18). These parameters have already been defined in (2.7) as a linear combination of the parameters of the theory. The stability of the theory in this background (tachyon-free condition or $M_\pm^2 \geq 0$) suggests that the allowed regions of parameter space are restricted. We have summarized our results in Table 1. Similar to the three dimensional case, as discussed in [7], here we have also special subspaces in our three-parameter space $\{\bar{\sigma}, \sigma_1, \sigma_2\}$ where we have two or three degenerate massless gravitons or two degenerate massive gravitons.

In the second approach in Sect. 3, we employ the auxiliary field formalism, which has been introduced in [31]. To find the graviton mass spectrum we consider the excitation modes around the background metric $g_{\mu\nu}$ and the auxiliary fields $(f_{\mu\nu}, \lambda_{\mu\nu})$ and we show that, similar to three dimensions [7], we can find a linear combination of three Pauli–Fierz spin-2 Lagrangians (3.13). The mass spectrum in this way confirms the results of the first approach.

On the other hand, the second approach shows that, for $\bar{\sigma} \neq 0$, in general it is impossible to avoid the rank-two ghost fields. Our computations confirm the known observation for pure gravitational theories with higher curvature terms that we cannot have both the tachyon-free condition and the ghost-free condition simultaneously. This statement can be verified by computing the energy of the linear excitations too. This results from the Hamiltonian formalism and by comparing the overall signs of energies. The results are given in Eqs. (3.16a) and (3.16b).

• Black hole solutions and conserved charges: The RCG as a theory of gravity with higher curvature terms admits different solutions for equations of motion. In this paper we have focused on two types of solutions: The Schwarzschild–AdS (5.1) and Lifshitz black holes (6.3). In Sect. 5 we have investigated different properties of SAdS solution such as the mass, Hawking temperature and entropy.

The mass has been computed in two different ways. In Sect. 3.2 we first use the Abbot–Deser method [33] to find the conserved charges corresponding to the symmetries of the solution. This can be done by using the linearized equation of motion following [7] and the conserved charge is given by Eq. (2.24). We can use it to compute the mass of the black hole simply by considering a time-like Killing vector.

On the other hand we can also compute the conserved charges in auxiliary field formalism. This can be done by computing the boundary stress tensor in this formalism. The energy-momentum tensor can be found by variation of the action with respect to the auxiliary fields and metric. To have a well-defined variational principle we need a generalized Gibbons–Hawking term (4.4). The final result is presented in Eq. (4.6) and the mass can be computed from (4.11).

The value of the mass in this way diverges, as one goes to the boundary at $r \to \infty$; see Eq. (5.11). To find a finite answer, we must renormalize the boundary terms by adding some proper counter-terms. The final value of the mass in this way is given in Eq. (5.13) and agrees with the mass from the first approach.

We have studied the thermodynamical properties of SAdS black holes in Sect. 5, where we have found the entropy of the black hole both by direct computation of the free energy (5.8) and Wald's entropy formula (5.9). The values of the mass, temperature and entropy satisfy the first law of thermodynamics for black holes, i.e. $\mathrm{d}M = T\,\mathrm{d}S$.

To complete our analysis for more complicated cases, in Sect. 6 we study the Lifshitz black hole and try to compute its mass from the boundary stress tensor which we found from the auxiliary field formalism. To have such a solution we need to restrict ourselves to the special values in parameter space of the RCG. In this case all constants can be written in terms of the two constants, b_2 and σ of the parameter space and also the dynamical exponent z (see Appendix B).

Computing the mass again gives a divergent answer (6.8), but unlike the SAdS black hole it contains four different divergent behaviors when one goes to the boundary. We show that in order to have a finite massive Lifshitz black hole, we have just three options for the dynamical exponent, $z = 1$ or SAdS, $z = d$ and $z = 2d - 1$.

As has been noted in [31] for three dimensional Lifshitz black holes, there is an ambiguity in choosing the counter-terms to renormalize the boundary terms. In our study this happens again and there are various possibilities to have a finite mass. Although it is not known whether Wald's entropy formula works here for Lifshitz black holes but one can use it naively to find a finite mass from the validity of the first law of thermodynamics for Lifshitz black holes. The value of this mass is given in Eq. (6.7). We fix the coefficients of the counter-terms (6.11) on the boundary so that the value of the mass is equal to its value in Eq. (6.7) (see Appendix C). We should note that we will recover the SAdS results at $z = 1$.

In Sect. 7, as an application of our results, we have studied the three dimensional RCG. For example we have computed the central charge associated to the dual CFT of AdS_3 space-time. We have also calculated the mass and angular momentum of the BTZ black holes. Our results confirm the well-known results in the literature when one considers special values of parameters in NMG or ENMG theories. By looking at the values of the mass and entropy of the BTZ black hole one can show that the stability condition holds when $\bar{\sigma}_{d=3} > 0$. This coincides with the unitarity condition of the dual CFT, because the value of central charge is also proportional to $\bar{\sigma}_{d=3}$.

As a general result in d dimensional space-time, we observe that the entropy (6.6) and mass (6.7) for the black holes in RCG we have considered in this paper are both proportional to a specific parameter $\tilde{\sigma}$, and stability of the solutions requires that this critical parameter of the theory must have a positive value i.e. $\tilde{\sigma} > 0$. For $z = 1$ the value of $\tilde{\sigma}$ reduces to $\bar{\sigma}$ for SAdS black holes. A similar behavior has been already reported for Gauss–Bonnet gravity in [54].

There are some open questions which we have postponed for further work [57]:

1. It would be interesting to solve the ambiguity in choosing the counter-terms which renormalize the boundary stress tensor. Our choice for these counter-terms in (6.11) is motivated by the holographic renormalization (see for example [55]). As we mentioned before, we have fixed the coefficients in an auxiliary field formalism so that $\lambda_{\mu\nu}$ becomes the Schouten tensor in d dimensions. By substituting (3.2a) into Eq. (6.11) we will have a Lagrangian with counter-terms constructed out of the Ricci tensors alone. These terms have been made out of the induced metric on the boundary. It would be interesting to build such a Lagrangian by the method of holographic renormalization and then translate it to the auxiliary field formalism.
2. One can consider the contribution of the Riemann tensor into our analysis. But as indicated in [31] one needs to introduce a rank-four auxiliary field into the game. This will make the analysis more complicated due to the existence of total derivative terms such as a Gauss–Bonnet term [57].

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Appendix A Lifshitz parameters

Coefficients related to the auxiliary fields in Eq. (6.4) are

$$\begin{aligned} f_{10} &= 4a_1 l^2\big(d-2+(d-2)z+2z^2\big)+8a_2 l^2\big(d^2\\ &\quad +2(z-1)^2+d(2z-3)\big)+8b_2(d-2)(z-1)z\\ &\quad -6c_1\big((d-2)^2+(d-2)dz^2+2(d-2)z^3+2z^4\big)\\ &\quad -4c_2\big(d^2+2(z-1)^2+d(2z-3)\big)\big(-4\\ &\quad +d(2+z)+z(3z-2)\big)-12c_3\big(d^2+2(z-1)^2\\ &\quad +d(2z-3)\big)^2, \end{aligned} \tag{A.1a}$$

$$\begin{aligned} f_{11} &= (z-1)\big(4a_1 l^2(2d-3-2z)+8a_2 l^2(d-1-2z)\\ &\quad -8b_1(d-1)(d-1-2z)(d-z)+b_2\big(4\\ &\quad -2d(1+d)+20z+8(d-2)dz-8(d-1)z^2\big)\\ &\quad -6c_1\big(10+d(3d-11)+6z+(d-5)dz+2(d\\ &\quad -1)z^2-4z^3\big)-4c_2\big(-2+d\big(7+d(2d-7)\big)+24z\\ &\quad +4(d-5)dz+2(1+d)z^2-12z^3\big)\\ &\quad +24c_3\big((2-d)(d-1)^2+2(d-3)z^2+4z^3\big), \end{aligned} \tag{A.1b}$$

$$\begin{aligned} f_{12} &= (z-1)\big(8b_1(d-1)(d-1-2z)(2d-1-z)\\ &\quad +2b_2\big(-8+d\big(17+d(4d-13)\big)-6z+4(3\\ &\quad -2d)dz+4(d-1)z^2\big)-3c_1(z-1)\big(13+5d^2\\ &\quad -8d(2+z)+4z(3+z)\big)-2c_2(z-1)\big(9d^2\\ &\quad -4d(6+5z)+(3+2z)(5+6z)\big)\\ &\quad -12c_3(d-1-2z)^2(z-1)\big), \end{aligned} \tag{A.1c}$$

$$\begin{aligned} f_{20} &= 4\big(2a_1 l^2\big(-2+d^2(z-1)-4(z-2)z\\ &\quad +d\big(3+(z-6)z\big)\big)-4a_2 l^2\big(d^2+2(z-1)^2+d(2z-3)\big) \end{aligned}$$

$$+4b_2(d-2)^2(z-1)z^2+3c_1\big((d-2)^2(d-1)+2(d-2)^2z-(d-2)\big(3+(d-5)d\big)z^2-2(d-3)(d-2)z^3-(d-4)z^4\big)-2c_2\big(d^2+2(z-1)^2+d(2z-3)\big)\big(d^2(z-1)+(8-5z)z+d\big(2+(z-6)z\big)\big)+6c_3\big(d^2+2(z-1)^2+d(2z-3)\big)^2\big), \tag{A.1d}$$

$$f_{21}=2(z-1)\big(2a_1(d-4)l^2(d-1-2z)-8a_2l^2(d-1-2z)+8b_1(d-1)(d-1-2z)\big(1+(d-2)z\big)+2b_2(d-1-2z)\big(5d-4-d^2+(d-2)(5d-8)z\big)+6c_1\big(2-3d+d^2-(d-2)\big(7+(d-4)d\big)z+\big(4+(d-3)d\big)z^2+2(d-4)z^3\big)+2c_2\big(-8-d\big(-26+d\big(27+(d-10)d\big)\big)+28z-2d\big(15+(d-6)d\big)z+4\big(11+(d-6)d\big)z^2+8(d-5)z^3\big)+24c_3(d-1-2z)\big(d^2+2(z-1)^2+d(2z-3)\big), \tag{A.1e}$$

$$f_{22}=(z-1)\big(8b_1(d-1)(d-1-2z)\big(1+d^2+4z-2d(2+z)\big)+2b_2(d-1-2z)\big(-4(1+4z)+d\big(17+d(4d-17-6z)+20z\big)\big)-3c_1(z-1)\big((d-11)(d-3)d-4(d-7)dz+4(d-4)z^2-4(5+8z)\big)-4c_2(z-1)\big(-13+d^3-d^2(11+4z)-4z(7+5z)+d\big(4z(7+z)+23\big)\big)+24c_3(d-1-2z)^2(z-1)\big), \tag{A.1f}$$

$$f_{30}=4\big(2a_1l^2\big(2-4(z-1)z+d\big((z-2)z-1\big)\big)-4a_2l^2\big(d^2+2(z-1)^2+d(2z-3)\big)-4b_2(d-2)^2(z-1)^2z-3c_1(d-2+z^2)\big(2-4(z-1)z+d\big(-1+(z-2)z\big)\big)-2c_2\big(d^2+2(z-1)^2+d(2z-3)\big)\big(4+(4-5z)z+d\big(-2+(z-2)z\big)\big)+6c_3\big(d^2+2(z-1)^2+d(2z-3)\big)^2\big), \tag{A.1g}$$

$$f_{31}=2(z-1)\big(2a_1l^2\big(12+d(3d-13-2z)+8z\big)-8a_2l^2(d-1-2z)-8b_1(d-1)(d-1-2z)\big((d-2)d+z\big)-2b_2\big(8+4(d-3)d+12z+d\big(-1+(d-4)d\big)z-2\big(12+d(3d-11)\big)z^2\big)+6c_1\big(22+d\big(-35-3(d-6)d\big)+18z+3(d-5)dz-(d-3)(3d-4)z^2+2(d-4)z^3\big)+2c_2\big(-16+d\big(50+d\big(-53+(22-3d)d\big)\big)+68z-2d\big(39+d(2d-15)\big)z-4(d-3)(1+d)z^2+8(d-5)z^3\big)-24c_3\big((2-d)(d-1)^2+2(d-3)z^2+4z^3\big)\big), \tag{A.1h}$$

$$f_{32}=(z-1)\big(8b_1(d-1)(d-1-2z)\big(3+d(3d-8)+2z\big)+2b_2\big(28+(d-3)d\big(21+d(4d-11)\big)+2d(4+d-d^2)z-4\big(4+(d-3)d\big)z^2\big)-3c_1(z-1)\big(9d^3-2d^2(29+6z)-4\big(17+4z(4+z)\big)+d\big(113+4z(15+z)\big)\big)-4c_2(z-1)\times\big(3(d-1)\big(7+(d-6)d\big)-4\big(11+d(2d-11)\big)z+4(d-5)z^2\big)+24c_3(d-1-2z)^2(z-1)\big). \tag{A.1i}$$

Appendix B Lifshitz couplings

The values of the couplings are fixed by the equations of motion for a Lifshitz black hole,

$$b_2=-\frac{2l^2(d-2)(z-1)(d-1-2z)^2}{\Xi\,(d-1)}\Big(b_1(d-1)^2\times\Big(144\,d^8+d^7(-1349+245\,z)+d^6(5069-1681\,z-28\,z^2)+d^5(-9631+4214\,z+503\,z^2-366\,z^3)-2d^4(-4699+2221\,z+1227\,z^2-1098z^3+69z^4)+d^3(-3841+1493z+5384z^2-6100z^3+704z^4+296z^5)+d^2(-229-269z-5798z^2+9584z^3-2028z^4-1284z^5+408z^6)+d(421+1128z+2425z^2-6150z^3+536z^4+2408z^5-640z^6-128z^7)-2(-9+344z-176z^2+422z^3-1775z^4+1398z^5+172z^6-504z^7+128z^8)\Big)+2l^4(z-1)\Big(-36d^5+39d^4(5+3z)-3d^3(134+169z+33z^2)+d^2(399+789z+316z^2+8z^3)+d(-192-549z-319z^2-20z^3+12z^4)+2(18+75z+69z^2-22z^3-4z^4+8z^5)\Big)\sigma\Big), \tag{B.1a}$$

$$a_1=\frac{2}{\Xi}(2z-d+1)\Big(b_1(d-1)^2\Big(96d^{12}-4d^{11}\times(313+155z)+d^{10}(5748+11168z-2084z^2)-2d^9(2380+43717z-14584z^2+623z^3)-2d^8(29538-194013z+83251z^2-2559z^3-1681z^4)+d^7(283749-1064198z+479036z^2+59252z^3-57337z^4+5306z^5)+d^6(-639651+1830052z-618548z^2-557360z^3+376829z^4-58148z^5+858z^6)-2d^5(-420065+932631z+98489z^2-999191z^3+623451z^4-105002z^5-8679z^6+2422z^7)-4(z-1)^3(729+2043z-18693z^2+43087z^3-23934z^4-2440z^5+5912z^6-1776z^7+256z^8)-2d^4(323943-431617z-920827z^2+1885347z^3-1080780z^4+88096z^5+104600z^6-27774z^7+1196z^8)+d^3(258053+198738z-2573016z^2+3774132z^3-1580157z^4-706838z^5$$

$$
\begin{aligned}
&+801404z^6-216300z^7+13088z^8+368z^9)\\
&+d^2(-22947-403568z+1508244z^2-1496736z^3\\
&-571491z^4+2071828z^5-1423398z^6\\
&+348044z^7+2936z^8-10224z^9+768z^{10})\\
&+2d(-7560+65220z-106625z^2-206748z^3\\
&+862502z^4-1090236z^5+606427z^6-96372z^7\\
&-45296z^8+21872z^9-3440z^{10}+256z^{11})\Big)\\
&+l^4(z-1)\Big(-48d^9+6d^8(73+91z)-d^7\\
&\times(1067+6051z+1090z^2)+d^6(-2216+26887z\\
&+12187z^2+126z^3)+d^5(17779-59698z-57313z^2\\
&-670z^3+206z^4)+2d^4(-21109+31661z\\
&+71906z^2+3861z^3-3173z^4+410z^5)-d^3\\
&\times(-51745+11323z+203100z^2+38588z^3-30914z^4\\
&+3736z^5+296z^6)+d^2(-34574-40589z\\
&+155707z^2+81754z^3-59310z^4+3716z^5+2824z^6\\
&-304z^7)-2(711+5151z-2031z^2-15167z^3\\
&+8904z^4+1520z^5-1984z^6+240z^7+64z^8)\\
&+d(11583+37208z-54777z^2-79318z^3\\
&+51256z^4+2336z^5-6368z^6+864z^7+64z^8)\Big)\sigma\Big),
\end{aligned}
\tag{B.1b}
$$

$$
\begin{aligned}
a_2=&\frac{1}{\Xi}\Big(b_1(d-1)^2\Big(96d^{12}(1+z)-4d^{11}(373+492z\\
&+215z^2)+2d^{10}(4439+10435z+5265z^2\\
&+309z^3)+2d^9(-9625-70978z-24276z^2\\
&-4618z^3+729z^4)+d^8(-45851+648636z\\
&+88120z^2+52314z^3-18061z^4+2330z^5)-2d^7\\
&\times(-213422+1010682z-20359z^2+62966z^3\\
&-47134z^4+13244z^5+999z^6)+d^6(-1351601\\
&+4315236z-407200z^2+7950z^3-271511z^4\\
&+102462z^5+44600z^6-6720z^7)+d^5(2505650\\
&-6248772z+411834z^2+756888z^3+398394z^4\\
&-70176z^5-321534z^6+68196z^7-896z^8)\\
&+d^4(-2971401+5919268z+764380z^2-2348514z^3\\
&+196217z^4-734178z^5+1145572z^6-236240z^7\\
&-21600z^8+6432z^9)+d^3(2264148-3324172z\\
&-2560626z^2+4208820z^3-2310448z^4+2819176z^5\\
&-2268746z^6+267960z^7+205232z^8\\
&-57184z^9+992z^{10})+d^2(-1057077+770046z\\
&+3165710z^2-5024144z^3+4741735z^4-4632974z^5\\
&+2404368z^6+278752z^7-609712z^8+151328z^9\\
&+2560z^{10}-512z^{11})-2d(-133650-79020z\\
&+999513z^2-1814186z^3+2177468z^4-1806584z^5\\
&+499333z^6+468702z^7-375688z^8+58112z^9\\
&+20944z^{10}-5088z^{11}+512z^{12})+4(-6561-23490z\\
&+134703z^2-290208z^3+381553z^4\\
&-254922z^5-27091z^6+160612z^7-82092z^8-4216z^9\\
&+15520z^{10}-4320z^{11}+512z^{12})\Big)\\
&+l^4(z-1)\Big(48d^{10}+6d^9(-213+257z)+d^8(13865\\
&-20391z-3434z^2)+d^7(-80167+109448z\\
&+46065z^2+758z^3)+d^6(277079-301787z\\
&-261672z^2-10010z^3+2654z^4)+d^5(-604465\\
&+422090z+816557z^2+71498z^3-32848z^4-1096z^5)\\
&+d^4(847481-156491z-1523076z^2\\
&-276702z^3+139852z^4+15200z^5-816z^6)\\
&-d^3(753973+386740z-1728795z^2-591394z^3\\
&+279236z^4+68456z^5-7952z^6+288z^7)\\
&+d^2(404505+625291z-1153920z^2-703870z^3\\
&+277954z^4+139184z^5-21376z^6-2080z^7+992z^8)\\
&+6(2133+14787z-8655z^2-18951z^3\\
&+2190z^4+8512z^5-704z^6-1392z^7+224z^8+128z^9)\\
&-d(115893+381684z-403191z^2\\
&-440606z^3+123980z^4+134112z^5-19936z^6\\
&-9344z^7+2688z^8+384z^9)\Big)\sigma\Big),
\end{aligned}
\tag{B.1c}
$$

$$
\begin{aligned}
c_1=&\frac{4l^2(d-1-2z)^2}{3\,\Xi\,(d-1)}\Big(b_1(d-1)^2\Big(192\,d^{10}-12d^9(167+9\,z)\\
&+d^8(8174+2276\,z-850\,z^2)\\
&-d^7(15125+16127z-7897z^2+261z^3)+d^6(6543\\
&+55761z-29143z^2+383z^3+1016z^4)\\
&+2d^5(11468-52082z+24603z^2+4179z^3-3953z^4\\
&+329z^5)-4(z-1)^3(-9-66z-481z^2\\
&+2424z^3-696z^4-584z^5+384z^6)-2d^4(21939\\
&-51140z+9653z^2+24198z^3-14750z^4\\
&+1320z^5+524z^6)+d^3(32965-40005z-58437z^2\\
&+116313z^3-65592z^4+4240z^5+7020z^6\\
&-1304z^7)+d^2(-10955-7665z+86779z^2-129523z^3\\
&+70072z^4+5084z^5-16680z^6+3272z^7\\
&+192z^8)+2d(594+3954z-18029z^2+22111z^3-471z^4\\
&-17497z^5+9594z^6+1536z^7-2304z^8\\
&+512z^9)\Big)-l^4(z-1)\Big(96d^7-48d^6(13+11z)\\
&+2d^5(701+1596z+487z^2)-d^4(973+7024z\\
&+5317z^2+606z^3)+d^3(-847+6324z+10917z^2\\
&+3006z^3-104z^4)+d^2(1693-880z-10235z^2\\
&-5566z^3+36z^4+168z^5)+4(36+228z+15z^2-593z^3\\
&+54z^4+84z^5-40z^6)+d(-891-1996z\\
&+3817z^2+5058z^3+100z^4-424z^5+96z^6)\Big)\sigma\Big),
\end{aligned}
\tag{B.1d}
$$

$$
\begin{aligned}
c_2=&\frac{2l^2(d-1-2z)}{\Xi(d-1)}\Big(-b_1(d-1)^2\Big(96d^{11}-124d^{10}\\
&\times(7+5z)+d^9(1748+8728z-1020z^2)\\
&+d^8(9153-51719z+9047z^2+1119z^3)+2d^7(-30663
\end{aligned}
$$

$$
\begin{aligned}
&+83293z-12584z^2-8167z^3+1169z^4)\\
&+d^6(159487-309623\,z-9071\,z^2+98107\,z^3\\
&-22348z^4+72z^5)+d^5(-226438+310048\,z\\
&+209684z^2-313896z^3+83778z^4+4168z^5-2496z^6)\\
&-d^4(-177839+93857z+521423z^2\\
&-570561z^3+136684z^4+47248z^5-21948z^6+752z^7)\\
&+2d^3(-31245-58491z+303532z^2\\
&-277487z^3+8835z^4+95976z^5-39536z^6+1112z^7\\
&+784z^8)+d^2(-4979+118111z-323489z^2\\
&+177297z^3+270180z^4-366208z^5+127992z^6+9176z^7\\
&-9552z^8+896z^9)+4(-378-1001z\\
&+8250\,z^2-26583\,z^3+41305\,z^4-25338\,z^5-3141z^6\\
&+9802z^7-2460z^8-840z^9+384z^{10})\\
&-2d(-4645+13334\,z-11392\,z^2-69538\,z^3\\
&+187549\,z^4-162044z^5+34752z^6+20800z^7\\
&-9072z^8-256z^9+512z^{10})\Big)+l^4(z-1)\Big(-495\\
&+48d^8-4272z-3303z^2+8030z^3+2248z^4\\
&-1584z^5+16z^6+224z^7-6d^7(41+91z)+d^6(-181\\
&+3891z+1618z^2)-d^5(-3910+9625z\\
&+11529z^2+1692z^3)+d^4(-11339+6684z+31675z^2\\
&+11036z^3+392z^4)+d^2(-11193-24191z\\
&+23170z^2+38126z^3+4240z^4-1112z^5)+2d^3(7780\\
&+5571z-20590z^2-14379z^3-1018z^4\\
&+136z^5)-d(-3936-16917z+707z^2+26502z^3\\
&+4444z^4-2024z^5+96z^6+128z^7)\Big)\sigma\Big), \qquad \text{(B.1e)}
\end{aligned}
$$

$$
\begin{aligned}
c_3=&\frac{l^2}{3\,\Xi\,(d-1)}\Big(2b_1(d-1)^2\Big(96d^{11}(2+z)-4d^{10}\\
&\times(542+647z+239z^2)+2d^9(4137+13375z\\
&+5821z^2+811z^3)+d^8(-575-148059z-67177z^2\\
&-14765z^3+272z^4)+d^7(-104799+495534z\\
&+254339z^2+51655z^3-7448z^4+1055z^5)-d^6\\
&\times(-422579+1048799z+705765z^2+92573z^3\\
&-66154z^4+6916z^5+4984z^6)+d^5(-883077\\
&+1391766z+1439961z^2+131129z^3-273580z^4\\
&-11363z^5+51308z^6+416z^7)+d^4(1130881\\
&-1050285z-2087619z^2-240425z^3+554776z^4\\
&+208454z^5-206350z^6-11640z^7+3680z^8)+d^3\\
&\times(-909813+246562z+2092313z^2+299465z^3\\
&-420636z^4-702839z^5+383884z^6+92536z^7\\
&-30160z^8+272z^9)+d^2(442189+266887z\\
&-1429703z^2-22723z^3-266418z^4+1043240z^5\\
&-257136z^6-275472z^7+75120z^8+6240z^9\\
&-2560z^{10})-2(-5787-30438z+75230z^2-86563z^3\\
&+125480z^4-49747z^5-85555z^6+67076z^7\\
&+7760z^8-14560z^9+1520z^{10}+768z^{11})+d(-115257\\
&-238740z+641505z^2-282383z^3+604464z^4\\
&-656757z^5-118152z^6+335960z^7-49600z^8\\
&-29968z^9+6400z^{10}+1024z^{11})\Big)\\
&+l^4(z-1)\Big(-5553+48d^9-44754z-25173z^2\\
&+48772z^3+38020z^4-6080z^5-4912z^6+1856z^7\\
&+704z^8+6d^8(-221+233z)+d^7(12479-12963z\\
&-5732z^2)+d^6(-58800+41258z+54524z^2\\
&+8306z^3)-d^5(-159342+32779z+204587z^2\\
&+71256z^3+4776z^4)+d^4(-262123-111044z\\
&+385791z^2+244444z^3+33276z^4+368z^5)+d^3\\
&\times(263275+325003z-374458z^2-433644z^3\\
&-94216z^4+800z^5+64z^6)+2d^2(-77539-184165z\\
&+75453z^2+212799z^3+68792z^4-3504z^5\\
&-528z^6+336z^7)+d(47736+202211z+18057z^2\\
&-222796z^3-107296z^4+11824z^5+4304z^6\\
&-2496z^7-384z^8)\Big)\sigma\Big), \qquad \text{(B.1f)}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_0=&\frac{(d-1)^2(z-1)(d-2)}{6l^4\,\Xi}\Big(b_1(d-1)^2\\
&\times\Big(96\,d^{13}-4\,d^{12}(301+167\,z)+2\,d^{11}(2321+4446\,z\\
&+649z^2)+d^{10}(4494-37194z-32138z^2+566z^3)\\
&-d^9(91761+46637z-332843z^2+19043z^3\\
&+4474z^4)+2d^8(142628+552660z-978547z^2\\
&+116805z^3+19033z^4+325z^5)+d^7(-209009\\
&-5243745z+7366309z^2-1506913z^3-54008z^4\\
&-29014z^5+10412z^6)-4d^6(242497-3500960z\\
&+4677846z^2-1425308z^3+74845z^4-22604z^5\\
&+15135z^6+577z^7)+d^5(3326409-23996307z\\
&+32437993\,z^2-12680361\,z^3+32802z^4+1250264\,z^5\\
&-354100\,z^6+111284\,z^7-12352\,z^8)\\
&-2d^4(2549650-13499190z+18601679z^2-7152261z^3\\
&-4002317z^4+5158255z^5-2070480z^6\\
&+492404z^7-64572z^8+2096z^9)+24(z-1)^2\\
&\times(-2673-6837z+59030z^2-194638z^3+184895z^4\\
&-51097z^5-20212z^6+20156z^7-6232z^8-16z^9\\
&+128z^{10})-4d(z-1)^2(-156789+27327z\\
&+869344z^2-3597020z^3+3281227z^4-1011643z^5\\
&-63478z^6+146288z^7-50544z^8+864z^9\\
&+384z^{10})+d^3(4507267-19319383z+25346473z^2\\
&-450439z^3-31360644z^4+33041610z^5
\end{aligned}
$$

$$
\begin{aligned}
&-14844876z^6+3466128z^7-362680z^8-33696z^9\\
&+13696z^{10})+2d^2(-1159953+3963453z\\
&-3307361z^2-9615401z^3+26992568z^4-27014098z^5\\
&+12737522z^6-2689998z^7-52956z^8\\
&+192432z^9-47072z^{10}+864z^{11})\Big)\\
&+l^4\Big(-48d^{11}-6d^{10}\\
&\times(-229+305z)+d^9(-16133+29463z-2026z^2)\\
&+d^8(103997-211556z+33497z^2-1754z^3)\\
&+d^7(-414271+888009z-213628z^2\\
&-9136z^3+14346z^4)+d^6(1079933-2392440z\\
&+665429z^2+252988z^3-157986z^4-10316z^5)\\
&+d^5(-1888171+4263303z-938908z^2-1474234z^3\\
&+694666z^4+104844z^5+668z^6)+d^4(2215571\\
&-4973808z-98209z^2+4201870z^3-1567856z^4\\
&-441772z^5-3404z^6+1056z^7)+d^3(-1703657\\
&+3591603z+2358120z^2-6831716z^3+2037678z^4\\
&+869616z^5+60772z^6-20904z^7+3456z^8)\\
&+12(z-1)^2(1755+7281z-37941z^2-1305z^3\\
&-9142z^4-1840z^5+3464z^6-704z^7-512z^8\\
&+64z^9)-d^2(-809505+1374322z+3511149z^2\\
&-6627432z^3+1817538z^4+654796z^5+243420z^6\\
&-48112z^7-8464z^8+6336z^9)+2d(-104580\\
&+68163z+1156005z^2-1866385z^3+658415z^4\\
&-36670z^5+137128z^6+15724z^7-27752z^8+7440z^9\\
&+288z^{10})\Big)\sigma\Big),
\end{aligned}
\tag{B.1g}
$$

$$
\begin{aligned}
\Xi=&\,(d-2)(d-1)l^2(z-1)\Big(48\,d^{10}+6\,d^9\\
&\times(-205+281\,z)+d^8(13283-23739\,z-848\,z^2)\\
&-2d^7(38719-70645z-6983z^2+1001z^3)\\
&+d^6(271048-460549z-106277z^2+32506z^3\\
&-2048z^4)+d^5(-599393+881958z+453483z^2\\
&-187002z^3+7594z^4+5752z^5)+d^4(851281\\
&-963709z-1153994z^2+518884z^3+35970z^4\\
&-48024z^5-2576z^6)-2d^3(382936-235875z\\
&-891966z^2+379545z^3+111998z^4-70298z^5\\
&-10128z^6+512z^7)+d^2(414374+97301z\\
&-1643993z^2+587670z^3+399372z^4-160156z^5\\
&-65600z^6+5104z^7+960z^8)+d(-119187\\
&-216572z+834955z^2-231586z^3-273774z^4\\
&+43412z^5+82736z^6-592z^7-6624z^8+1280z^9)\\
&+2(6543+35292z-90804z^2+21142z^3+28009z^4\\
&+6090z^5-10736z^6-6656z^7+3120z^8\\
&+736z^9-512z^{10})\Big).
\end{aligned}
\tag{B.1h}
$$

Appendix C Counter-term coefficients

The coefficients of the counter-terms in Eq. (6.11) are as follows:

$$
\begin{aligned}
\alpha_1=&\,(-b_1(d-1)^3(-395016343175136\\
&+2348550175021544d-3316571455059268d^2\\
&-9312466632690362d^3+41832587177120467d^4\\
&-59950661474513668d^5\\
&+s46523853371804 73d^6+128519961747254140d^7\\
&-255642279986199850d^8\\
&+284182844961412260d^9-209960379268229286d^{10}\\
&+106564661277557906d^{11}\\
&-36670939831292789d^{12}+8074817985963160d^{13}\\
&-943970979676719d^{14}\\
&+10294748426892d^{15}+7149242979468d^{16})\\
&+2(-2507607366432+9370485047140d\\
&-15322963757344d^2+29194387485479d^3\\
&-34599990417180d^4-101018121331169d^5\\
&+569000982666250d^6-1443232620917570d^7\\
&+2383139360258160d^8\\
&-2637960123419686d^9+1920556277058316d^{10}\\
&-892060727009845d^{11}\\
&+251884406261660d^{12}-38888689855581d^{13}\\
&+1851979430970d^{14}\\
&+344868283728d^{15})l^4\sigma)/\Upsilon,
\end{aligned}
\tag{C.1a}
$$

$$
\begin{aligned}
\alpha_2=&\,4l^2\Big(2b_1(d-1)^3(-76757505990096\\
&+452153505555884d-624980807053248d^2\\
&-1809192261855057d^3+7952875768988962d^4\\
&-11226865296348523d^5\\
&+7407559584712 53d^6+23887391502121540d^7\\
&-46975129179219225d^8\\
&+51879608281499860d^9-38210420906572496d^{10}\\
&+19391626011252941d^{11}\\
&-6690472887584954d^{12}+1480665214857635d^{13}\\
&-174392189027309d^{14}\\
&+1988579461512d^{15}+1327387196073d^{16})\\
&+(1990194128808-7386025312660d\\
&+11742213076986d^2-21594332747301d^3
\end{aligned}
$$

$$
\begin{aligned}
&+23694835045670d^4+83844566027911d^5\\
&-437291091342750d^6+1076981245232830d^7\\
&-1754567637812540d^8\\
&+1934727144371834d^9-1409181677959554d^{10}\\
&+655519789230055d^{11}\\
&-185368649436290d^{12}+28733073480339d^{13}\\
&-1405718397930d^{14}\\
&-254604527232d^{15})l^4\sigma\Bigg)/(d-1)\Upsilon,
\end{aligned}\tag{C.1b}
$$

$$
\begin{aligned}
\alpha_3=4l^4\Bigg(&-b_1(d-1)^3(1408481241557088\\
&-7962097900340552d+8702328896412548d^2\\
&+40621533255118130d^3-144622783539983159d^4\\
&+152594357275463212d^5\\
&+118911670846630178d^6-560396683643759318d^7\\
&+764990748437182203d^8\\
&-490234816648922540d^9-27828284889708812d^{10}\\
&+361280304561738662d^{11}\\
&-362359563639109409d^{12}\\
&+202400102173512036d^{13}-71325089681437902d^{14}\\
&+15535905541976574d^{15}-1742569596681603d^{16}\\
&+10872378426212d^{17}\\
&+13031320359348d^{18})+2(9079919253456\\
&-31647743507020d+41985077042700d^2\\
&-71966579675217d^3+59972441260106d^4\\
&+475565980123096d^5-2033918977126280d^6\\
&+4370244951580351d^7-5843504482948030d^8\\
&+4135073914235468d^9\\
&+309968734691532d^{10}-3588169908956611d^{11}\\
&+3470610662200846d^{12}\\
&-1717595387287272d^{13}+484209291110800d^{14}\\
&-72910129367115d^{15}+3042549337270d^{16}\\
&+609885548208d^{17})l^4\sigma\Bigg)/9(d-1)^4\Upsilon,
\end{aligned}\tag{C.1c}
$$

$$
\begin{aligned}
\alpha_4=8(d-2)l^4\Bigg(&b_1(d-1)^3(-528808080422496\\
&+3130553990186984d\\
&-4379020454475448d^2-12459226694726482d^3\\
&+55439333803203762d^4\\
&-78953331948719473d^5+5806789128993578d^6\\
&+168554151710199165d^7\\
&-333816422809221600d^8+370185524282685110d^9\\
&-273220870869987396d^{10}\\
&+138699652880065216d^{11}-47789659274303054d^{12}\\
&+10546137780003635d^{13}\\
&-1236609482027534d^{14}+13691674656437d^{15}\\
&+9391169673148d^{16})\\
&+2(3384337452252-12612711742540d\\
&+20346088117159d^2-38096175355894d^3\\
&+43891199458605d^4+138169028535484d^5\\
&-755344603837500d^6+1895200175897020d^7\\
&-3114206365072010d^8+3442492641342796d^9\\
&-2506736133351601d^{10}\\
&+1164981189763170d^{11}-329152871207135d^{12}\\
&+50916198362516d^{13}-2456245454170d^{14}\\
&-450654150008d^{15})l^4\sigma\Bigg)/9(d-1)^3\Upsilon,
\end{aligned}\tag{C.1d}
$$

$$
\begin{aligned}
\alpha_5=-4l^6\Bigg(&-b_1(d-1)^3(359778834028416\\
&-4953309366088160d+21513244670246864d^2\\
&-25733295796545048d^3-88916513146380432d^4\\
&+392638707702220032d^5\\
&-618350454473661033d^6+191645008663521032d^7\\
&+1108939951725441017d^8\\
&-2628098535293123038d^9+3328476896217706846d^{10}\\
&-2826396397716689052d^{11}\\
&+1683301856093304062d^{12}-693105132653871760d^{13}\\
&+181474277097201271d^{14}\\
&-20572142542150452d^{15}-3854293317416223d^{16}\\
&+1889784234492878d^{17}\\
&-262274701598268d^{18}+2960158017632d^{19}\\
&+1786518075960d^{20})+4(1138565977896\\
&-13135644442796d+45147119557134d^2\\
&-84986822962523d^3+146870086724791d^4\\
&-145372529469655d^5-498268517295789d^6\\
&+2798356123102759d^7\\
&-7314741050355708d^8+12688604682534093d^9\\
&-15370507537963346d^{10}\\
&+12921839882975879d^{11}-7328568420878773d^{12}\\
&+2630599453118935d^{13}\\
&-488404739538533d^{14}-11229759183123d^{15}\\
&+27825947094802d^{16}\\
&-5713942549473d^{17}+262503065366d^{18}\\
&+42373630080d^{19})l^4\sigma\Bigg)/9(d-3)(d-1)^6\Upsilon,
\end{aligned}\tag{C.1e}
$$

$$
\begin{aligned}
\alpha_6 = -2(d-2)l^6\Bigg(& b_1(d-1)^3(902871864042048 \\
&- 2611066522935696d \\
&- 10000441381807568d^2 + 5159685561414849 6d^3 \\
&- 41200165261664328d^4 \\
&- 181595179962421999d^5 + 533465796657080410d^6 \\
&- 513376878534607377d^7 \\
&- 279078582910266796d^8 + 149198911530177159 0d^9 \\
&- 224728510007827491 2d^{10} \\
&+ 206667383168458401 8d^{11} \\
&- 129419066375853077 2d^{12} + 563666594089214817d^{13} \\
&- 167708947843161610d^{14} + 31807182182030383d^{15} \\
&- 3067722379531768d^{16} \\
&- 15879451896872d^{17} + 22671972030304d^{18}) \\
&- 4(2899692106488 - 2065011520884d \\
&- 19172437911422d^2 + 34519082576341d^3 \\
&- 83448423628318d^4 + 277332351302717d^5 \\
&- 353449049883778d^6 - 467932776789536d^7 \\
&+ 3107362543650320d^8 \\
&- 7360760025239886d^9 + 10493493512297994d^{10} \\
&- 9626241270064687d^{11} \\
&+ 5748085802510894d^{12} - 2210046214383931d^{13} \\
&+ 525872565846518d^{14} \\
&- 68068023018246d^{15} + 1803501433576d^{16} \\
&+ 540861289792d^{17})l^4\sigma\Bigg)/9(d-3)(d-1)^5\Upsilon,
\end{aligned}
\tag{C.1f}
$$

$$
\begin{aligned}
\alpha_7 = 2(d-2)^2l^6\Bigg(& b_1(d-1)^3(-786581801966112 \\
&+ 4658820413439448d \\
&- 6529543654029956d^2 - 18497282599374254d^3 \\
&+ 82484042277274539d^4 \\
&- 117690486380092256d^5 + 91492257186929 41d^6 \\
&+ 250315027315438380d^7 \\
&- 496717867127149950d^8 + 551374708000114420d^9 \\
&- 407208903035254662d^{10} \\
&+ 206801753285171702d^{11} - 71269711008561613d^{12} \\
&+ 15728365006707220d^{13} \\
&- 1844046118745923d^{14} + 20374203626764d^{15} \\
&+ 14008411441856d^{16}) \\
&- 4(-2512500801972 + 9353193417940d \\
&- 15028567971649d^2 + 28141140546484d^3 \\
&- 32887608597405d^4 - 10090867607097 4d^5 \\
&+ 558339673947000d^6 - 1407114644223220d^7 \\
&+ 2317072816104110d^8 - 2563402773401356d^9 \\
&+ 1866929648661211d^{10} \\
&- 867565615650120d^{11} + 245083387747235d^{12} \\
&- 37904656710026d^{13} + 1827835429870d^{14} \\
&+ 335120714888d^{15})l^4\sigma\Bigg)/9(d-3)(d-1)^4\Upsilon,
\end{aligned}
\tag{C.1g}
$$

$$
\begin{aligned}
\Upsilon = 1000\Big(&-4362 + 3115d + 17033d^2 - 32164d^3 \\
&+ 21034d^4 - 5675d^5 + 631d^6 + 68d^7\Big)\Big(-120984 \\
&+ 504826d - 662169d^2 + 182970d^3 + 106438d^4 \\
&+ 252706d^5 - 415237d^6 + 154042d^7\Big)l^5.
\end{aligned}
\tag{C.1h}
$$

References

1. K.S. Stelle, Renormalization of higher derivative quantum gravity. Phys. Rev. D **16**, 953 (1977)
2. B. Zwiebach, Curvature squared terms and string theories. Phys. Lett. B **156**, 315 (1985)
3. R.R. Metsaev, A.A. Tseytlin, Curvature cubed terms in string theory effective actions. Phys. Lett. B **185**, 52 (1987)
4. A. Buchel, J. Escobedo, R.C. Myers, M.F. Paulos, A. Sinha, M. Smolkin, Holographic GB gravity in arbitrary dimensions. JHEP **1003**, 111 (2010). arXiv:0911.4257 [hep-th]
5. R.C. Myers, A. Sinha, Holographic c-theorems in arbitrary dimensions. JHEP **1101**, 125 (2011). arXiv:1011.5819 [hep-th]
6. E.A. Bergshoeff, S. de Haan, W. Merbis, M. Porrati, J. Rosseel, Unitary truncations and critical gravity: a toy model. JHEP **1204**, 134 (2012). arXiv:1201.0449 [hep-th]
7. E.A. Bergshoeff, S. de Haan, W. Merbis, J. Rosseel, T. Zojer, On three-dimensional tricritical gravity. Phys. Rev. D **86**, 064037 (2012). arXiv:1206.3089 [hep-th]
8. T. Nutma, Polycritical gravities. Phys. Rev. D **85**, 124040 (2012). arXiv:1203.5338 [hep-th]
9. A. Kleinschmidt, T. Nutma, A. Virmani, On unitary subsectors of polycritical gravities. Gen. Relativ. Gravity **45**, 727 (2013). arXiv:1206.7095 [hep-th]
10. H. Lü, Y. Pang, C.N. Pope, Black Holes in six-dimensional conformal gravity. Phys. Rev. D **87**(10), 104013 (2013). arXiv:1301.7083 [hep-th]
11. L. Apolo, M. Porrati, Nonlinear dynamics of parity-even tricritical gravity in three and four dimensions. JHEP **1208**, 051 (2012). arXiv:1206.5231 [hep-th]
12. S. Deser, R. Jackiw, S. Templeton, Topologically massive gauge theories. Ann. Phys. **140**, 372 (1982). [Erratum-ibid. **185**, 406.1988 APNYA,281,409 (1988 APNYA,281,409-449.2000)]
13. W. Li, W. Song, A. Strominger, Chiral gravity in three dimensions. JHEP **0804**, 082 (2008). arXiv:0801.4566 [hep-th]
14. E.A. Bergshoeff, O. Hohm, P.K. Townsend, Massive gravity in three dimensions. Phys. Rev. Lett. **102**, 201301 (2009). arXiv:0901.1766 [hep-th]
15. E.A. Bergshoeff, O. Hohm, P.K. Townsend, More on massive 3D gravity. Phys. Rev. D **79**, 124042 (2009). arXiv:0905.1259 [hep-th]
16. H.R. Afshar, E.A. Bergshoeff, W. Merbis, Extended massive gravity in three dimensions. JHEP **1408**, 115 (2014). arXiv:1405.6213 [hep-th]
17. A. Sinha, On the new massive gravity and AdS/CFT. JHEP **1006**, 061 (2010). arXiv:1003.0683 [hep-th]

18. M.F. Paulos, New massive gravity extended with an arbitrary number of curvature corrections. Phys. Rev. D **82**, 084042 (2010). arXiv:1005.1646 [hep-th]
19. A. Ghodsi, M. Moghadassi, Charged black holes in new massive gravity. Phys. Lett. B **695**, 359 (2011). arXiv:1007.4323 [hep-th]
20. A. Ghodsi, D.M. Yekta, On asymptotically AdS-like solutions of three dimensional massive gravity. JHEP **1206**, 131 (2012). arXiv:1112.5402 [hep-th]
21. A. Ghodsi, D.M. Yekta, Stability of vacua in new massive gravity in different gauges. JHEP **1308**, 095 (2013). arXiv:1212.6876 [hep-th]
22. R.C. Myers, B. Robinson, Black holes in quasi-topological gravity. JHEP **1008**, 067 (2010). arXiv:1003.5357 [gr-qc]
23. H. Lu, Y. Pang, C.N. Pope, Conformal gravity and extensions of critical gravity. Phys. Rev. D **84**, 064001 (2011). arXiv:1106.4657 [hep-th]
24. I. Gullu, B. Tekin, Massive higher derivative gravity in D-dimensional Anti-de sitter spacetimes. Phys. Rev. D **80**, 064033 (2009). arXiv:0906.0102 [hep-th]
25. J. Oliva, S. Ray, Classification of six derivative lagrangians of gravity and static spherically symmetric solutions. Phys. Rev. D **82**, 124030 (2010). arXiv:1004.0737 [gr-qc]
26. T.C. Sisman, I. Gullu, B. Tekin, All unitary cubic curvature gravities in D dimensions. Class. Quantum Gravity **28**, 195004 (2011). arXiv:1103.2307 [hep-th]
27. P. Bueno, P.A. Cano, Einsteinian cubic gravity. Phys. Rev. D **94**(10), 104005 (2016). arXiv:1607.06463 [hep-th]
28. P. Bueno, P.A. Cano, V.S. Min, M.R. Visser, Aspects of general higher-order gravities. Phys. Rev. D **95**(4), 044010 (2017). arXiv:1610.08519 [hep-th]
29. L.F. Abbott, S. Deser, Stability of gravity with a cosmological constant. Nucl. Phys. B **195**, 76 (1982)
30. S. Deser, B. Tekin, Gravitational energy in quadratic curvature gravities. Phys. Rev. Lett. **89**, 101101 (2002). arXiv:hep-th/0205318
31. O. Hohm, E. Tonni, A boundary stress tensor for higher-derivative gravity in AdS and Lifshitz backgrounds. JHEP **1004**, 093 (2010). arXiv:1001.3598 [hep-th]
32. S. Deser, H. Liu, H. Lu, C.N. Pope, T.C. Sisman, B. Tekin, Critical points of D-dimensional extended gravities. Phys. Rev. D **83**, 061502 (2011). arXiv:1101.4009 [hep-th]
33. S. Deser, B. Tekin, Energy in generic higher curvature gravity theories. Phys. Rev. D **67**, 084009 (2003). arXiv:hep-th/0212292
34. E.A. Bergshoeff, O. Hohm, J. Rosseel, P.K. Townsend, Modes of log gravity. Phys. Rev. D **83**, 104038 (2011). arXiv:1102.4091 [hep-th]
35. E. Dyer, K. Hinterbichler, Boundary terms, variational principles and higher derivative modified gravity. Phys. Rev. D **79**, 024028 (2009). arXiv:0809.4033 [gr-qc]
36. A. Balcerzak, M.P. Dabrowski, Gibbons-Hawking boundary terms and junction conditions for higher-order brane gravity models. JCAP **0901**, 018 (2009). arXiv:0804.0855 [hep-th]
37. T. Jacobson, G. Kang, R.C. Myers, On black hole entropy. Phys. Rev. D **49**, 6587 (1994). arXiv:gr-qc/9312023
38. S. Kachru, X. Liu, M. Mulligan, Gravity duals of Lifshitz-like fixed points. Phys. Rev. D **78**, 106005 (2008). arXiv:0808.1725 [hep-th]
39. R.B. Mann, Lifshitz topological black holes. JHEP **0906**, 075 (2009). arXiv:0905.1136 [hep-th]
40. G. Bertoldi, B.A. Burrington, A. Peet, Black holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent. Phys. Rev. D **80**, 126003 (2009). arXiv:0905.3183 [hep-th]
41. E. Ayon-Beato, A. Garbarz, G. Giribet, M. Hassaine, Lifshitz black hole in three dimensions. Phys. Rev. D **80**, 104029 (2009). arXiv:0909.1347 [hep-th]
42. M.H. Dehghani, R.B. Mann, Lovelock-Lifshitz black holes. JHEP **1007**, 019 (2010). arXiv:1004.4397 [hep-th]
43. M.H. Dehghani, R.B. Mann, Thermodynamics of Lovelock-Lifshitz black branes. Phys. Rev. D **82**, 064019 (2010). arXiv:1006.3510 [hep-th]
44. W.G. Brenna, M.H. Dehghani, R.B. Mann, Quasi-topological Lifshitz black holes. Phys. Rev. D **84**, 024012 (2011). arXiv:1101.3476 [hep-th]
45. G.G. Anastasiou, M.R. Setare, E.C. Vagenas, Searching for AdS_3 waves and Asymptotically Lifshitz black holes in R^3 new massive gravity. Phys. Rev. D **88**(6), 064054 (2013). arXiv:1309.4704 [hep-th]
46. A. Giacomini, G. Giribet, M. Leston, J. Oliva, S. Ray, Scalar field perturbations in asymptotically Lifshitz black holes. Phys. Rev. D **85**, 124001 (2012). arXiv:1203.0582 [hep-th]
47. W.G. Brenna, R.B. Mann, M. Park, Mass and thermodynamic volume in Lifshitz spacetimes. Phys. Rev. D **92**(4), 044015 (2015). arXiv:1505.06331 [hep-th]
48. R.G. Cai, Y. Liu, Y.W. Sun, A Lifshitz black hole in four dimensional R**2 gravity. JHEP **0910**, 080 (2009). arXiv:0909.2807 [hep-th]
49. E. Ayon-Beato, A. Garbarz, G. Giribet, M. Hassaine, Analytic Lifshitz black holes in higher dimensions. JHEP **1004**, 030 (2010). arXiv:1001.2361 [hep-th]
50. E. Ayn-Beato, M. Bravo-Gaete, F. Correa, M. Hassane, M.M. Jurez-Aubry, J. Oliva, First law and anisotropic Cardy formula for three-dimensional Lifshitz black holes. Phys. Rev. D **91**(6), 064006 (2015). arXiv:1501.01244 [gr-qc]
51. V. Balasubramanian, P. Kraus, A Stress tensor for Anti-de Sitter gravity. Commun. Math. Phys. **208**, 413 (1999). arXiv:hep-th/9902121
52. J.D. Brown, M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity. Commun. Math. Phys. **104**, 207 (1986)
53. M. Banados, C. Teitelboim, J. Zanelli, The black hole in three-dimensional space-time. Phys. Rev. Lett. **69**, 1849 (1992). arXiv:hep-th/9204099
54. M. Cvetic, S. Nojiri, S.D. Odintsov, Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity. Nucl. Phys. B **628**, 295 (2002). arXiv:hep-th/0112045
55. S. de Haro, S.N. Solodukhin, K. Skenderis, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence. Commun. Math. Phys. **217**, 595 (2001). arXiv:hep-th/0002230
56. T. Nutma, xTras: a field-theory inspired xAct package for mathematica. Comput. Phys. Commun. **185**, 1719 (2014). arXiv:1308.3493 [cs.SC]
57. A. Ghodsi, F. Najafi, Work in progress