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S-matrix element of two R-R and one NS states

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We explicitly calculate the disk-level *S*-matrix element of two closed string R-R and one open string NS vertex operators in RNS formalism. We show that the amplitude satisfies various duality Ward identities. In particular, when one of the R-R is zero form, the other one is two form, and the NS state is a gauge boson, the amplitude transforms under an S-duality Ward identity to the amplitude of one dilaton, one B-field, and one gauge boson, which has recently been calculated explicitly. We have also proposed a soft theorem for the disk-level scattering amplitude of an ar bitrary number of hard closed strings and one soft open string at the leading order of soft momentum, and we have shown that the above amplitude satisfies the soft theorem.

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I. INTRODUCTION

The perturbative spectrum of type II closed superstrings in flat spacetime consists of a tower of bosonic states in the NS-NS and R-R sectors and their corresponding fermionic states in the R-NS and NS-R sectors (see, e.g., [1]). The nonperturbative spectrum of the type II superstring theory includes dynamical D_p-brane objects [2]. The perturbative excitations of the D_p-branes are given by the open string spectrum, which consists of bosonic states in the NS sector and their corresponding fermionic states in the R sector. In perturbative theory, the leading interaction of excited D_p-branes with the closed string states is given by the *S*-matrix elements of the corresponding closed and open string vertex operators on the disk world sheet [3,4]. The type II theory has various dualities [5,6] that appear in the *S*-matrix elements through the corresponding Ward identities [7,8].

A duality of type II theory is T-duality, which appears when one considers the theory on a compact manifold. In the simplest case that the compact manifold is a circle, the closed string spectrum of type IIA theory on the circle with radius ρ transforms under T-duality to the closed string spectrum of type IIB on a circle with radius α'/ρ. Moreover, the D_p-brane along the circle in type IIA theory transforms to a D_{p-1}-brane orthogonal to the dual circle in the type IIB theory. The T-duality Ward identity indicates that the disk-level *S*-matrix elements on the world volume of the D_p-brane in type IIA theory on a circle transform under linear T-duality to the corresponding disk-level *S*-matrix elements on the world volume of a D_{p-1}-brane. The T-duality Ward identity has been used in [9–11] to generate various disk-level *S*-matrix elements.

The type IIB theory also enjoys S-duality, which indicates that the spectrum of type IIB in flat spacetime transforms covariantly under $SL(2,R)$ transformation. In particular, the D_3-brane is invariant under the $SL(2,R)$ transformation, the NS-NS antisymmetric B-field, and the R-R two-form transforms as a doublet under the $SL(2,R)$ transformation. The S-duality Ward identity indicates that the disk-level S-matrix elements on the world volume of the D_3-brane in type IIB transforms under linear $SL(2,R)$ transformation to the corresponding disk-level S-matrix elements on the world volume of the D_3-brane. This Ward identity may be used to generate the complicated S-matrix elements of R-R vertex operators that involve the spin operator [12] from the corresponding *S*-matrix elements of NS-NS vertex operators, which are straightforward to calculate.

As an example of the *S*-matrix elements of the R-R vertex operators, in this paper, we explicitly calculate the disk-level *S*-matrix element of two R-R and one NS vertex operators in RNS formalism. Such amplitude has recently been predicted by the S-duality Ward identity [13]. We observe that the explicit calculations produce exactly the amplitude predicted by the S-duality. The S-duality Ward identity indicates that apart from the overall dilaton factor of a background dilaton, the disk-level *S*-matrix elements must combine into S-dual multiplets which are invariant under the linear $SL(2,R)$ transformation [13]. This indicates that the amplitudes involving two R-R and one NS states, which cannot be written in the S-dual multiplet, must be zero. We observe that the amplitudes that are predicted by S-duality to be zero, e.g., the $C^{(0)}C^{(0)}F$-amplitude, are in fact zero by explicit calculation.

A consistency check of the *S*-matrix elements in string theory is that they must satisfy the soft graviton/photon theorems [14–27] in which one graviton/photon is soft. In the soft theorems [14–27], however, the external states are all either closed string states or all open string states, which we are not interested in for this paper. When one string state is soft open string and all the other states are hard closed string states, one can easily find the corresponding soft therm at the leading order of the soft open string momentum.

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The Ward identity corresponding to the gauge boson transformation indicates that the disk-level S-matrix element of n closed strings and one open string gauge field must be in the following form[1]:

$$\mathcal{A}_{n+1} = f^{ab}[\mathcal{A}_n(k, p_1, p_1\cdot D, p_2, p_2\cdot D, \ldots, p_n)]_{ab},$$

where f^{ab} is the gauge field strength in momentum space and $(\mathcal{A}_n)_{[ab]}$ is the factor which does not involve the open string gauge field polarization.[2] In general, this factor is a complicated function of the momentum of the gauge field k^a. However, using the fact that there is only one open string state, one observes that there is no pole $1/k\cdot k$ in the amplitude. As a result, when the gauge field is soft, i.e., $k^a \to 0$, one finds that the above amplitude defines a soft theorem, which involves the soft factor f^{ab} at the leading order, and the hard factor $[\mathcal{A}_n]_{ab}$, which involves only polarizations and the momenta of the n closed string states, i.e.,

$$\mathcal{A}_{n+1} = f^{ab}[\mathcal{A}_n(p_1, p_1\cdot D, p_2, p_2\cdot D, \ldots, p_n)]_{ab}. \quad (1)$$

The trace of $[\mathcal{A}_n]_{ab}$ is the disk-level scattering amplitude of n closed strings, which is zero because $[\mathcal{A}_n]_{ab}$ is antisymmetric. The above relation exists at any order of α', so one expects the disk-level S-matrix elements to satisfy this theorem for the soft gauge field.

There is a similar theorem when the open string state is a transverse scalar field. To find such a theorem at the leading order of the scalar momentum, we consider the observation that the closed string fields in effective world-volume action must be the Taylor expansion of the transverse scalar fields [28], i.e., $C(\Phi^i) = C + \Phi^i\partial_i C + \cdots$, where C is a closed string field. Using the coupling $\Phi^i\partial_i C$, one can easily write the S-matrix element of n closed strings and one transverse scalar field at the leading order of k^a and at the leading order of α' to be

$$\mathcal{A}_{n+1} = \zeta^i(p_1 + p_2 + \cdots + p_n)_i\mathcal{A}_n, \quad (2)$$

where ζ^i is the polarization of the scalar fields and $\mathcal{A}_n$ is the scattering amplitude of n closed string states at low energy. The above relation must be valid for any order of α'. As a result, the disk-level S-matrix element of one transverse scalar and n closed string states must satisfy the above soft theorem. We will show that the scattering amplitude of two R-R and one NS states satisfies the above soft theorems.

An outline of the paper is as follows: We begin Sec. II by explicitly calculating the disk-level scattering amplitude of two R-R and one NS vertex operators in RNS formalism. We use the $(-1/2,-1/2)$-picture for the R-R vertex operators. In this picture, the field strengths of the R-R fields appear in the vertex operators; as a result, the amplitude satisfies the R-R gauge symmetry Ward identity from the onset. We show that the final amplitude satisfies the open string gauge symmetry Ward identity as well. In Sec. III, we show that the amplitude satisfies the S-duality Ward identity. In particular, the amplitude of one R-R zero-form, one R-R two-form, and one NS gauge field transforms under the S-duality Ward identity to the amplitude of one NS-NS dilaton, one B-field, and one NS gauge field that has recently been calculated explicitly. In this section, we also show that the amplitudes that are predicted by the S-duality Ward identity to be zero are in fact zero. In Sec. IV, we explicitly write the D_p-brane amplitudes that are nonzero for $p = 0, 1, 2, 3$. In Sec. V, we show that these amplitudes satisfy the T-duality Ward identity. In Sec. VI, we show that the amplitudes that we have found in Sec. IV satisfy the soft scalar theorem (2). In Sec. VII, we show that the amplitudes satisfy the soft-photon theorem and find the kinematic factors in $[\mathcal{A}_2]_{ab}$.

II. EXPLICIT CALCULATION OF THE AMPLITUDE

The tree-level scattering amplitude of two R-R closed string and one NS open string states on the world volume of a D_p-brane is given by the correlation function of their corresponding vertex operators on the disk (see, e.g., [8]). Since the background charge of the world sheet with the topology of a disk is $Q_\phi = 2$, one has to choose the vertex operators in the appropriate pictures to produce the compensating charge $Q_\phi = -2$. The scattering amplitude may then be given by the following correlation function:

$$\mathcal{A}(\varepsilon_1, p_1; \varepsilon_2, p_2; \zeta_3, k_3) \sim \langle V_{RR}^{(-1/2,-1/2)}(\varepsilon_1, p_1) V_{RR}^{(-1/2,-1/2)}(\varepsilon_2, p_2) V_{NS}^{(0)}(\zeta_3, k_3)\rangle, \quad (3)$$

where the vertex operators, after using the doubling trick [3], are

$$\begin{aligned} V_{RR}^{(-1/2,-1/2)} &= (P_-\Gamma_{1(n)}M_p)^{AB}\int d^2z_1 \\ &\quad :e^{-\phi(z_1)/2}S_A(z_1)e^{ip_1\cdot X} \\ &\quad :e^{-\phi(\bar{z}_1)/2}S_B(\bar{z}_1)e^{ip_1\cdot D\cdot X}: \\ V_{RR}^{(-1/2,-1/2)} &= (P_-\Gamma_{2(m)}M_p)^{CD}\int d^2z_2 \\ &\quad :e^{-\phi(z_2)/2}S_C(z_2)e^{ip_2\cdot X} \\ &\quad :e^{-\phi(\bar{z}_2)/2}S_D(\bar{z}_2)e^{ip_2\cdot D\cdot X}: \\ V_{NS}^{(0)} &= (\zeta_3)_\mu\int dz_3 :(\partial X^\mu + 2ik_3\cdot\psi\psi^\mu)e^{2ik_3\cdot X}, \end{aligned} \quad (4)$$

[1]Using conservation of momentum along the D-brane, i.e., $2k + p_1 + p_1\cdot D + p_2 + p_2\cdot D + \cdots + p_n + p_n\cdot D = 0$, one may write $p_n\cdot D$ in terms of other momenta.

[2]Our index convention is that the greek letters $(\mu, \nu, \ldots)$ are the indices of the space-time coordinates; the latin letters $(a, d, c, \ldots)$ are the world-volume indices; and the letters $(i, j, k, \ldots)$ are the normal bundle indices.

where z_3 is along the real axis and z_1, z_2 are in upper-half z-plane. The index μ is the world volume index a when the NS state is a gauge boson and is the transverse index i when the NS state is the transverse scalar fields. The indices $A, B, \ldots$ are the Dirac spinor indices and $P_- = \frac{1}{2}(1-\gamma_{11})$ is the chiral projection operator so our calculations are always done with the full 32×32 Dirac matrices of the ten dimensions. The R-R polarization appears in $\Gamma_{(n)}$ and the world-volume Levi-Cività tensor appears in M_p, i.e.,

$$\Gamma_{(n)} = \frac{1}{n!} F_{\mu_1 \ldots \mu_n} \gamma^{\mu_1} \ldots \gamma^{\mu_n}$$
$$M_p = \frac{\pm 1}{(p+1)!} \epsilon_{a_0 \ldots a_p} \gamma^{a_0} \ldots \gamma^{a_p}, \tag{5}$$

where $F^{(n)}$ is the linearized field strength of the R-R potential $C^{(n-1)}$. The matrix $D_{\mu\nu}$ is a diagonal matrix which is the world-volume flat metric when the indices are the world-volume indices, and is minus the flat metric of the transverse space when the indices are the transverse indices. We have used the integral form for all vertex operators; as a result, the integrand of the amplitude must be invariant under the conformal transformation of the upper-half plane, which is the $SL(2,R)$ transformation.

Using the standard upper-half z-plane propagators

$$\langle X^\mu(x) X^\nu(y) \rangle = -\eta^{\mu\nu} \log(x-y)$$
$$\langle \phi(x) \phi(y) \rangle = -\log(x-y), \tag{6}$$

one can easily calculate the ghost ϕ and X^μ correlators in (3). To calculate the correlation functions between the spin operators and the world-sheet field ψ^μ, we use the following operator product expansion:

$$:S_A(z_1):\psi^\mu\psi^\nu(z_2) \simeq -\frac{(\Gamma^{\mu\nu})_A{}^E}{2z_{21}} :S_E(z_1): \tag{7}$$

where $z_{ij} = z_i - z_j$ and $\Gamma^{\mu\nu}$ is $\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, to reduce the correlators to the following known correlation function [12]:

$$\langle :S_A(z_1):S_B(z_2):S_C(z_3):S_D(z_4): \rangle = I_{ABCD}, \tag{8}$$

where

$$I_{ABCD} = \frac{z_{14}z_{23}(\gamma^\mu)_{AB}(\gamma_\mu)_{CD} - z_{12}z_{34}(\gamma^\mu)_{AD}(\gamma_\mu)_{BC}}{2(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{\frac{3}{4}}}. \tag{9}$$

This calculation produces the following 14 terms,

$$\mathcal{A}(\varepsilon_1, p_1; \varepsilon_2, p_2; \zeta_3, k_3) \sim \int d^2z_1 d^2z_2 dz_3 K \sum_{i=1}^{14} A_i, \tag{10}$$

where $A_1, \ldots, A_{14}$ in the integrand are the following:

$$A_1 = \frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma^\mu)_{AB}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{CD}(\zeta_3)^\chi p_2 \cdot D_\chi}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{3\bar{1}}z_{3\bar{2}}}$$
$$A_2 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma_\mu)_{BC}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma^\mu)_{AD}(\zeta_3)^\chi p_2 \cdot D_\chi}{z_{12}z_{1\bar{2}}z_{3\bar{1}}z_{3\bar{2}}z_{\bar{1}2}}$$
$$A_3 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\Gamma^{b\chi})_A{}^E(\gamma^\mu)_{EB}(k_3)_b(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{CD}}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{31}z_{\bar{1}\,\bar{2}}}$$
$$A_4 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma^\mu)_{AB}(k_3)_b(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\Gamma^{b\chi})_C{}^E(\gamma^\mu)_{ED}}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{32}z_{\bar{1}\,\bar{2}}}$$
$$A_5 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\Gamma^{b\chi})_B{}^E(\gamma^\mu)_{AE}(k_3)_b(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{CD}}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{3\bar{1}}z_{\bar{3}\,\bar{2}}}$$
$$A_6 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma^\mu)_{AB}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{CD}(\zeta_3)^\chi p_{1\chi}}{z_{12}z_{2\bar{2}}z_{31}z_{3\bar{1}}z_{\bar{1}\,\bar{2}}}$$
$$A_7 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma_\mu)_{BC}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma^\mu)_{AD}(\zeta_3)^\chi p_{2\chi}}{z_{12}z_{1\bar{2}}z_{32}z_{3\bar{1}}z_{\bar{1}\,\bar{2}}}$$
$$A_8 = -\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma^\mu)_{AB}(P_-\Gamma_{2(m)}M_p)^{CD}(k_3)_b(\gamma_\mu)_{CE}(\Gamma^{b\chi})_D{}^E(\zeta_3)_\chi}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{3\bar{2}}z_{\bar{1}\,\bar{2}}}$$
$$A_9 = \frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\Gamma^{a\chi})_A{}^E(\gamma^\mu)_{ED}(k_3)_a(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{BC}}{z_{12}z_{1\bar{2}}z_{31}z_{\bar{1}2}z_{\bar{1}\,\bar{2}}}$$

$$A_{10}=\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\Gamma^{a\chi})_C{}^E(\gamma^\mu)_{AD}(k_3)_a(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{BE}}{z_{12}z_{1\bar{2}}z_{32}z_{\bar{1}2}z_{\bar{1}\bar{2}}}$$
$$A_{11}=\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\Gamma^{a\chi})_B{}^E(\gamma^\mu)_{AD}(k_3)_a(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{EC}}{z_{12}z_{1\bar{2}}z_{3\bar{1}}z_{\bar{1}2}z_{\bar{1}\bar{2}}}$$
$$A_{12}=\frac{iz_{1\bar{1}}(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma_\mu)_{BC}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma^\mu)_{AD}(\zeta_3)^\chi p_{1\chi}}{z_{12}z_{1\bar{2}}z_{31}z_{3\bar{1}}z_{\bar{1}2}z_{\bar{1}\bar{2}}}$$
$$A_{13}=\frac{i(P_-\Gamma_{1(n)}M_p)^{AB}(\Gamma^{a\chi})_D{}^E(\gamma^\mu)_{AE}(k_3)_a(\zeta_3)_\chi(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{BC}}{z_{12}z_{1\bar{2}}z_{3\bar{2}}z_{\bar{1}2}z_{\bar{1}\bar{2}}}$$
$$A_{14}=\frac{iz_{\bar{1}2}(P_-\Gamma_{1(n)}M_p)^{AB}(\gamma^\mu)_{AB}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{CD}(\zeta_3)^\chi p_{2\chi}}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{32}z_{3\bar{1}}z_{\bar{1}\bar{2}}}, \tag{11}$$

and the overall kinematic factor K is

$$K=z_{1\bar{1}}^{p_1.D.p_1}|z_{12}|^{2p_1.p_2}|z_{1\bar{2}}|^{2p_1.D.p_2}|z_{13}|^{4p_1.k_3}z_{2\bar{2}}^{p_2.D.p_2}|z_{23}|^{4p_2.k_3}. \tag{12}$$

One can easily verify that each integrand is invariant under $SL(2,R)$ transformation.

For subsequent discussions, we rewrite the amplitude as

$$\mathcal{A}(\varepsilon_1,p_1;\varepsilon_2,p_2;\zeta_3,k_3)\sim\sum_{i=1}^{14}a_iq_i, \tag{13}$$

where q_i's are integrals of some function of z_i's and momenta, e.g.,

$$q_1=\int d^2z_1d^2z_2dz_3\frac{K}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{3\bar{1}}z_{3\bar{2}}}, \tag{14}$$

and a_i's are the remaining terms in A_i's, e.g.,

$$a_1=i(P_-\Gam_{1(n)}M_p)^{AB}(\gamma^\mu)_{AB}(P_-\Gamma_{2(m)}M_p)^{CD}(\gamma_\mu)_{CD}(\zeta_3)^\chi p_2\cdot D_\chi. \tag{15}$$

Contracting the Dirac indices, one finds each of a_2, a_7, a_9, a_{10}, a_{11}, a_{12}, a_{13} produces one trace over the gamma matrices, i.e.,

$$\begin{aligned}
&a_2=ib_2(\zeta_3)^\chi p_2\cdot D_\chi, \qquad a_7=ib_2(\zeta_3)^\chi p_{2\chi},\\
&a_{12}=ib_2(\zeta_3)^\chi p_{1\chi} \qquad a_9=ib_9^{a\chi}(k_3)_a(\zeta_3)_\chi,\\
&a_{10}=a_9(1\leftrightarrow 2) \qquad a_{13}=-ib_{13}^{a\chi}(k_3)_a(\zeta_3)_\chi,\\
&a_{11}=a_{13}(1\leftrightarrow 2),
\end{aligned} \tag{16}$$

where b_2, b_9, b_{13} are the following one-trace terms:

$$b_2=\mathrm{Tr}(P_-\Gamma_{1(n)}M_p\gamma_\mu\Gamma_{2(m)}M_p\gamma^\mu) \tag{17}$$

$$b_9^{a\chi}=\mathrm{Tr}(P_-\Gamma_{1(n)}M_p\gamma_\mu\Gamma_{2(m)}M_p\gamma^\mu\Gamma^{a\chi}) \tag{18}$$

$$b_{13}^{a\chi}=\mathrm{Tr}(P_-\Gamma_{1(n)}M_p\gamma_\mu\Gamma_{2(m)}M_p\Gamma^{a\chi}\gamma^\mu). \tag{19}$$

One also finds that each of a_1, a_3, a_4, a_5, a_6, a_8, a_{14} produces two traces over the gamma matrices, i.e.,

$$\begin{aligned}
&a_1=ib_1(\zeta_3)^\chi p_2\cdot D_\chi, \qquad a_6=ib_1(\zeta_3)^\chi p_{1\chi},\\
&a_{14}=ib_1(\zeta_3)^\chi p_{2\chi} \qquad a_3=ib_3^{b\chi}(k_3)_b(\zeta_3)_\chi,\\
&a_4=a_3(1\leftrightarrow 2) \qquad a_5=-ib_5^{b\chi}(k_3)_a(\zeta_3)_\chi,\\
&a_8=b_5(1\leftrightarrow 2),
\end{aligned}$$

where b_1, b_3, b_5 are the following two-trace terms:

$$\begin{aligned}
b_1&=\mathrm{Tr}(P_-\Gamma_{1(n)}M_p\gamma^\mu)\mathrm{Tr}(P_-\Gamma_{2(m)}M_p\gamma_\mu)\\
b_3^{b\chi}&=\mathrm{Tr}(P_-\Gamma_{1(n)}M_p\gamma^\mu\Gamma^{b\chi})\mathrm{Tr}(P_-\Gamma_{2(m)}M_p\gamma_\mu)\\
b_5^{b\chi}&=\mathrm{Tr}(P_-\Gamma_{1(n)}M_p\Gamma^{b\chi}\gamma^\mu)\mathrm{Tr}(P_-\Gamma_{2(m)}M_p\gamma_\mu).
\end{aligned} \tag{20}$$

Each trace in b_1, $b_3^{b\chi}$, $b_5^{b\chi}$ has one factor of M_p, whereas in b_2, $b_9^{a\chi}$, $b_{13}^{a\chi}$ each trace has two factors of M_p. This makes the calculation of the traces in b_2, $b_9^{a\chi}$, $b_{13}^{a\chi}$ difficult for a general D_p-brane. Using the GAMMA package [29], we have found the following relations,

$$\begin{aligned}
b_2&=c_2-b_1\\
b_9^{a\chi}&=c_9^{a\chi}-b_3^{a\chi}\\
b_{13}^{a\chi}&=c_{13}^{a\chi}-b_5^{a\chi},
\end{aligned} \tag{21}$$

where c_2, $c_9^{a\chi}$, $c_{11}^{a\chi}$ are the following one-trace terms,

$$c_2 \equiv \mathrm{Tr}(P_-\Gamma_{1(n)}D_\mu{}^\nu\gamma_\nu M_p C^{-1}M_p^T\Gamma^T_{2(m)}C\gamma^\mu)$$
$$c_9^{a\chi} \equiv \mathrm{Tr}(P_-\Gamma_{1(n)}D_\mu{}^\nu\gamma_\nu M_p C^{-1}M_p^T\Gamma^T_{2(m)}C\gamma^\mu\Gamma^{a\chi})$$
$$c_{13}^{a\chi} \equiv \mathrm{Tr}(P_-\Gamma_{1(n)}D_\mu{}^\nu\gamma_\nu M_p C^{-1}M_p^T\Gamma^T_{2(m)}C\Gamma^{a\chi}\gamma^\mu). \quad (22)$$

The first relation in (21) has been found in [30] by using the Kawai-Lewellen-Tye relation that maps the amplitude of four open string fermions to the amplitude of two closed string R-R bosons. This relation has been found by using two different amplitudes for four fermions that are produced by different ways of fixing the $SL(2,R)$ symmetry of the disk amplitude. Now using the relation $M_pC^{-1}M_p^TC = 1$ [3], one finds the traces in c_2, $c_9^{a\chi}, c_{13}^{a\chi}$ have in fact no factor of M_p, i.e.,

$$c_2 = \mathrm{Tr}(P_-\Gamma_{1(n)}D_\mu{}^\nu\gamma_\nu\Gamma_{2(m)}\gamma^\mu)(-1)^{\frac{1}{2}m(m+1)}$$
$$c_9^{a\chi} = \mathrm{Tr}(P_-\Gamma_{1(n)}D_\mu{}^\nu\gamma_\nu\Gamma_{2(m)}\gamma^\mu\Gamma^{a\chi})(-1)^{\frac{1}{2}m(m+1)}$$
$$c_{13}^{a\chi} = \mathrm{Tr}(P_-\Gamma_{1(n)}D_\mu{}^\nu\gamma_\nu\Gamma_{2(m)}\Gamma^{a\chi}\gamma^\mu)(-1)^{\frac{1}{2}m(m+1)}. \quad (23)$$

So c_2, $c_9^{a\chi}, c_{13}^{a\chi}$ are independent of the dimension of the D_p-brane. One has to perform the traces in (20) and (23) to find the amplitude (13) in terms of momenta and polarizations of the external states.

Before performing the traces, we show that the amplitude (13) satisfies the Ward identity corresponding to the gauge boson. To this end, we have to replace the gauge boson polarization tensor ζ_3^a with its momentum ik_3^a and show that the result vanishes. Using the antisymmetric property of the function Γ^{ab}, one easily observes that $a_3, a_4, a_5, a_8, a_9, a_{10}, a_{11}, a_{13}$ become zero after replacing ζ_3^a with ik_3^a. Using the on-shell relation $p_1\cdot k_3 = -p_2\cdot k_3$, one finds the remaining terms to be

$$\mathcal{A} \sim -ib_1 p_1\cdot k_3[-q_1+q_6-q_{14}] + ib_2 p_1\cdot k_3[-q_2-q_7+q_{12}]. \quad (24)$$

Since b_1, b_2 are not zero generally, we should prove that the integrals in each bracket are zero, i.e.,

$$M_1 \equiv -q_1+q_6-q_{14} = \int \frac{K(z_{1\bar1}z_{32}z_{3\bar2}+z_{31}z_{3\bar2}z_{\bar12}+z_{31}z_{32}z_{\bar1\bar2})}{z_{12}z_{1\bar1}z_{2\bar2}z_{31}z_{32}z_{3\bar1}z_{3\bar2}z_{\bar1\bar2}} \times d^2z_1d^2z_2dx_3 = 0$$
$$M_2 \equiv -q_2-q_7+q_{12} = \int \frac{K(z_{1\bar1}z_{32}z_{3\bar2}+z_{31}z_{3\bar2}z_{\bar12}+z_{31}z_{32}z_{\bar1\bar2})}{z_{12}z_{1\bar2}z_{31}z_{32}z_{3\bar1}z_{3\bar2}z_{\bar12}z_{\bar1\bar2}} \times d^2z_1d^2z_2dx_3 = 0. \quad (25)$$

Using the fact that the integrands are invariant under the $SL(2,R)$ transformation, one may map the integrands to the unit disk by the following transformations:

$$z_i \to -i\frac{z_i-1}{z_i+1}$$
$$\bar z_i \to i\frac{\bar z_i-1}{\bar z_i+1}. \quad (26)$$

To fix the $SL(2,R)$ symmetry, we use the Faddeev-Popov gauge-fixing mechanism to fix $z_1 = \bar z_1 = 0$ and $x_3 = 1$. The Jacobian of this transformation is $2i$. Writing $z_2 = re^{i\theta}$ and $\bar z_2 = re^{-i\theta}$, one finds

$$M1 = \int_0^1 rdr\int_0^{2\pi}d\theta\frac{4\sin(\theta)(r^2)^{p_1\cdot p_2}(1-r^2)^{p_2.D.p_2-1}}{r(r^2-2r\cos(\theta)+1)^{2p_1.k_3+1}}$$
$$M2 = \int_0^1 rdr\int_0^{2\pi}d\theta\frac{4\sin(\theta)(r^2)^{p_1\cdot p_2}(1-r^2)^{p_2.D.p_2}}{r(r^2-2r\cos(\theta)+1)^{2p_1.k_3+1}}. \quad (27)$$

The θ integration then gives zero result. So the amplitude (13) satisfies the Ward identity corresponding to the gauge boson transformation, as expected.

A. Performing the traces

We now calculate the traces in (20) and (23). Defining the one-trace terms $d_1^\mu, d_3^{\mu b\chi}, d_5^{\mu b\chi}$ as

$$d_1^\mu(n) \equiv \mathrm{Tr}(P_-\Gamma_{(n)}M_p\gamma^\mu)$$
$$d_3^{\mu b\chi}(n) \equiv \mathrm{Tr}(P_-\Gamma_{1(n)}M_p\gamma^\mu\Gamma^{b\chi})$$
$$d_5^{\mu b\chi}(n) \equiv \mathrm{Tr}(P_-\Gamma_{1(n)}M_p\Gamma^{b\chi}\gamma^\mu), \quad (28)$$

the two-trace terms in (20) can be written as

$$b_1 = d_1^\mu(n)d_{1\mu}(m), \qquad b_3^{b\chi} = d_3^{\mu b\chi}(n)d_{1\mu}(m),$$
$$b_5^{b\chi} = d_5^{\mu b\chi}(n)d_{1\mu}(m). \quad (29)$$

The explicit calculation of the traces in $d_1^\mu, d_3^{\mu b\chi}, d_3^{\mu b\chi}$ gives the following result:

$$
\begin{aligned}
d_1^{\mu}(n) &= \frac{16}{n!}(-1)^{\frac{1}{2}n(n+1)}\epsilon_{a_0\ldots a_p}[\delta_{np}\delta_{\mu}^{a_0}F^{a_1\ldots a_p} + n\delta_{n,p+2}F^{a_0\ldots a_p\mu}] + (n\to n', F\to *F)\\
d_3^{\mu b\chi}(n) &= -(-1)^{\frac{1}{2}n(n-1)}\frac{16}{n!}\{\delta_{n,p-2}F_{a_0a_1\ldots a_{p-3}}\varepsilon^{a_0\ldots a_{p-3}\mu b\chi} + \delta_{np}[nF_{a_0a_1\ldots a_{p-2}}{}^{\chi}\varepsilon^{a_0\ldots a_{p-2}b\mu}\\
&\quad - nF_{a_0\ldots a_{p-2}}{}^{\mu}\varepsilon^{a_0a_1a_2\ldots b\chi} + F_{a_0\ldots a_{p-1}}\varepsilon^{a_0\ldots a_{p-1}\chi}\delta^{b\mu} - nF_{a_0\ldots a_{p-2}}{}^{b}\varepsilon^{a_0\ldots a_{p-2}\chi\mu}\\
&\quad - F_{a_1\ldots a_p}\varepsilon^{a_1\ldots a_pb}\delta^{\chi\mu}] - \delta_{n,p+2}n[(n-1)\varepsilon^{\chi}{}_{a_0a_1\ldots a_{p-1}}F^{ba_0\ldots a_{p-1}\mu}\\
&\quad - (n-1)\varepsilon_{a_0\ldots a_{p-1}}{}^{\mu}F^{a_0\ldots a_{p-1}b\chi} - \varepsilon_{a_0\ldots a_p}F^{a_0\ldots a_p\chi}\delta^{b\mu}\\
&\quad - (n-1)\varepsilon_{a_0\ldots a_{p-1}}{}^{b}F^{a_0\ldots a_{p-1}\chi\mu} + \varepsilon_{a_0\ldots a_p}F^{a_0\ldots a_pb}\delta^{\chi\mu}]\} + (n\to n', F\to *F)\\
d_5^{\mu b\chi}(n) &= -(-1)^{\frac{1}{2}n(n-1)}\frac{16}{n!}\{\delta_{n,p-2}F_{a_0a_1\ldots a_{p-3}}\varepsilon^{a_0\ldots a_{p-3}\mu b\chi} + \delta_{n,p}[nF_{a_0a_1\ldots a_{p-2}}{}^{\chi}\varepsilon^{a_0\ldots a_{p-2}b\mu}\\
&\quad - nF_{a_0\ldots a_{p-2}}{}^{\mu}\varepsilon^{a_0a_1a_2\ldots b\chi} - F_{a_0\ldots a_{p-1}}\varepsilon^{a_0\ldots a_{p-1}\chi}\delta^{b\mu} - nF_{a_0\ldots a_{p-2}}{}^{b}\varepsilon^{a_0\ldots a_{p-2}\chi\mu}\\
&\quad + F_{a_1\ldots a_p}\varepsilon^{a_1\ldots a_pb}\delta^{\chi\mu}] - n\delta_{n,p+2}[(n-1)\varepsilon_{a_0a_1\ldots a_{p-1}}{}^{\chi}F^{a_0\ldots a_{p-1}b\mu}\\
&\quad - (n-1)\varepsilon_{a_0\ldots a_{p-1}}{}^{\mu}F^{a_0\ldots a_{p-1}b\chi} + \varepsilon_{a_0\ldots a_p}F^{a_0\ldots a_p\chi}\delta^{b\mu}\\
&\quad - (n-1)\varepsilon_{a_0\ldots a_{p-1}}{}^{b}F^{a_0\ldots a_{p-1}\chi\mu} - \varepsilon_{a_0\ldots a_p}F^{a_0\ldots a_pb}\delta^{\chi\mu}]\} + (n\to n', F\to *F),
\end{aligned}
\tag{30}
$$

where $n' = 10 - n$ and F is the R-R field strength. When we replace them in (29), one of the R-R field strengths is F_1 and the other one is F_2. Replacing the above one-trace terms in (29), one finds b_1, $b_3^{b\chi}, b_3^{b\chi}$ in terms of momenta and polarizations of the external states. With the replacement of the resulting b_1, $b_3^{b\chi}, b_3^{b\chi}$ in (19), one then finds a_1, a_3, a_4, a_5, a_6, a_8, a_{14}.

The explicit calculation of the one-trace terms c_2, $c_9^{b\chi}, c_{13}^{b\chi}$ gives the following result:

$$
\begin{aligned}
c_2 &= \frac{16}{n!}\delta_{n,m}[\mathrm{Tr}(D)F_1^{(n)}.F_2^{(n)} - 2nD^{\lambda}{}_{\rho}F_{1\lambda\mu_2\ldots\mu_n}F_2^{\rho\mu_2\ldots\mu_n}] + (n\to n', F_1\to *F_1)\\
c_9^{b\chi} &= \frac{16}{n!}\delta_{m,n+2}[\mathrm{Tr}(D)F_{1(n)}^{\mu_1\ldots\mu_n}F_{2(m)\mu_1\ldots\mu_n}^{b\chi} + 2D^{b\mu}F_1^{\mu_1\ldots\mu_n}F_2{}^{\chi}{}_{\mu\mu_1\ldots\mu_n} - 2D^{\mu\chi}F_1^{\mu_1\ldots\mu_n}F_{2\mu,\mu_1,\ldots,\mu_n}^{b} - 2nD_{\mu}{}^{\nu}F_1^{\mu,\mu_2,\ldots,\mu_n}F_{2\nu,a,\chi,\mu_2,\ldots,\mu_n}]\\
&\quad + \frac{16}{(n-1)!}\delta_{mn}[\mathrm{Tr}(D)F_{1\mu_2\ldots\mu_n}^{b}F_2^{\chi\mu_2\ldots\mu_n} - \mathrm{Tr}(D)F_1^{\chi\mu_2\ldots,\mu_n}F_{2\mu_2\ldots\mu_n}^{b} - 2D^{b\mu}F_{1\mu\mu_2\ldots\mu_n}F_2^{\chi\mu_2\ldots\mu_n} + 2D^{\chi}{}_{\mu}F_1^{\mu\mu_2\ldots\mu_n}F_{2\mu_2\ldots\mu_n}^{b}\\
&\quad + 2(n-1)D^{\mu}{}_{\nu}(F_{1\mu\mu_3\ldots\mu_n}^{b}F_2^{\chi\nu\mu_3\ldots\mu_n} - F_1^{\chi\nu\mu_2\ldots\mu_n}F_{2\mu\mu_3\ldots\mu_n}^{b})] + (n\to n', F_1\to *F_1)\\
c_{13}^{b\chi} &= \frac{16}{n!}\delta_{m,n+2}[\mathrm{Tr}(D)F_1^{\mu_1\ldots\mu_n}F_2^{a\chi}{}_{\mu_1\ldots\mu_n} - 2D^{b\mu}F_1^{\mu_1\ldots\mu_n}F_{2\mu\mu_1\ldots\mu_n}^{\chi} + 2D^{\mu\chi}F_1^{\mu_1\ldots\mu_n}F_{2\mu\mu_1\ldots\mu_n}^{b} - 2nD_{\mu}{}^{\nu}F_1^{\mu\mu_2\ldots\mu_n}F_2^{b\chi}{}_{\nu\mu_2\ldots\mu_n}]\\
&\quad + \frac{16}{(n-1)!}\delta_{mn}[\mathrm{Tr}(D)F_{1\mu_2\ldots\mu_n}^{b}F_2^{\chi\mu_2\ldots\mu_n} - \mathrm{Tr}(D)F_1^{\chi\mu_2\ldots\mu_n}F_{2\mu_2\ldots\mu_n}^{b} + 2D^{b\mu}F_{1\mu\mu_2\ldots\mu_n}F_2^{\chi\mu_2\ldots\mu_n} - 2D^{\chi}{}_{\mu}F_1^{\mu\mu_2\ldots\mu_n}F_{2\mu_2\ldots\mu_n}^{b}\\
&\quad + 2(n-1)D^{\mu}{}_{\nu}(F_{1\mu\mu_3\ldots\mu_n}^{b}F_2^{\chi\nu\mu_3\ldots\mu_n} - F_1^{\chi\nu\mu_3\ldots\mu_n}F_{2\mu\mu_3\ldots\mu_n}^{b})] + (n\to n', F_1\to *F_1).
\end{aligned}
\tag{31}
$$

Replacing the above results for c_2, $c_9^{b\chi}, c_{13}^{b\chi}$ and b_1, $b_3^{b\chi}, b_5^{b\chi}$ in (21), one finds b_2, $b_9^{b\chi}, b_{13}^{b\chi}$ and hence one finds a_2, a_7, a_9, a_{10}, a_{11}, a_{12}, a_{13} upon replacing them in (16). Having performed the traces in (13), one now has the amplitude (13) in terms of momenta and polarization of the external states. To check our results, in the next section, we study the S-duality Ward identity of the amplitude.

III. S-DUALITY WARD IDENTITY

The D_p-brane S-matrix element in the previous section is valid for any p. It is known that the D_3-brane is invariant under S-duality transformation, so the S-matrix elements of the D_3-brane should satisfy the Ward identity corresponding to the $SL(2,R)$ transformation, which is a linear transformation on quantum fluctuations and a nonlinear transformation on the background fields [13]. The B-field and RR two-form transform as a doublet, i.e.,

$$
\begin{aligned}
\mathcal{B} &\equiv \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} B \\ C^{(2)} \end{pmatrix};\\
\Lambda &= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,R).
\end{aligned}
\tag{32}
$$

The gauge boson field strength F_{ab} and its magnetic dual $(*F)_{ab} = \epsilon_{abcd}F^{cd}/2$ also transform as a doublet, i.e.,

$$\mathcal{F} \equiv \begin{pmatrix} (*f) \\ e^{-\phi_0} f - C_0 (*f) \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} (*f) \\ e^{-\phi_0} f - C_0 (*f) \end{pmatrix},$$

where ϕ_0 and C_0 are the background dilaton and R-R scalar, respectively. The transformation of the background dilaton and R-R scalar is [31–33]

$$\mathcal{M}_0 \to \Lambda \mathcal{M}_0 \Lambda^T, \tag{33}$$

where the matrix $\mathcal{M}_0$ is

$$\mathcal{M}_0 = e^{\phi_0} \begin{pmatrix} |\tau_0|^2 & C_0 \\ C_0 & 1 \end{pmatrix} \tag{34}$$

where $\tau_0 = C_0 + i e^{-\phi_0}$. Quantum fluctuations of the dilaton and the R-R scalar appear in $\delta\mathcal{M}$, i.e.,

$$\delta\mathcal{M} = \begin{pmatrix} -(e^{-\phi_0} - C_0^2 e^{\phi_0})\delta\phi + 2C_0 e^{\phi_0}\delta C & C_0 e^{\phi_0}\delta\phi + e^{\phi_0}\delta C \\ C_0 e^{\phi_0}\delta\phi + e^{\phi_0}\delta C & e^{\phi_0}\delta\phi \end{pmatrix}. \tag{35}$$

It also transforms as

$$\delta\mathcal{M} \to \Lambda \delta\mathcal{M} \Lambda^T. \tag{36}$$

The transverse scalar fields, the graviton in the Einstein frame, and the R-R four-form are invariant under the S-duality.

Using the above transformations, one should be able to construct a set of S-matrix elements in terms of the product of the above $SL(2,R)$ tensors such that they make an invariant under the $SL(2,R)$ transformations. For example, $(*\mathcal{F})^T \delta\mathcal{M}\mathcal{B}$ is invariant under the linear $SL(2,R)$ transformations. It has the following six elements [13]:

$$(*\mathcal{F})^T \delta\mathcal{M}\mathcal{B} = e^{-\phi_0}\delta\phi f B + \delta\phi (*f) C^{(2)} + C_0 \delta\phi (*f) B + \delta C (*f) B - e^{\phi_0} C_0 \delta C f B - e^{\phi_0}\delta C f C^{(2)}. \tag{37}$$

As a result, the S-matrix of the above six terms should have the same structure. For flat spacetime with no R-R background field, the above S-dual multiplet simplifies to

$$(*\mathcal{F})^T \delta\mathcal{M}\mathcal{B} = e^{-\phi_0}\delta\phi f B + \delta\phi (*f) C^{(2)} + \delta C (*f) B - e^{\phi_0}\delta C f C^{(2)}. \tag{38}$$

By explicit calculation, it has been shown in [13] that the S-matrix elements of the first three terms in the above multiplet have identical structure. We shall show that the S-matrix element of the last term also has the same structure. On the other hand, if an S-matrix element could not be combined with some other S-matrix element to be written in terms of an $SL(2,R)$ invariant, that S-matrix element should be zero. In the following subsections, we fix $p = 3$ and examine the S-duality Ward identity of the amplitude (13) for various R-R fields.

A. $C^{(0)} C^{(2)} f$

When $n = 1$ and $m = 3$, the trace parts of a_1, a_2, a_4, a_6, a_7, a_8, a_{12}, and a_{14} are zero. The nonzero terms yield

$$\begin{aligned} a_3 &= 128 C^{(0)} p_1.V.F^{(3)}_{ba} f^{ab} \\ a_5 &= -128 C^{(0)} p_1.V.F^{(3)}_{ba} f^{ab} \\ a_9 &= -64 C^{(0)} p_1.V.F^{(3)}_{ba} f^{ab} + 32 C^{(0)} p_1.N.F^{(3)}_{ba} f^{ab} \\ a_{10} &= 32 C^{(0)} p_1.V.F^{(3)}_{ba} f^{ab} \\ a_{11} &= 64 C^{(0)} p_1.V.F^{(3)}_{ba} f^{ab} - 32 C^{(0)} p_1.N.F^{(3)}_{ba} f^{ab} \\ a_{13} &= 32 C^{(0)} p_1.V.F^{(3)}_{ba} f^{ab}, \end{aligned} \tag{39}$$

where $f^{ab} = i(k_3^a \zeta_3^b - k_3^b \zeta_3^a)$ is the gauge boson field strength in momentum space, $F^{(3)}_{\mu\nu\alpha}$ is the R-R two-form field strength, and $C^{(0}$ is the polarization of the R-R scalar. The matrix V is the world-volume metric and N is the transverse space metric, i.e., $\eta_{\mu\nu} = V_{\mu\nu} + N_{\mu\nu}, D_{\mu\nu} = V_{\mu\nu} - N_{\mu\nu}$. Replacing them in (13), one finds

$$\mathcal{A}_{C^{(0)}C^{(2)}F} \sim \int d^2 z_1 d^2 z_2 dz_3 e^{-3\phi_0/2} C^{(0)} K f^{ab} \left[p_1.N.F^{(3)}_{ba} \frac{z_{\bar{1}1}}{z_{12} z_{13} z_{2\bar{1}} z_{3\bar{1}} z_{1\bar{2}} z_{\bar{1}\bar{2}}} + p_1.V.F^{(3)}_{ba} \times \frac{(-2 z_3 z_{\bar{1}} + z_2(z_{3\bar{2}} + z_{\bar{1}\bar{2}}) + (z_3 + z_{\bar{1}}) z_{\bar{2}})(z_2(z_3 - 2z_{\bar{2}}) + z_3 z_{\bar{2}} + z_1(z_{23} + z_{\bar{2}3}))}{z_{12} z_{13} z_{23} z_{2\bar{1}} z_{3\bar{1}} z_{1\bar{2}} z_{2\bar{2}} z_{3\bar{2}} z_{\bar{1}\bar{2}}} \right], \tag{40}$$

where we have also transformed the amplitude to the Einstein frame. This amplitude should have the same structure as the amplitude of the first three terms in (38).

The amplitude of one dilaton, one B-field, and one gauge boson in the Einstein frame has been calculated in [13] to be

$$A_{\phi BF} \sim \int d^2 z_1 d^2 z_2 dz_3 \phi_1 e^{-3\phi_0/2} f^{ab} \left(I_{11} \left[\frac{p_1.D.p_1}{p_1.p_2} p_1.V.H_{ba} + \frac{p_1.k_3 p_1.D.p_1}{(p_1.p_2)^2} p_1.H_{ba} \right] - I_2 \left[4 p_1.V.H_{ba} + \frac{p_2.D.p_2}{p_1.p_2}(2 p_1.H_{ba} - p_1.N.H_{ba}) + \frac{p_1.k_3 p_2.D.p_2}{(p_1.p_2)^2} p_1.H_{ba} \right] \right), \tag{41}$$

where ϕ_1 is the polarization of the dilaton, H is the B-field strength, and

$$I_2 = \frac{K}{z_{31}z_{2\bar{1}}z_{3\bar{1}}z_{1\bar{2}}z_{2\bar{2}}} \qquad I_{11} = \frac{K}{z_{32}z_{1\bar{1}}z_{2\bar{1}}z_{1\bar{2}}z_{3\bar{2}}}. \tag{42}$$

These integrals satisfy the following two relations:

$$\begin{aligned} 4I_2 k_3.p_1 + (2I_2 + I_3)p_1.p_2 + (I_2 - I_{11})p_1.D.p_1 = 0, \\ 2I_3 k_3.p_1 + (-2I_1 + I_3)p_1.p_2 - (I_2 + I_{11})p_1.D.p_1 = 0, \end{aligned} \tag{43}$$

where

$$I_1 = \frac{K}{z_{12}z_{31}z_{3\bar{1}}z_{2\bar{2}}z_{\bar{1}\bar{2}}} \qquad I_3 = \frac{K}{z_{12}z_{32}z_{1\bar{1}}z_{3\bar{1}}z_{1\bar{2}}} - \frac{2K}{z_{12}z_{31}z_{2\bar{1}}z_{3\bar{1}}z_{2\bar{2}}} + \frac{K}{z_{12}z_{31}z_{2\bar{1}}z_{3\bar{1}}z_{3\bar{2}}} + \frac{K}{z_{31}z_{32}z_{1\bar{1}}z_{2\bar{1}}z_{\bar{1}\bar{2}}}. \tag{44}$$

To compare the amplitude (41) with (40), we solve the above two relations to find

$$\frac{p_1.D.p_1}{p_1.p_2} = \frac{(2I_2 + I_3) \times 2I_3 - (-2I_1 + I_3) \times 4I_2}{-(I_2 + I_{11}) \times 4I_2 - (I_2 - I_{11}) \times 2I_3} \qquad \frac{p_1.k_3}{p_1.p_2} = \frac{(-2I_1 + I_3)(I_2 - I_{11}) + (2I_2 + I_3)(I_2 + I_{11})}{-(I_2 + I_{11}) \times 4I_2 - (I_2 - I_{11}) \times 2I_3}. \tag{45}$$

Now using the on-shell relation

$$p_2.D.p_2 = p_1.D.p_1 + 4p_1.k_3 \tag{46}$$

to write $p_2.D.p_2$ in (41) in terms of $p_1.D.p_1$ and $p_1.k_3$, and then using the relations in (45), one can simplify the amplitude (41) as

$$A_{\phi BF} \sim \int d^2z_1 d^2z_2 dz_3 \phi_1 e^{-3\phi_0/2} f^{ab}[p_1.V.H_{ba}(I_1 - I_2 + I_3) + p_1.N.H_{ba}(I_1 + I_2)].$$

Using (42) and (44), one finds exactly the structure in (40). The extra factor of $e^{2\phi_0}$ in the last term in (38) compared to the first term is that in the study of the S-duality the R-R fields should be rescaled as $C \to e^{\phi}C$.

B. $C^{(0)}C^{(0)}f$

Since the R-R scalar transforms as (36) and the gauge boson field strength transforms as a doublet, it is impossible to construct an $SL(2,R)$ invariant combination from two $\delta\mathcal{M}$'s and one gauge boson. As a result, the amplitude of two R-R scalars and one gauge boson must be zero. Using the identity (25), one finds that the contribution of A_1, A_2, A_6, A_7, A_{12}, and A_{14} to (10) is zero. The trace d_1^{μ} is zero, so b_1, $b_3^{b\chi}$, $b_5^{b\chi}$ are zero. So the contributions of A_1, A_3, A_4, A_6, A_8, and A_{14} are zero. Moreover, using the identity (25), one finds that the contributions of A_2, A_7, A_{12} to (10) are also zero. The traces in the remaining four terms produce $p_1^a p_2^b f_{ab}$, which is zero using momentum conservation $p_2^a = -(p_1^a + k^a)$ and the on-shell relation $k_3^a f_{ab} = 0$.

C. $C^{(2)}C^{(2)}f$

One cannot construct an $SL(2,R)$ invariant from three doublets, so the amplitude of two R-R two-forms and one gauge boson must be zero. Using the identity (25), one finds zero contribution from A_1, A_2, A_6, A_7, A_{12}, and A_{14}. The nonzero terms are

$$\begin{aligned} A_3 &= 128 f^{ab} F_1^{(3)}{}_{acd} F_2^{(3)}{}_{b}{}^{cd} q_3 \\ A_4 &= -128 f^{ab} F_1^{(3)}{}_{acd} F_2^{(3)}{}_{b}{}^{cd} q_4 \\ A_5 &= 128 f^{ab} F_1^{(3)}{}_{acd} F_2^{(3)}{}_{b}{}^{cd} q_5 \\ A_8 &= -128 f^{ab} F_1^{(3)}{}_{acd} F_2^{(3)}{}_{b}{}^{cd} q_8 \\ A_9 &= 64 f^{ab} F_1^{(3)}{}_{ac\mu} F_2^{(3)}{}_{b}{}^{c\mu} q_9 \\ A_{10} &= -64 f^{ab} F_1^{(3)}{}_{ac\mu} F_2^{(3)}{}_{b}{}^{c\mu} q_{10} \\ A_{11} &= 64 f^{ab} F_1^{(3)}{}_{ac\mu} F_2^{(3)}{}_{b}{}^{c\mu} q_{11} \\ A_{13} &= -64 f^{ab} F_1^{(3)}{}_{ac\mu} F_2^{(3)}{}_{b}{}^{c\mu} q_{13}. \end{aligned} \tag{47}$$

The amplitude (13) simplifies to

$$\mathcal{A} \sim 128 f^{ab} F_1^{(3)}{}_{acd} F_2^{(3)}{}_{b}{}^{cd} M_3 + 64 f^{ab} F_1^{(3)}{}_{ac\mu} F_2^{(3)}{}_{b}{}^{c\mu} M_4, \tag{48}$$

where

$$M_3 \equiv q_3 - q_4 + q_5 - q_8 = \int \frac{K[z_{32}z_{3\bar{1}}z_{3\bar{2}} - z_{31}(z_{32}(z_{3\bar{1}} - z_{3\bar{2}}) + z_{3\bar{1}}z_{3\bar{2}})]}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{31}z_{32}z_{3\bar{1}}z_{3\bar{2}}z_{\bar{1}\bar{2}}} \times d^2z_1 d^2z_2 dx_3$$

$$M_4 \equiv q_9 - q_{10} + q_{11} - q_{13} = \int \frac{K[z_{32}z_{3\bar{1}}z_{3\bar{2}} - z_{31}(z_{32}(z_{3\bar{1}} - z_{3\bar{2}}) + z_{3\bar{1}}z_{3\bar{2}})]}{z_{12}z_{1\bar{2}}z_{31}z_{32}z_{3\bar{1}}z_{3\bar{2}}z_{\bar{1}2}z_{\bar{1}\bar{2}}} \times d^2z_1 d^2z_2 dx_3. \quad (49)$$

Using the same step that we have done for M_1 and M_2 integrals, one finds

$$M_3 = \int_0^1 rdr \int_0^{2\pi} d\theta \frac{2\sin(\theta)(r^2)^{p_1\cdot p_2}(1-r^2)^{p_2\cdot D\cdot p_2 - 1}}{r(r^2 - 2r\cos(\theta) + 1)^{2p_1\cdot k_3 + 1}}$$

$$M_4 = \int_0^1 rdr \int_0^{2\pi} d\theta \frac{2\sin(\theta)(r^2)^{p_1\cdot p_2}(1-r^2)^{p_2\cdot D\cdot p_2}}{r(r^2 - 2r\cos(\theta) + 1)^{2p_1\cdot k_3 + 1}}. \quad (50)$$

The θ integration again gives zero result. Therefore, as the gauge symmetry Ward identity predicts the constraint (25) between the q_i's, the above S-duality Ward identity produces the following constraints:

$$q_3 - q_4 + q_5 - q_8 = 0$$
$$q_9 - q_{10} + q_{11} - q_{13} = 0. \quad (51)$$

One may use the above constraints and the constraints in (25) to simplify the amplitude (13).

D. $C^{(4)}C^{(4)}f$

The R-R four-form is invariant under the S-duality and f transforms as a doublet, so the amplitude of two R-R four-forms and one gauge field cannot be combined with any S-matrix element to be invariant under the $SL(2,R)$ transformation. As a result, this amplitude must be zero for a D_3-brane. The explicit calculations produce the following nonzero terms for A_i's:

$$A_3 = \frac{128}{3} f^{ab} F_1^{(5)}{}_{acde\mu} F_2^{(5)}{}_b{}^{cde\mu} q_3 \qquad A_4 = -\frac{128}{3} f^{ab} F_1^{(5)}{}_{acde\mu} F_2^{(5)}{}_b{}^{cde\mu} q_4 \qquad A_5 = \frac{128}{3} f^{ab} F_1^{(5)}{}_{acde\mu} F_2^{(5)}{}_b{}^{cde\mu} q_5$$

$$A_8 = -\frac{128}{3} f^{ab} F_1^{(5)}{}_{acde\mu} F_2^{(5)}{}_b{}^{cde\mu} q_8 \qquad A_9 = -\frac{16}{6} f^{ab}[F_1^{(5)}{}_{a\mu\nu\rho\sigma} F_2^{(5)}{}_b{}^{\mu\nu\rho\sigma} + 2D^{\mu\nu} F_1^{(5)}{}_{a\mu\rho\sigma\delta} F_2^{(5)}{}_{b\nu}{}^{\rho\sigma\delta}] q_9$$

$$A_{10} = \frac{16}{6} f^{ab}[F_1^{(5)}{}_{a\mu\nu\rho\sigma} F_2^{(5)}{}_b{}^{\mu\nu\rho\sigma} + 2D^{\mu\nu} F_1^{(5)}{}_{a\mu\rho\sigma\delta} F_2^{(5)}{}_{b\nu}{}^{\rho\sigma\delta}] q_{10}$$

$$A_{11} = -\frac{16}{6} f^{ab}[F_1^{(5)}{}_{a\mu\nu\rho\sigma} F_2^{(5)}{}_b{}^{\mu\nu\rho\sigma} + 2D^{\mu\nu} F_1^{(5)}{}_{a\mu\rho\sigma\delta} F_2^{(5)}{}_{b\nu}{}^{\rho\sigma\delta}] q_{11}$$

$$A_{13} = \frac{16}{6} f^{ab}[F_1^{(5)}{}_{a\mu\nu\rho\sigma} F_2^{(5)}{}_b{}^{\mu\nu\rho\sigma} + 2D^{\mu\nu} F_1^{(5)}{}_{a\mu\rho\sigma\delta} F_2^{(5)}{}_{b\nu}{}^{\rho\sigma\delta}] q_{13}. \quad (52)$$

Replacing them in (13), one finds

$$\mathcal{A} \sim \frac{128}{3} f^{ab} F_1^{(5)}{}_{acde\mu} F_2^{(5)}{}_b{}^{cde\mu} M_3 + \frac{16}{6} f^{ab}[F_1^{(5)}{}_{a\mu\nu\rho\sigma} F_2^{(5)}{}_b{}^{\mu\nu\rho\sigma} + 2D^{\mu\nu} F_1^{(5)}{}_{a\mu\rho\sigma\delta} F_2^{(5)}{}_{b\nu}{}^{\rho\sigma\delta}] M_4,$$

where M_3, M_4 are given in (49). Since they are zero, i.e., (51), the above amplitude is zero, as expected from the S-duality Ward identity.

E. $C^{(0)}C^{(2)}\Phi$

The amplitude of one R-R scalar, one R-R two-form, and one transverse scalar must be zero because $C^{(0)}$ transforms as a modular, $C^{(2)}$ transforms as a doublet, and Φ transforms as a scalar under S-duality transformations. The only nonzero A_i's are the following:

$$A_9 = -128iF^{(1)\mu} F^{(3)}{}_{\mu ai} k^a \zeta^i q_9$$
$$A_{10} = 64iF^{(1)\mu} F^{(3)}{}_{\mu ai} k^a \zeta^i q_{10}$$
$$A_{11} = -128iF^{(1)\mu} F^{(3)}{}_{\mu ai} k^a \zeta^i q_{11}$$
$$A_{13} = 64iF^{(1)\mu} F^{(3)}{}_{\mu ai} k^a \zeta^i q_{13}. \quad (53)$$

The amplitude (13) then becomes

$$\mathcal{A} \sim 64iF^{(1)\mu} F^{(3)}{}_{\mu ai} k^a \zeta^i M_5, \quad (54)$$

where

$$M_5 \equiv -2q_9 + q_{10} - 2q_{11} + q_{13} = \int \frac{K[z_{31}z_{32}z_{3\bar{1}} - 2z_{31}z_{32}z_{3\bar{2}} + z_{31}z_{3\bar{1}}z_{3\bar{2}} - 2z_{32}z_{3\bar{1}}z_{3\bar{2}}]}{z_{12}z_{1\bar{2}}z_{31}z_{32}z_{3\bar{1}}z_{3\bar{2}}z_{\bar{1}2}z_{\bar{1}\bar{2}}} \times d^2z_1 d^2z_2 dx_3. \quad (55)$$

Transforming it to the unit disk and fixing the position of one closed string state and the open string states, one finds

$$M_5 = \int_0^1 rdr \int_0^{2\pi} d\theta \frac{4\sin(\theta)(r^2)^{p_1\cdot p_2}(1-r^2)^{p_2.D.p_2}}{r(r^2-2r\cos(\theta)+1)^{2p_1.k_3+1}}. \tag{56}$$

Again the θ integration gives zero result. So the above S-duality Ward identity produces the following constraint,

$$-2q_9 + q_{10} - 2q_{11} + q_{13} = 0, \tag{57}$$

which is verified explicitly.

F. $C^{(0)}C^{(4)}\Phi$

Since $C^{(0)}$ transforms as a modular and $C^{(4)}$ and Φ transform as scalars under the S-duality transformation, the S-duality Ward identity predicts that the amplitude of one R-R scalar, one R-R four-form, and one transverse scalar must be zero. In this case, all trace parts of the amplitude are in fact zero.

G. $C^{(2)}C^{(4)}\Phi$

The S-duality Ward identity also predicts that the amplitude of one R-R scalar, one R-R two-form, and one transverse scalar is zero. $C^{(4)}$ and Φ transform as a scalar but $C^{(2)}$ transform as a doublet under S-duality transformation. The nonzero A_i's are the following:

$$\begin{aligned}
A_3 &= -\frac{128i}{3}F^{(3)bcd}F^{(5)}_{abcdi}k^a\zeta^i q_3\\
A_4 &= -\frac{128i}{3}F^{(3)bcd}F^{(5)}_{abcdi}k^a\zeta^i q_4\\
A_5 &= -\frac{128i}{3}F^{(3)bcd}F^{(5)}_{abcdi}k^a\zeta^i q_5\\
A_8 &= -\frac{128i}{3}F^{(3)bcd}F^{(5)}_{abcdi}k^a\zeta^i q_8\\
A_9 &= ik^a\zeta^i[32F^{(3)\mu\nu\rho}F^{(5)}_{\mu\nu\rho ai} + 96F^{(3)b\mu j}F^{(5)}_{b\mu jai}]q_9\\
A_{10} &= -\frac{2i}{3}k^a\zeta^i[32F^{(3)\mu\nu\rho}F^{(5)}_{\mu\nu\rho ai} + 96F^{(3)b\mu j}F^{(5)}_{b\mu jai}]q_{10}\\
A_{11} &= ik^a\zeta^i[32F^{(3)\mu\nu\rho}F^{(5)}_{\mu\nu\rho ai} + 96F^{(3)b\mu j}F^{(5)}_{b\mu jai}]q_{11}\\
A_{13} &= -\frac{2i}{3}k^a\zeta^i[32F^{(3)\mu\nu\rho}F^{(5)}_{\mu\nu\rho ai} + 96F^{(3)b\mu j}F^{(5)}_{b\mu jai}]q_{13}.
\end{aligned} \tag{58}$$

Using the constraints (51) and (57), one finds that the amplitude simplifies to

$$\mathcal{A} \sim \frac{128}{3} iF^{(3)bcd}F^{(5)}_{abcdi}k^a\zeta^i M_6, \tag{59}$$

where

$$\begin{aligned}
M_9 &\equiv q_3 + q_4 + q_5 + q_8\\
&= \int -\frac{K[z_{31}(z_{32}(z_{3\bar{1}} + z_{3\bar{2}}) + z_{3\bar{1}}z_{3\bar{2}}) + z_{32}z_{3\bar{1}}z_{3\bar{2}}]}{z_{12}z_{1\bar{1}}z_{2\bar{2}}z_{31}z_{32}z_{3\bar{1}}z_{3\bar{2}}z_{\bar{1}\bar{2}}}\\
&\quad \times d^2z_1 d^2z_2 dx_3\\
&\to \int_0^1 rdr \int_0^{2\pi} d\theta \frac{4\sin(\theta)(r^2)^{p_1\cdot p_2}(1-r^2)^{p_2.D.p_2-1}}{r(r^2-2r\cos(\theta)+1)^{2p_1.k_3+1}},
\end{aligned} \tag{60}$$

which is zero upon integrating over the θ variable. So the Ward identity predicts another constraint, i.e.,

$$q_3 + q_4 + q_5 + q_8 = 0, \tag{61}$$

which is verified by the explicit calculation.

We have seen that the amplitudes that are constrained by the S-duality Ward identity to be zero are in fact zero by explicit calculations. However, there are other amplitudes in the D_3-brane that the Ward identity does not predict to be zero. They are either invariant under the $SL(2,R)$ transformations or they are related to the other amplitudes that involve NS-NS closed string states. In the next section we are going to write the explicit form of these amplitudes and the amplitudes for $p \neq 3$.

IV. NONZERO AMPLITUDES

The Ward identities predict the six constraints (25), (51), (57), and (61) between the integrals that appear in (13). Examining the integrals, we have also found the following four relations between them:

$$\begin{aligned}
q_1 + q_5 - q_8 &= 0\\
q_4 - q_5 - q_{14} &= 0\\
q_6 + 2q_5 &= 0\\
q_{12} + 2q_{11} &= 0.
\end{aligned} \tag{62}$$

Using these ten constraints, one can express all integrals in (13) in terms of the following four integrals:

$$\begin{aligned}
Q_1 &\equiv q_6 \to \int_0^1 2rdr \int_0^{2\pi} d\theta \frac{(r^2)^{p_1\cdot p_2-1}(1-r^2)^{p_2.D.p_2-1}}{(r^2-2r\cos(\theta)+1)^{2p_1.k_3}}\\
Q_2 &\equiv q_{14} - q_1 \to \int_0^1 2rdr \int_0^{2\pi} d\theta \frac{(r^2)^{p_1\cdot p_2-1}(1-r^2)^{p_2.D.p_2}}{(r^2-2r\cos(\theta)+1)^{2p_1.k_3+1}}\\
Q_3 &\equiv q_{12} \to \int_0^1 2rdr \int_0^{2\pi} d\theta \frac{(r^2)^{p_1\cdot p_2-1}(1-r^2)^{p_2.D.p_2}}{(r^2-2r\cos(\theta)+1)^{2p_1.k_3}}\\
Q_4 &\equiv q_7 - q_2 \to \int_0^1 2rdr \int_0^{2\pi} d\theta \frac{(r^2)^{p_1\cdot p_2-1}(1-r^2)^{p_2.D.p_2+1}}{(r^2-2r\cos(\theta)+1)^{2p_1.k_3+1}}.
\end{aligned}$$

The θ-integral can be evaluated using the following formula [34]:

$$\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{(1+x^2-2x\cos(\theta))^b} = 2\pi x^n \frac{\Gamma(b+n)}{n!\Gamma(b)} {}_2F_1\left[\begin{matrix} b, & n+b \\ & n+1 \end{matrix}; x^2\right], \tag{63}$$

where $|x|<1$. The r-integral can be evaluated using the following formula [34]:

$$\int_0^1 dx x^a (1-x)^b {}_2F_1\left[\begin{matrix} a_1, & a_2 \\ & b_1 \end{matrix}; x\right] = {}_3F_2\left[\begin{matrix} 1+a, & a_1, & a_2 \\ 2+a+b, & & b_1 \end{matrix}; 1\right] B(1+a, 1+b).$$

Then the Q_i's become

$$\begin{aligned}
Q_1 &= 2\pi B(p_1.p_2, p_2.D.p_2) {}_3F_2\left[\begin{matrix} p_1.p_2, 2p_1.k_3, 2p_1.k_3 \\ p_1.p_2+p_2.D.p_2, 1 \end{matrix}; 1\right] \\
Q_2 &= 2\pi B(p_1.p_2, p_2.D.p_2+1) {}_3F_2\left[\begin{matrix} p_1.p_2, 2p_1.k_3+1, 2p_1.k_3+1 \\ p_1.p_2+p_2.D.p_2+1, 1 \end{matrix}; 1\right] \\
Q_3 &= 2\pi B(p_1.p_2, p_2.D.p_2+1) {}_3F_2\left[\begin{matrix} p_1.p_2, 2p_1.k_3, 2p_1.k_3 \\ p_1.p_2+p_2.D.p_2+1, 1 \end{matrix}; 1\right] \\
Q_4 &= 2\pi B(p_1.p_2, p_2.D.p_2+2) {}_3F_2\left[\begin{matrix} p_1.p_2, 2p_1.k_3+1, 2p_1.k_3+1 \\ p_1.p_2+p_2.D.p_2+2, 1 \end{matrix}; 1\right].
\end{aligned} \tag{64}$$

Using the package [35], one may expand the integrals at low energy, i.e.,

$$\begin{aligned}
Q_1 &= \frac{1}{p_1.p_2} + \frac{1}{p_2.D.p_2} + \frac{\pi^2}{6}\left(-p_1.p_2 + \frac{4(p_2.k_3)^2}{p_2.D.p_2} - p_2.D.p_2\right) + \cdots \\
Q_2 &= \frac{1}{p_1.p_2} + \frac{1}{p_1.D.p_1} + \frac{\pi^2}{6}\left(-p_1.p_2 + \frac{4(p_1.k_3)^2}{p_1.D.p_1} - p_1.D.p_1\right) + \cdots \\
Q_3 &= \frac{1}{p_1.p_2} + \frac{\pi^2}{6} p_2.D.p_2 + \cdots \qquad Q_4 = \frac{1}{p_1.p_2} + \frac{\pi^2}{6} p_1.D.p_1 + \cdots,
\end{aligned}$$

where we have also used the on-shell relation (46). One may use the above expansions to find four-derivative couplings of two R-R and one NS states, which we are not interested in for this paper. By studying the above low energy expansions, we have observed that the integrals Q_1, Q_2 interchange under the changing of the R-R labels, i.e., $Q_1(p_1,p_2) \leftrightarrow Q_2(p_2,p_1)$. Similarly, $Q_3(p_1,p_2) \leftrightarrow Q_4(p_2,p_1)$.

Since the results of the traces in (30) depend on p, it is convenient to write the nonzero amplitudes for explicit p. In the next subsections, we write the amplitudes for $p=0$, 1, 2, 3.

A. $p=0$

When $p=0$, the nonzero amplitudes are

$$\begin{aligned}
\mathcal{A}_{C^{(1)}C^{(1)}\Phi} \sim{}& \frac{256}{1!}[p_1.N.\zeta(Q_1-Q_3) + p_2.N.\zeta(Q_2-Q_4)]F_1^{(2)\,ia}F_{2\,ia}^{(2)} + \frac{16}{2!}[p_1.N.\zeta Q_3 + p_2.N.\zeta Q_4](-8F_1^{(2)\,\mu\nu}F_{2\,\mu\nu}^{(2)} \\
&- 4D^\lambda{}_\rho F_1^{(2)\,\rho\nu}F_{2\,\lambda\nu}^{(2)}) + [64(4Q_1-2Q_3-Q_4)F_1^{(2)\,ij}F_{2\,aj}^{(2)} + 64(4Q_2-Q_3-2Q_4)F_{1\,aj}^{(2)}F_2^{(2)\,ij}]k^a\zeta_i \\
\mathcal{A}_{C^{(3)}C^{(3)}\Phi} \sim{}& \frac{16}{4!}[p_1.N.\zeta Q_3 + p_2.N.\zeta Q_4](-8F_1^{(4)\,\mu\nu\sigma\delta}F_{2\,\mu\nu\sigma\delta}^{(4)} - 8D^\lambda{}_\rho F_1^{(4)\,\rho\nu\sigma\delta}F_{2\,\lambda\nu\sigma\delta}^{(4)}) \\
&- \left[\frac{32}{3}Q_3 F_1^{(4)}{}_i{}^{jkl}F_{2\,ajkl}^{(4)} + \frac{32}{3}Q_4 F_{1\,ajkl}^{(4)}F_2^{(4)}{}_i{}^{jkl}\right]k^a\zeta^i.
\end{aligned} \tag{65}$$

The amplitudes are symmetric under interchanging the particle labels 1,2, as expected.

There is no nonzero amplitude when the open string is the gauge field, which is consistent with the gauge transformation Ward identity because the world-volume indices can take only one value. As a result, f^{ab} is zero.

B. $p=1$

When $p=1$, the nonzero amplitudes are

$$\begin{aligned}
\mathcal{A}_{C^{(0)}C^{(0)}\Phi} \sim & \frac{256}{1!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)]F_1^{(1)\,a}F_2^{(1)}{}_a \\
& +\frac{16}{1!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](-6F_1^{(1)\,\mu}F_2^{(1)}{}_\mu-2D^\lambda{}_\rho F_1^{(1)\,\rho}F_2^{(1)}{}_\lambda)+[64(4Q_1-2Q_3-Q_4)F_1^{(1)\,i}F_2^{(1)\,a} \\
& +64(4Q_2-Q_3-2Q_4)F_1^{(1)\,a}F_2^{(1)\,i}]k_a\zeta_i \\
\mathcal{A}_{C^{(2)}C^{(2)}\Phi} \sim & \frac{256}{2!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)]F_1^{(3)\,iab}F_2^{(3)}{}_{iab} \\
& +\frac{16}{3!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](-6F_1^{(3)\,\mu\nu\rho}F_2^{(3)}{}_{\mu\nu\rho}-6D^\lambda{}_\rho F_1^{(3)\,\rho\mu\nu}F_2^{(3)}{}_{\lambda\mu\nu}) \\
& +[64(4Q_1-2Q_3-Q_4)F_1^{(3)\,bij}F_2^{(3)}{}_{abj}+64(4Q_2-Q_3-2Q_4)F_1^{(3)}{}_{abj}F_2^{(3)\,bij}]k^a\zeta_i \\
& -[32Q_3F_1^{(3)\,ijk}F_2^{(3)}{}_{ajk}+32Q_4F_1^{(3)}{}_{ajk}F_2^{(3)\,ijk}]k^a\zeta_i \\
\mathcal{A}_{C^{(4)}C^{(4)}\Phi} \sim & -\frac{16}{5!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](6F_1^{(5)\,\mu\nu\rho\sigma\delta}F_2^{(5)}{}_{\mu\nu\rho\sigma\delta}+10D^\lambda{}_\rho F_1^{(5)\,\rho\mu\nu\sigma\delta}F_2^{(5)}{}_{\lambda\mu\nu\sigma\delta}) \\
& -\left[\frac{32}{3}Q_3F_1^{(5)\,b}{}_i{}^{jkl}F_2^{(5)}{}_{abjkl}+\frac{32}{3}Q_4F_1^{(5)}{}_a{}^{bjkl}F_2^{(5)}{}_{bijkl}\right]k^a\zeta^i-\left[\frac{8}{3}Q_4F_1^{(5)}{}_i{}^{jklm}F_2^{(5)}{}_{ajklm}+\frac{8}{3}Q_3F_1^{(5)}{}_a{}^{jklm}F_2^{(5)}{}_{ijklm}\right]k^a\zeta^i \\
\mathcal{A}_{C^{(0)}C^{(2)}f} \sim & -32i(4Q_1-2Q_3-Q_4)F_1^{(1)\,i}F_2^{(3)}{}_{iab}f^{ab} \\
\mathcal{A}_{C^{(2)}C^{(4)}f} \sim & \frac{16i}{3}Q_3F_1^{(3)}{}_{ijk}F_2^{(5)}{}_{ab}{}^{ijk}f^{ab}.
\end{aligned} \tag{66}$$

The first three amplitudes in which the degrees of the R-R fields are identical are symmetric under the interchange of the particle labels 1,2. The last two amplitudes, in which the degrees of the R-R fields are not identical, are not symmetric. When the particle labels are changed, the amplitudes become

$$\mathcal{A}_{C^{(2)}C^{(0)}f} \sim -32i(4Q_2-2Q_4-Q_3)F_2^{(1)\,i}F_1^{(3)}{}_{iab}f^{ab} \qquad \mathcal{A}_{C^{(4)}C^{(2)}f} \sim \frac{16i}{3}Q_4F_2^{(3)}{}_{ijk}F_1^{(5)}{}_{ab}{}^{ijk}f^{ab}. \tag{67}$$

C. $p=2$

For the case that $p=2$, the nonzero amplitudes are

$$\begin{aligned}
\mathcal{A}_{C^{(1)}C^{(1)}\Phi} \sim & \frac{256}{2!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)]F_1^{(2)\,ab}F_2^{(2)}{}_{ab} \\
& +\frac{16}{2!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](-4F_1^{(2)\,\mu\nu}F_2^{(2)}{}_{\mu\nu}-4D^\lambda{}_\rho F_1^{(2)\,\rho\nu}F_2^{(2)}{}_{\lambda\nu}) \\
& +[64(4Q_1-2Q_3-Q_4)F_1^{(2)\,ib}F_2^{(2)}{}_{ab}+1\leftrightarrow 2]k^a\zeta_i-[32Q_3F_1^{(2)\,ij}F_2^{(2)}{}_{aj}+32Q_4F_1^{(2)}{}_{aj}F_2^{(2)\,ij}]k^a\zeta_i \\
\mathcal{A}_{C^{(3)}C^{(3)}\Phi} \sim & \frac{256}{3!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)]F_1^{(4)\,iabc}F_2^{(4)}{}_{iabc} \\
& -\frac{64}{4!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](F_1^{(4)\,\mu\nu\sigma\delta}F_2^{(4)}{}_{\mu\nu\sigma\delta}+2D^\lambda{}_\rho F_1^{(4)\,\rho\nu\sigma\delta}F_2^{(4)}{}_{\lambda\nu\sigma\delta}) \\
& +[32(4Q_1-2Q_3-Q_4)F_1^{(4)\,bc}{}_i{}^jF_2^{(4)}{}_{abcj}+1\leftrightarrow 2]k^a\zeta^i-[32Q_3F_1^{(4)\,b}{}_i{}^{jk}F_2^{(4)}{}_{abjk}+32Q_4F_1^{(4)}{}_a{}^{bjk}F_2^{(4)}{}_{bijk}]k^a\zeta^i \\
& -[\frac{32}{3}Q_3F_1^{(4)}{}_a{}^{jkl}F_2^{(4)}{}_{ijkl}+\frac{32}{3}Q_4F_1^{(4)}{}_i{}^{jkl}F_2^{(4)}{}_{ajkl}]k^a\zeta^i \\
\mathcal{A}_{C^{(1)}C^{(3)}f} \sim & -32i(4Q_1-2Q_3-Q_4)F^{(2)\,ci}F^{(4)}{}_{abci}f^{ab}+16iQ_3F^{(2)\,ij}F^{(4)}{}_{abij}f^{ab}.
\end{aligned} \tag{68}$$

Here also the first two amplitudes are symmetric under an interchange of the particle labels 1,2, as expected.

D. $p = 3$

Finally, for $p = 3$ the amplitudes are

$$\begin{aligned}
\mathcal{A}_{C^{(0)}C^{(0)}\Phi} \sim & \frac{16}{1!}[p_1.N.\zeta Q_3 + p_2.N.\zeta Q_4](-2F_1^{(1)\,\mu}F_{2\,\mu}^{(1)} - 2D^{\lambda}{}_{\rho}F_1^{(1)\,\rho}F_{2\,\lambda}^{(1)}) \\
& - [64Q_3F_1^{(1)}{}_iF_2^{(1)}{}_a + 64Q_4F_1^{(1)}{}_aF_2^{(1)}{}_i]k^a\zeta^i \\
\mathcal{A}_{C^{(2)}C^{(2)}\Phi} \sim & \frac{256}{3!}[p_1.N.\zeta(Q_1 - Q_3) + p_2.N.\zeta(Q_2 - Q_4)]F_1^{(3)\,abc}F_{2\,abc}^{(3)} \\
& + \frac{16}{3!}[p_1.N.\zeta Q_3 + p_2.N.\zeta Q_4](-2F_1^{(3)\,\mu\nu\rho}F_{2\,\mu\nu\rho}^{(3)} - 6D^{\lambda}{}_{\rho}F_1^{(3)\,\rho\mu\nu}F_{2\,\lambda\mu\nu}^{(3)}) \\
& + [32(4Q_1 - 2Q_3 - Q_4)F_1^{(3)\,ibc}F_{2\,abc}^{(3)} + 1 \leftrightarrow 2]k^a\zeta_i \\
\mathcal{A}_{C^{(4)}C^{(4)}\Phi} \sim & \frac{256}{4!}[p_1.N.\zeta(Q_1 - Q_3) + p_2.N.\zeta(Q_2 - Q_4)]F_1^{(5)\,iabcd}F_{2\,iabcd}^{(5)} \\
& - \frac{32}{5!}[p_1.N.\zeta Q_3 + p_2.N.\zeta Q_4](F_1^{(5)\,\mu\nu\rho\sigma\delta}F_{2\,\mu\nu\rho\sigma\delta}^{(5)} + 5D^{\lambda}{}_{\rho}F_1^{(5)\,\rho\mu\nu\sigma\delta}F_{2\,\lambda\mu\nu\sigma\delta}^{(5)}) \\
& + \left[\frac{32}{3}(4Q_1 - 2Q_3 - Q_4)F_1^{(5)\,bcd\,j}{}_iF_{2\,abcdj}^{(5)} + 1 \leftrightarrow 2\right]k^a\zeta^i \\
& - [16Q_3F_1^{(5)\,bc\,jk}{}_iF_{2\,abcjk}^{(5)} + 16Q_4F_1^{(5)}{}_a{}^{bcjk}F_{2\,bcijk}^{(5)}]k^a\zeta^i \\
& - \left[\frac{32}{3}Q_3F_1^{(5)}{}_a{}^{bjkl}F_{2\,bijkl}^{(5)} + \frac{32}{3}Q_4F_1^{(5)}{}_{bi}{}^{jkl}F_{2\,abjkl}^{(5)}\right]k^a\zeta^i \\
\mathcal{A}_{C^{(0)}C^{(2)}f} \sim & -32i(4Q_1 - 2Q_3 - Q_4)F_1^{(1)\,a}F_{2\,abc}^{(3)}f^{bc} + 32iQ_3F_1^{(1)\,i}F_{2\,ibc}^{(3)}f^{bc} \\
\mathcal{A}_{C^{(2)}C^{(4)}f} \sim & -16i(4Q_1 - 2Q_3 - Q_4)F_1^{(3)\,cdi}F_{2\,abcdi}^{(5)}f^{ab} + 16iQ_3F_1^{(3)\,cij}F_{2\,abcij}^{(5)}f^{ab} + \frac{16i}{3}Q_4F_1^{(3)\,ijk}F_{2\,abijk}^{(5)}f^{ab}. \qquad (69)
\end{aligned}$$

We have already shown in the previous section that the S-duality Ward identity transforms the amplitude $\mathcal{A}_{C^{(0)}C^{(2)}f}$ to three other amplitudes, e.g., the amplitude of one dilaton, one B-field and one gauge boson, which are consistent with explicit calculations. The amplitude $\mathcal{A}_{C^{(4)}C^{(4)}\Phi}$ is invariant under the S-duality Ward identity; the amplitude $\mathcal{A}_{C^{(0)}C^{(0)}\Phi}$ transforms to the amplitude of two dilatons and one transverse scalar; the amplitude $\mathcal{A}_{C^{(2)}C^{(2)}\Phi}$ transforms to the amplitude of two B-fields and one transverse scalar; and the amplitude $\mathcal{A}_{C^{(2)}C^{(4)}f}$ transforms to the amplitude of one B-field, one R-R four-form, and one gauge boson. It would be interesting to calculate these amplitudes explicitly and compare them with the S-duality Ward identity predictions.

V. T-DUALITY WARD IDENTITY

The S-matrix elements in string theory must satisfy the T-duality Ward identity. We now verify that the amplitudes that we have found in the previous subsections satisfy the T-duality Ward identity. If one reduces the theory on a circle with coordinate y and if the D_p-brane is alone on the circle, then after T-duality the brane transforms to the reduction of the D_{p-1}-brane on the dual circle. The D_{p-1}-brane is also orthogonal to the dual circle. The y index in the D_p-brane, which is a world-volume index, becomes a transverse index in the T-dual D_{p-1}-brane.

For example, consider the reduction of the amplitude $\mathcal{A}_{C^{(1)}C^{(1)}\Phi}$ in (65) when it is orthogonal to the circle; i.e., the y index is a transverse coordinate,

$$\begin{aligned}
\mathcal{A}_{C^{(1)}C^{(1)}\Phi} \sim & \frac{256}{1!}[p_1.N.\zeta(Q_1 - Q_3) + p_2.N.\zeta(Q_2 - Q_4)]F_1^{(2)\,\tilde{i}a}F_{2\,\tilde{i}a}^{(2)} \\
& + \frac{16}{2!}[p_1.N.\zeta Q_3 + p_2.N.\zeta Q_4](-16F_1^{(2)\,\tilde{i}a}F_{2\,\tilde{i}a}^{(2)} - 4F_1^{(2)\,\tilde{i}\tilde{j}}F_{2\,\tilde{i}\tilde{j}}^{(2)}) \\
& + [64(4Q_1 - 2Q_3 - Q_4)F_1^{(2)\,\tilde{i}\tilde{j}}F_{2\,a\tilde{j}}^{(2)} + 64(4Q_2 - Q_3 - 2Q_4)F_1^{(2)}{}_{a\tilde{j}}F_2^{(2)\,\tilde{i}\tilde{j}}]k^a\zeta_{\tilde{i}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{256}{1!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)](F_1^{(2)\,ya}F_2^{(2)}{}_{ya})\\
&+\frac{16}{2!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](-16F_1^{(2)\,ya}F_2^{(2)}{}_{ya}-8F_1^{(2)\,\tilde{i}y}F_2^{(2)}{}_{\tilde{i}y})\\
&+[64(4Q_1-2Q_3-Q_4)F_1^{(2)\,\tilde{i}y}F_2^{(2)}{}_{ay}+64(4Q_2-Q_3-2Q_4)F_1^{(2)}{}_{ay}F_2^{(2)\,\tilde{i}y}]k^a\zeta_{\tilde{i}}\\
&+[64(4Q_1-2Q_3-Q_4)F_1^{(2)\,y\tilde{j}}F_2^{(2)}{}_{a\tilde{j}}+64(4Q_2-Q_3-2Q_4)F_1^{(2)}{}_{a\tilde{j}}F_2^{(2)\,y\tilde{j}}]k^a\zeta_y,
\end{aligned}
\tag{70}
$$

where $\tilde{i}$, $\tilde{j}$ are the transverse indices which do not include the y-index and ζ_y is the polarization of the transverse scalar fields along the y-direction. In the above, we have used the implicit assumption in the reduction that fields do not depend on the y-direction. The above reduction of the D_0-brane amplitude should be reproduced by the linear T-duality of the reduction of the D_1-brane amplitude when it is along the circle. So consider the reduction of the D_1-brane amplitudes $\mathcal{A}_{C^{(0)}C^{(0)}\Phi}$, $\mathcal{A}_{C^{(2)}C^{(2)}\Phi}$, $\mathcal{A}_{C^{(0)}C^{(2)}f}$, and $\mathcal{A}_{C^{(2)}C^{(0)}f}$, i.e.,

$$
\begin{aligned}
\mathcal{A}_{C^{(0)}C^{(0)}\Phi}\sim&\frac{256}{1!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)]F_1^{(1)\,\tilde{a}}F_2^{(1)}{}_{\tilde{a}}\\
&+\frac{16}{1!}[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](-8F_1^{(1)\,\tilde{a}}F_2^{(1)}{}_{\tilde{a}}-4F_1^{(1)\,i}F_2^{(1)}{}_{i})\\
&+[64(4Q_1-2Q_3-Q_4)F_1^{(1)\,i}F_2^{(1)\,\tilde{a}}+64(4Q_2-Q_3-2Q_4)F_1^{(1)\,\tilde{a}}F_2^{(1)\,i}]k_{\tilde{a}}\zeta_i\\
\mathcal{A}_{C^{(2)}C^{(2)}\Phi}\sim&\frac{256}{2!}[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)](2F_1^{(3)\,iy\tilde{a}}F_2^{(3)}{}_{iy\tilde{a}})\\
&-32[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4](F_1^{(3)\,ijy}F_2^{(3)}{}_{ijy}+4F_1^{(3)\,iy\tilde{a}}F_2^{(3)}{}_{iy\tilde{a}}+3F_1^{(3)\,y\tilde{a}\tilde{b}}F_2^{(3)}{}_{y\tilde{a}\tilde{b}})\\
&+[64(4Q_1-2Q_3-Q_4)F_1^{(3)\,yij}F_2^{(3)}{}_{\tilde{a}yj}+64(4Q_2-Q_3-2Q_4)F_1^{(3)}{}_{\tilde{a}yj}F_2^{(3)\,yij}]k^{\tilde{a}}\zeta_i\\
\mathcal{A}_{C^{(0)}C^{(2)}f}\sim&\,64(4Q_1-2Q_3-Q_4)F_1^{(1)\,i}F_2^{(3)}{}_{i\tilde{a}y}k^{\tilde{a}}\zeta^y\\
\mathcal{A}_{C^{(2)}C^{(0)}f}\sim&\,64(4Q_2-2Q_4-Q_3)F_2^{(1)\,i}F_1^{(3)}{}_{i\tilde{a}y}k^{\tilde{a}}\zeta^y,
\end{aligned}
\tag{71}
$$

where $\tilde{a}$, $\tilde{b}$ are the world-volume indices that do not include the y-index and ζ^y is the polarization of the gauge field along the y-direction. In above reduction, we have discarded the $F^{(3)}$-terms that have no y-index because they are transformed under T-duality to $F^{(4)}$-terms that are not included in (70). Using the linear T-duality for the R-R fields, i.e.,

$$
C^{(n)}_{\mu\nu\ldots y}\to C^{(n-1)}_{\mu\nu\ldots};\qquad C^{(n-1)}_{\mu\nu\ldots}\to C^{(n)}_{\mu\nu\ldots y},
\tag{72}
$$

and the T-duality for the gauge field along the circle, i.e.,

$$
A^y\to\Phi_y,
\tag{73}
$$

one finds that the T-duality of the amplitudes in (71) are exactly that of the amplitude in (70). We have done similar calculations for all other amplitudes and found exact agreement with the T-duality Ward identity.

VI. SOFT SCALAR THEOREM

The S-matrix elements should satisfy the corresponding soft theorems as well. We have proposed in the Introduction a soft theorem for the scattering amplitude of n closed strings and one soft open string transverse scalar field, i.e., Eq. (2). We now check this theorem explicitly for the scattering amplitude of two closed string R-R states and one scalar field.

When the open string vertex operator is the transverse scalar, the amplitudes that we have found can be written as

$$
\mathcal{A}_{2+1}\sim[p_1.N.\zeta(Q_1-Q_3)+p_2.N.\zeta(Q_2-Q_4)]d_2+[p_1.N.\zeta Q_3+p_2.N.\zeta Q_4]d_1+\text{terms proportional to }k_3,
\tag{74}
$$

where d_1, d_2 are

$$d_1 = \frac{16}{n!}[\mathrm{Tr}(D)F_1^{(n)}\cdot F_2^{(n)} - 2nD^{\lambda}{}_{\rho}F_1^{(n)\,\rho\mu\ldots\nu}F_2^{(n)}{}_{\lambda\mu\ldots\nu}]$$
$$d_2 = \frac{256}{n!}[\delta_{n,p}F_1^{(n)}\cdot V\cdot F_2^{(n)} + \delta_{n,p+2}F_1^{(n)\,a_0a_1\ldots a_p i}F_2^{(n)}{}_{a_0a_1\ldots a_p i}]. \tag{75}$$

When the scalar field is soft, i.e., $k_3 \to 0$, at the leading order of k_3, the terms in the second line of (74) vanish and the integrals in the first line become

$$Q_1 - Q_3 = 2\pi B(1+p_1.p_2, p_2.D.p_2){}_3F_2\left[\begin{matrix}1+p_1.p_2, 0, 0\\ 1+p_1.p_2+p_2.D.p_2, 1\end{matrix};1\right]$$
$$Q_2 - Q_4 = 2\pi B(1+p_1.p_2, p_2.D.p_2+1){}_3F_2\left[\begin{matrix}1+p_1.p_2, 1, 1\\ 1+p_1.p_2+p_2.D.p_2+1, 1\end{matrix};1\right]$$
$$Q_3 = 2\pi B(p_1.p_2, p_2.D.p_2+1){}_3F_2\left[\begin{matrix}p_1.p_2, 0, 0\\ p_1.p_2+p_2.D.p_2+1, 1\end{matrix};1\right]$$
$$Q_4 = 2\pi B(p_1.p_2, p_2.D.p_2+2){}_3F_2\left[\begin{matrix}p_1.p_2, 1, 1\\ p_1.p_2+p_2.D.p_2+2, 1\end{matrix};1\right]. \tag{76}$$

Using the following identities,

$${}_3F_2\left[\begin{matrix}a, 0, 0\\ b, c\end{matrix};1\right] = 1$$
$${}_3F_2\left[\begin{matrix}a, 1, 1\\ b, c\end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{77}$$

one finds

$$Q_1 - Q_3 = Q_2 - Q_4 = 2\pi B(1+p_1.p_2, p_2.D.p_2)$$
$$Q_3 = Q_4 = 2\pi B(p_1.p_2, p_2.D.p_2+1). \tag{78}$$

Replacing them in (74), one finds that the amplitude at the leading order of k_3 becomes

$$\mathcal{A}_{2+1} = \zeta^i(p_1+p_2)_i\mathcal{A}_2, \tag{79}$$

where

$$\mathcal{A}_2 \sim K(1,2)\frac{\Gamma(p_1.p_2)\Gamma(p_2.D.p_2)}{\Gamma(1+p_1.p_2+p_2.D.p_2)} \tag{80}$$

and the kinematic factor $K(1,2)$ is

$$K(1,2) = (p_2.D.p_2 d_1 + p_1.p_2 d_2). \tag{81}$$

The amplitude (80) is exactly the disk-level scattering amplitude of two R-R vertex operators [36]. So the amplitude that we have found satisfies the soft theorem at the leading order, when the transverse scalar field is soft. It would be interesting to find the subleading terms in the soft theorem (2) and check them by explicit calculations.

VII. SOFT PHOTON THEOREM

We have proposed in the Introduction a soft theorem for the scattering amplitude of n closed strings and one soft open string gauge field, i.e., Eq. (1). This relation can be used to find for the antisymmetric matrix $[\mathcal{A}_n]_{ab}$ that its trace is the scattering amplitude of n closed string vertex operators, which is zero. Using this theorem, one can find $[\mathcal{A}_n]_{ab}$ explicitly.

Using the soft limit of the integrals (78), one finds that the matrix $[\mathcal{A}_2]_{ab}$ has the same structure as the scattering amplitude of the two closed strings from the D-brane, i.e., (80), with the following kinematic factors:

$$[K^{p=1}(C^{(0)}, C^{(2)})]_{ab} \sim p_1\cdot N\cdot p_2 F_1^{(1)\,i}F_2^{(3)}{}_{iab}$$
$$[K^{p=1}(C^{(2)}, C^{(4)})]_{ab} \sim \frac{1}{3!}p_1\cdot V\cdot p_2 F_1^{(3)}{}_{ijk}F_2^{(5)}{}_{ab}{}^{ijk}$$
$$[K^{p=2}(C^{(1)}, C^{(3)})]_{ab} \sim p_1\cdot N\cdot p_2 F^{(2)\,ci}F^{(4)}{}_{abci} + \frac{1}{2!}p_1\cdot V\cdot p_2 F^{(2)\,ij}F^{(4)}{}_{abij}$$
$$[K^{p=3}(C^{(0)}, C^{(2)})]_{ab} \sim p_1\cdot N\cdot p_2 F_1^{(1)\,c}F_2^{(3)}{}_{cab} + p_1\cdot V\cdot p_2 F_1^{(1)\,i}F_2^{(3)}{}_{iab}$$
$$[K^{p=3}(C^{(2)}, C^{(4)})]_{ab} \sim \frac{1}{2!}p_1\cdot N\cdot p_2 F_1^{(3)\,cdi}F_2^{(5)}{}_{abcdi} + \frac{1}{2!}p_1\cdot V\cdot p_2 F_1^{(3)\,cij}F_2^{(5)}{}_{abcij} + \frac{1}{3!}p_1\cdot V\cdot p_2 F_1^{(3)\,ijk}F_2^{(5)}{}_{abijk}. \tag{82}$$

The trace of the above matrices is zero. One may use a stringy recursion relation similar to the

Britto-Cachazo-Feng-Witten field theory recursion relation [37] to construct the amplitude of four R-R states, i.e., $\mathcal{A}_{C^{(n)}C^{(n)}C^{(n+2)}C^{(n+2)}}$, from the above two-point functions.

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[1] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-theory: A Modern Introduction*, (Cambridge University Press, Cambridge, 2007).
[2] J. Polchinski, Dirichlet Branes and Ramond-Ramond Charges, Phys. Rev. Lett. **75**, 4724 (1995).
[3] M. R. Garousi and R. C. Myers, Superstring scattering from D-branes, Nucl. Phys. **B475**, 193 (1996).
[4] A. Hashimoto and I. R. Klebanov, Decay of excited D-branes, Phys. Lett. B **381**, 437 (1996).
[5] A. Giveon, M. Porrati, and E. Rabinovici, Target space duality in string theory, Phys. Rep. **244**, 77 (1994).
[6] J. H. Schwarz, Lectures on superstring and M theory dualities: Given at ICTP Spring School and at TASI Summer School, Nucl. Phys. B, Proc. Suppl. **55**, 1 (1997).
[7] M. R. Garousi, S-duality of S-matrix, J. High Energy Phys. 11 (2011) 016.
[8] M. R. Garousi, Duality constraints on effective actions, arXiv:1702.00191.
[9] K. B. Velni and M. R. Garousi, S-matrix elements from T-duality, Nucl. Phys. **B869**, 216 (2013).
[10] K. B. Velni and M. R. Garousi, Ramond-Ramond S-matrix elements from the T-dual Ward identity, Phys. Rev. D **89**, 106002 (2014).
[11] A. Jalali and M. R. Garousi, Higher derivative corrections to the Wess-Zumino action, Phys. Rev. D **94**, 086002 (2016).
[12] J. Cohn, D. Friedan, Z. a. Qiu, and S. H. Shenker, Covariant quantization of supersymmetric string theories: The spinor field of the Ramond-Neveu-Schwarz Model, Nucl. Phys. **B278**, 577 (1986).
[13] M. R. Garousi, Tree-level S-matrix elements from S-duality, J. High Energy Phys. 04 (2012) 140.
[14] M. Ademollo, A. D'Adda, R. D'Auria, F. Gliozzi, E. Napolitano, S. Sciuto, and P. Di Vecchia, Soft dilations and scale renormalization in dual theories, Nucl. Phys. **B94**, 221 (1975).
[15] J. A. Shapiro, On the renormalization of dual models, Phys. Rev. D **11**, 2937 (1975).
[16] B. U. W. Schwab, Subleading soft factor for string disk amplitudes, J. High Energy Phys. 08 (2014) 062.
[17] M. Bianchi, S. He, Y. t. Huang, and C. Wen, More on soft theorems: Trees, loops and strings, Phys. Rev. D **92**, 065022 (2015).
[18] B. U. W. Schwab, A note on soft factors for closed string scattering, J. High Energy Phys. 03 (2015) 140.
[19] P. Di Vecchia, R. Marotta, and M. Mojaza, Soft theorem for the graviton, dilaton and the Kalb-Ramond field in the bosonic string, J. High Energy Phys. 05 (2015) 137.
[20] M. Bianchi and A. L. Guerrieri, On the soft limit of open string disk amplitudes with massive states, J. High Energy Phys. 09 (2015) 164.
[21] S. G. Avery and B. U. W. Schwab, Burg-Metzner-Sachs symmetry, string theory, and soft theorems, Phys. Rev. D **93**, 026003 (2016).
[22] A. L. Guerrieri, Soft behavior of string amplitudes with external massive states, Nuovo Cimento Soc. Ital. Fis. **39C**, 221 (2016).
[23] P. Di Vecchia, R. Marotta, and M. Mojaza, Soft theorems from string theory, Fortschr. Phys. **64**, 389 (2016).
[24] M. Bianchi and A. L. Guerrieri, On the soft limit of tree-level string amplitudes, arXiv:1601.03457.
[25] P. Di Vecchia, R. Marotta, and M. Mojaza, Subsubleading soft theorems of gravitons and dilatons in the bosonic string, J. High Energy Phys. 06 (2016) 054.
[26] P. Di Vecchia, R. Marotta, and M. Mojaza, Soft behavior of a closed massless state in superstring and universality in the soft behavior of the dilaton, J. High Energy Phys. 12 (2016) 020.
[27] P. Di Vecchia, R. Marotta, and M. Mojaza, The B-field soft theorem and its unification with the graviton and dilaton, arXiv:1706.02961.
[28] M. R. Garousi and R. C. Myers, World volume interactions on D-branes, Nucl. Phys. **B542**, 73 (1999).
[29] U. Gran, GAMMA: A Mathematica package for performing gamma matrix algebra and Fierz transformations in arbitrary dimensions, arXiv:hep-th/0105086.
[30] M. R. Garousi, Superstring scattering from Dirichlet branes, Report No. UMI-0-612-19725-5.
[31] G. W. Gibbons and D. A. Rasheed, Sl(2,R) invariance of nonlinear electrodynamics coupled to an axion and a dilaton, Phys. Lett. B **365**, 46 (1996).
[32] A. A. Tseytlin, Selfduality of Born-Infeld action and Dirichlet three-brane of type IIB superstring theory, Nucl. Phys. **B469**, 51 (1996).
[33] M. B. Green and M. Gutperle, Comments on three-branes, Phys. Lett. B **377**, 28 (1996).
[34] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1994).
[35] T. Huber and D. Maitre, HypExp: A Mathematica package for expanding hypergeometric functions around integer-valued parameters, Comput. Phys. Commun. **175**, 122 (2006).
[36] M. R. Garousi, S-duality of D-brane action at order $O(\alpha'^2)$, Phys. Lett. B **701**, 465 (2011).
[37] R. Britto, F. Cachazo, B. Feng, and E. Witten, Direct Proof of Tree-Level Recursion Relation in Yang-Mills Theory, Phys. Rev. Lett. **94**, 181602 (2005).