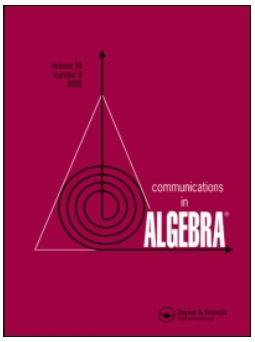

Taylor & Francis
Taylor & Francis Group

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

Lie algebras with few centralizers

Masoomeh Hoseiniravesh, Mohammad Reza Rajabzadeh Moghaddam & Mohammad Farrokhi Derakhshandeh Ghouchan

COMMUNICATIONS IN ALGEBRA®
2017, VOL. 45, NO. 7, 2867–2874
http://dx.doi.org/10.1080/00927872.2016.1233225

Taylor & Francis
Taylor & Francis Group

Lie algebras with few centralizers

Masoomeh Hoseiniravesh[a], Mohammad Reza Rajabzadeh Moghaddam[b], and Mohammad Farrokhi Derakhshandeh Ghouchan[c]

[a]Faculty of Mathematical Sciences, International Campus, Ferdowsi University of Mashhad, Mashhad, Iran; [b]Department of Mathematics, Khayyam University, International Campus and Centre of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran; [c]Mathematical Science Research Unit, College of Liberal Arts, Muroran Institute of Technology, Mizumoto, Muroran, Hokkaido, Japan

ABSTRACT
We will characterize all finite dimensional Lie algebras with at most $|F|^2+|F|+2$ centralizers, where F is the underlying field of Lie algebras under consideration.

ARTICLE HISTORY
Received 17 February 2016
Revised 10 April 2016
Communicated by
A. Elduque

KEYWORDS
Centralizer; Lie algebra

2010 MATHEMATICS SUBJECT CLASSIFICATION
Primary: 17B05; Secondary: 17B99

1. Introduction

Given a finite group G, the number of its element centralizers is known to have strong influence on its structure. As the first example, abelian groups are exactly those groups with only one element centralizer. The first nontrivial results are due to Belcastro and Sherman [6] who showed that

- There are no groups with 2 or 3 centralizers;
- A finite group G has 4 centralizers if and only if $\frac{G}{Z(G)} \cong C_2 \times C_2$; and
- A finite group G has 5 centralizers if and only if $\frac{G}{Z(G)} \cong C_3 \times C_3$ or S_3.

The results of Belcastro and Sherman are extended by several authors in [1–3] to include all finite groups with at most 8 centralizers. Also, in the case of simple groups, Ashrafi and Taeri in [4] describe all finite simple groups with at most 22 centralizers and later Zarrin [9] characterizes all finite semi-simple groups with at most 73 centralizers.

In case of Lie algebras, the only results on centralizers we are aware of are due to Barnea and Isaacs [5] who studied the relationships between the size of centralizers in a Lie algebra L with the dimension of $L/Z(L)$ as well as with the nilpotency of L. The works of Barnea and Isaacs are further continued by Jaikin-Zapirain [7] and Mann [8]. We note that the same problem for groups was already studied extensively by Ito in a series of papers within 1953–1973 and later continued by many authors.

The aim of this paper is to study Lie algebras with few centralizers. Indeed, we shall determine which Lie algebras over a finite field F have at most $|F|^2+|F|+2$ centralizers. We note that the underlying field of a Lie algebra with finitely many centralizers is always finite (see Theorem 3.1). To achieve our aim, we also count the number of centralizers of all Lie algebras L for which $L/Z(L)$ has dimension at most 4.

In what follows, $\mathrm{Cent}(L)$ denotes the set of centralizers of elements of a given Lie algebra L. Throughout this paper, L denotes a Lie algebra over a finite field F and Z stands for the centre of L. To ease notations, the set of non-central elements of L will be denoted by L^*. Also, the map $^- : L \longrightarrow L/Z(L)$

CONTACT Mohammad Farrokhi Derakhshandeh Ghouchan m.farrokhi.d.g@gmail.com Mathematical Science Research Unit, College of Liberal Arts, Muroran Institute of Technology, 27-1, Mizumoto, Muroran, Hokkaido 050-8585, Japan.

denotes the natural epimorphism. Accordingly, $\overline{\mathrm{Cent}}(L) = \{C/Z(L) : C \in \mathrm{Cent}(L)\}$. Moreover, if S is a subspace of L including $Z(L)$, then $\dim S/Z(L)$ is called the *bar-dimension* of S for convenience.

2. Counting centralizers in Lie algebras of low bar-dimensions

In this section, we count the number of centralizers of Lie algebras with central factors of dimension at most 4. The following two theorems are straightforward and we omit their proofs.

Theorem 2.1. *If* $\dim \overline{L} = 2$, *then* $|\mathrm{Cent}(L)| = |F| + 2$.

Theorem 2.2. *If* $\dim \overline{L} = 3$, *then one of the following holds:*
(1) $|\mathrm{Cent}(L)| = |F|^2 + 2$ *if and only if all but one of the proper centralizers of L have bar-dimension* 1; *and*
(2) $|\mathrm{Cent}(L)| = |F|^2 + |F| + 2$ *if and only if all of the proper centralizers of L have bar-dimension* 1.

The rest of this section is devoted to the evaluation of the number of centralizers of a Lie algebra with bar-dimension four.

Theorem 2.3. *If* $\dim \overline{L} = 4$, *then one of the following holds:*
(1) *if* $\mathrm{Cent}(L^*)$ *has an abelian element with bar-dimension* 3, *then* $|\mathrm{Cent}(L)| = |F|^3 + 2$;
(2) *if* $\mathrm{Cent}(L^*)$ *has elements with bar-dimension* 3 *but none of which are abelian, then either*
  (i) *L has a unique centralizer of bar-dimension* 3 *and* $\mathrm{Cent}(L) = |F|^3 + |F| + 3$;
  (ii) $L = \langle a, b, c, d : [a, c] = c, [b, d] = d\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^3 + |F|^2 + |F| + 2$;
  (iii) $L = \langle a, b, c, d, e : [a, c] = c, [b, d] = e\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^3 + |F|^2 + |F| + 2$;
  (iv) $L = \langle a, b, c, d, e, f : [a, c] = e, [b, d] = f\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^3 + |F|^2 + |F| + 2$;
  (v) $L = \langle a, b, c, d, e : [a, c] = [b, d] = e\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^3 + |F|^2 + |F| + 2$;
  (vi) $L = \langle a, b, c, d, e, f^* : [a, c] = [b, d] = e, [c, d] = \alpha a + \beta b + f^* \neq 0, \alpha, \beta \in \{0, 1\}\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^2 + 2|F| + 2$;
  (vii) $L = \langle a, b, c, d : [a, c] = [b, d] = \alpha a + \beta b, [c, d] = \beta c - \alpha d, (\alpha, \beta) \in \{(1, 0), (0, 1), (1, 1)\}\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^2 + 2|F| + 2$; *or*
  (viii) $L = \langle a, b, c, d, e^*, f^*, g^* : [a, c] = a \text{ or } e^*, [b, d] = b \text{ or } f^*, [c, d] = \alpha a + \beta b + g^* \neq 0, \alpha, \beta \in \{0, 1\}\rangle \oplus A$ *and* $|\mathrm{Cent}(L)| = |F|^3 - |F|^2 + 2|F| + 4$;
(3) *if* $\mathrm{Cent}(L^*)$ *has* $k \leq |F| + 1$ *centralizers of bar-dimension* 2, *at least one element of bar-dimension* 1 *and no elements of bar-dimension* 3, *then* $|\mathrm{Cent}(L)| = |F|(|F| + 1 - k) + |F|^3 + 2$; *and*
(4) *if all elements of* $\mathrm{Cent}(L^*)$ *have bar-dimension* 2, *then* $|\mathrm{Cent}(L)| = |F|^2 + 2$,
where A is an abelian Lie algebra and by e^*, f^*, g^* *it means that they can be omitted from the generators as well as relators.*

Proof. First assume that L has centralizers with bar-dimension 3. If L has an abelian centralizer C of bar-dimension 3, then one can easily see that $C_L(l) = C$, for all $l \in C\backslash Z$ and $C_L(l) = \langle Z, l\rangle$, for all $l \in L\backslash C$. Hence

$$|\mathrm{Cent}(L)| = 2 + \frac{|F|^4 - |F|^3}{|F| - 1} = |F|^3 + 2,$$

which is part (1). Now, assume that all centralizers with bar-dimension 3 are non-abelian. If L has a unique centralizer $C_L(a)$ of bar-dimension 3, then we observe that $C_L(l) = \langle Z, a, l\rangle$, for all $l \in C_L(a)\backslash\langle Z, a\rangle$ and $C_L(l) = \langle Z, l\rangle$, for all $l \in L\backslash C_L(a)$. Thus

$$|\mathrm{Cent}(L)| = 2 + \frac{|F|^3 - |F|}{|F|^2 - |F|} + \frac{|F|^4 - |F|^3}{|F| - 1} = |F|^3 + |F| + 3,$$

and we have part (2i). Hence we may assume that L has at least two centralizers with bar-dimension 3. It is easy to see that there must exist commuting elements a and b such that $C_L(a) \neq C_L(b)$ and both $C_L(a)$

and $C_L(b)$ have bar-dimension 3. Let $C_L(a) = \langle Z, a, b, d\rangle$ and $C_L(b) = \langle Z, a, b, c\rangle$. Since centralizers are subalgebras, we must have $[a, c] \in C_L(b)$ so that $[a, c] = \alpha a + \beta b + \gamma c + z$, for some $\alpha, \beta, \gamma \in F$ and $z \in Z$. First suppose that $\gamma \neq 0$, then by applying the transformations $a \mapsto \gamma^{-1}a$ and $c \mapsto \gamma^{-1}\alpha a + \gamma^{-1}\beta b + c + \gamma^{-1}z$, we may also assume that $[a, c] = c$. From the Jacobi identity

$$[a, d, c] + [d, c, a] + [c, a, d] = 0,$$

it follows that $[c, d] = [a, [c, d]]$. Assume $[c, d] = \alpha' a + \beta' b + \gamma' c + \delta' d + z'$ with $\alpha', \beta', \gamma', \delta' \in F$ and $z' \in Z$. Then $[c, d] = [a, [c, d]] = \gamma' c$. Hence either $[c, d] = 0$ or $[a - \gamma'^{-1}d, c] = 0$ so that, by using the transformation $d \mapsto a - \gamma'^{-1}d$ when $\gamma' \neq 0$, we may assume that $[c, d] = 0$. Finally, $[b, d] \in C_L(a) \cap C_L(c)$, which implies that $[b, d] = \beta'' b + \delta'' d + z''$ with $\beta'', \delta'' \in F$ and $z'' \in Z$. But then by suitable transformations of b and d, we may assume that either $[b, d] = d$ or $[b, d] = z'' \neq 0$. Now, a simple computation shows, in both cases, that $\dim \overline{C_L(l)} = 3$ if and only if $\bar{l} \in \langle \bar{a}, \bar{c}\rangle \cup \langle \bar{b}, \bar{d}\rangle \setminus \{0\}$ and that the map $\langle \bar{l}\rangle \mapsto C_L(l)$ is a bijection for such elements l. So we have $2(|F|+1)$ centralizers of bar-dimension 3. Also, all other non-central elements have centralizers of bar-dimension 2 sharing two subspaces of bar-dimension 1 with those centralizers of bar-dimension 3. Hence we have $(|F|^4 - (2|F|^2 - 1))/(|F| - 1) = (|F|^2 - 1)(|F| + 1)$ centralizers of dimension 2. Therefore $|\mathrm{Cent}(L)| = |F|^3 + |F|^2 + |F| + 2$ and parts (2ii) and (2iii) follow.

Next assume that $\gamma = 0$ for any choice of a, b, c, d as above. Thus $[a, c] = \alpha a + \beta b + z$. Similarly, we have $[b, d] = \alpha' a + \beta' b + z'$ for some $\alpha', \beta' \in F$ and $z' \in Z$. Let $[c, d] = \alpha'' a + \beta'' b + \gamma'' c + \delta'' d + z''$ with $\alpha'', \beta'', \gamma'', \delta'' \in F$ and $z'' \in Z$. From the Jacobi identity, it follows that $\beta[b, d] = \gamma''[a, c]$ and $\alpha'[a, c] = -\delta''[b, d]$. Now, we have two cases.

(I) There exists an element $x \in L\setminus\langle Z, a, b\rangle$ with centralizer of bar-dimention 3. Then $C_L(x)$ has non-trivial intersections with $C_L(a)\setminus\langle Z, a, b\rangle$ and $C_L(b)\setminus\langle Z, a, b\rangle$, from which by a suitable choice of c and d we may assume that $C_L(c) = \langle Z, b, c, d\rangle$ and $C_L(d) = \langle Z, a, c, d\rangle$ have bar-dimension 3 so that $L' \subseteq \langle Z, a, b\rangle$. But then the same argument with $C_L(a)$ and $C_L(d)$, and $C_L(b)$ and $C_L(b)$ shows that $L' \subseteq \langle Z, a, d\rangle$ and $L' \subseteq \langle Z, b, c\rangle$. Therefore $L' \subseteq Z$, which implies that $[a, c] = z$ and $[b, d] = z'$. If $\langle z\rangle \neq \langle z'\rangle$, then as before we may show that $|\mathrm{Cent}(L)| = |F|^3 + |F|^2 + |F| + 2$, which gives part (2iv). Now, assume that $\langle z\rangle = \langle z'\rangle$. Then, by replacing d with a suitable multiple, we can further assume that $z = z'$. Since $\dim L' = 1$, all proper centralizers have bar-dimension 3 and the map $\langle \bar{l}\rangle \mapsto C_L(l)$ is bijective, when $\bar{l}$ runs over non-zero elements of $\bar{L}$ so that $|\mathrm{Cent}(L)| = 1 + (|F|^4 - 1)/(|F| - 1) = |F|^3 + |F|^2 + |F| + 2$ and this is part (2v).

(II) If $C_L(l)$ has bar-dimension 3, then $l \in \langle Z, a, b\rangle$. If $\langle [a, c]\rangle = \langle [b, d]\rangle$, then by replacing d with a suitable multiple, we can assume that $[a, c] = [b, d]$. Hence $\gamma'' = \beta$ and $\delta'' = -\alpha$. Then $C_L(l) = \langle Z, a, b, vc - ud\rangle$ has bar-dimension 3 for every $l = ua + vb \in \langle a, b\rangle\setminus\{0\}$ so that L has exactly $|F| + 1$ centralizers of bar-dimension 3. Since any element with centralizer of bar-dimension 2 commutes with some $l \in \langle a, b\rangle\setminus\{0\}$, it follows that L has exactly $|F|^2 + |F|$ centralizers of bar-dimension 2. On the other hand, one can easily check that $L = \bigcup_{0\neq l\in\langle a,b\rangle} C_L(l)$ so that there is no element with a centralizer of bar-dimension 1. Therefore $|\mathrm{Cent}(L)| = |F|^2 + 2|F| + 2$. If $(\alpha, \beta) = (0, 0)$, then $[a, c] = [b, d] = z \neq 0$ and $[c, d] = \alpha'' a + \beta'' b + z'' \neq 0$. Also, by replacing a, b with suitable multiples, one can assume that $\alpha'', \beta'' = 0, 1$, which gives part (2vi). Otherwise, by using the transformations $c \mapsto ua + vb + c$ and $d \mapsto u'a + v'b + d$ with a suitable choice of u, v, u', v', one can assume that $\alpha'' = \beta'' = 0$. Now, by transformations $(a, d) \mapsto (a + \alpha^{-1}z, d - \alpha^{-1}z'')$ when $\alpha \neq 0$ and $(b, c) \mapsto (b + \beta^{-1}z, c + \beta^{-1}z'')$ when $\beta \neq 0$ we may further assume that $z = z'' = 0$. Finally, by replacing a, b, c, d with suitable multiples, we may also assume that $\alpha, \beta = 0, 1$ so that we obtain part (2vii).

Next assume that $\langle [a, c]\rangle \neq \langle [b, d]\rangle$. Then $\beta = \alpha' = \gamma'' = \delta'' = 0$ and one can easily check that $C_L(a)$ and $C_L(b)$ are the only centralizers of bar-dimension 3, $C_L(l)$ for $x \in (\langle \bar{a}, \bar{b}, \bar{c}\rangle \cup \langle \bar{a}, \bar{b}, \bar{d}\rangle)\setminus(\langle \bar{a}\rangle \cup \langle \bar{b}\rangle)$ are the only centralizer of bar-dimension 2 and all other proper centralizers have bar-dimension 1. Therefore $|\mathrm{Cent}(L)| = |F|^3 - |F|^2 + 2|F| + 4$. Finally, by suitable transformations $a \mapsto ua + wz$ and $b \mapsto v'b + w'z'$, we may assume that $\alpha'', \beta'' = 0, 1$, $[a, c] = a, z$ and $[b, d] = b, z'$, from which we get part (2viii).

Finally, suppose that $\mathrm{Cent}(L)$ has no elements of bar-dimension 3. If all elements of $\overline{\mathrm{Cent}}(L^*)$ are 2-dimensional, then we obtain case (4). Hence we may assume $\overline{\mathrm{Cent}}(L^*)$ contains 1-dimensional elements. First suppose that $\overline{\mathrm{Cent}}(L^*)$ contains two distinct elements $\overline{C}$ and $\overline{D}$ of dimension 2. Clearly, $\overline{L} = \overline{C} + \overline{D}$. If $\overline{C} = \langle \overline{c}_1, \overline{c}_2 \rangle$, $\overline{D} = \langle \overline{d}_1, \overline{d}_2 \rangle$ and $\dim \overline{C_L(c_1 + d_1)} = \dim \overline{C_L(c_1 + d_2)} = 2$, then there exist $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ $(i = 1, 2)$ with $(\beta_1, \delta_1), (\beta_2, \gamma_2) \neq (0, 0)$ such that $[c_1 + d_i, \alpha_i c_1 + \beta_i c_2 + \gamma_i d_1 + \delta_i d_2] = 0$. Hence

$$(\gamma_1 - \alpha_1)[c_1, d_1] + \delta_1[c_1, d_2] - \beta_1[c_2, d_1] = 0,$$
$$\gamma_2[c_1, d_1] + (\delta_2 - \alpha_2)[c_1, d_2] - \beta_2[c_2, d_2] = 0.$$

One can easily see that $\beta_1\beta_2 \neq 0$ so that $L' = \langle [c_1, d_1], [c_1, d_2] \rangle$ is 2-dimensional. But then all elements of $\overline{\mathrm{Cent}}(L^*)$ are 2-dimensional contradicting our assumption. Thus, for every element $\overline{c} \in \overline{C} \backslash \{\overline{0}\}$ and a basis $\{\overline{d}, \overline{d}'\}$ of $\overline{D}$, either $\overline{C_L(c + d)}$ or $\overline{C_L(c + d')}$ is 1-dimensional. Hence, to every 1-dimensional subspace $\overline{C}_0$ of $\overline{C}$ there corresponds a 1-dimensional subspace $\overline{D}_0$ such that $\overline{C_L(c + d)} = 1$ for all $\overline{c} \in \overline{C}_0 \backslash \{\overline{0}\}$ and $\overline{d} \in \overline{D} \backslash \overline{D}_0$. Therefore $\overline{\mathrm{Cent}}(L^*)$ contains at least $(|F| + 1)(|F|^2 - |F|) = |F|^3 - |F|$ elements with dimension 1. Now, if k denotes the number of 2-dimensional elements of $\overline{\mathrm{Cent}}(L^*)$, then we observe that

$$k \leq \frac{(|F|^4 - 1) - (|F|^3 - |F|)(|F| - 1)}{|F|^2 - 1} = |F| + 1.$$

This yields case (3) and the proof is complete. □

Example. Let $L = \langle x, y, z, w : [x, z] = w, [x, w] = [y, z] = z, [y, w] = y \rangle$ be a 4-dimensional Lie algebra with $Z(L) = 0$. Then L has exactly $|F| + 1$ centralizers of bar-dimension 2 and all other proper centralizers have bar-dimension 1. Hence the bound given in part (3) of the above theorem is sharp.

Note that Lie algebras satisfying part (4) of the above theorem always exist (see the example to Theorem 3.4 for instance).

3. Lie algebras with few centralizers

Utilizing the results of Section 2, we are now able to determine all Lie algebras L with at most $|F|^2 + |F| + 2$ centralizers. In addition, in this case, the possible number of centralizers of L will be computed. We begin with the characterization of those non-abelian Lie algebras which admit the smallest possible number of centralizers.

Theorem 3.1. *Let L be a non-abelian Lie algebra. Then $|\mathrm{Cent}(L)| \geq |F| + 2$ and the equality holds if and only if $\dim \overline{L} = 2$.*

Proof. Since L is not abelian, there exist two elements a and b in L such that $[a, b] \neq 0$. Let $A_{a,b} = \{\alpha a + b : \alpha \in F\} \cup \{0, a\}$. A simple verification shows that $C_L(x) \neq C_L(y)$, for all distinct elements x and y of $A_{a,b}$. Therefore $|\mathrm{Cent}(L)| \geq |F| + 2$. On the other hand, by Theorem 2.1, $|\mathrm{Cent}(L)| = |F| + 2$ whenever $\dim \overline{L} = 2$. Hence it remains to show that $\dim \overline{L} = 2$, when $|\mathrm{Cent}(L)| = |F| + 2$.

Suppose that $|\mathrm{Cent}(L)| = |F| + 2$ and let $a, b \in L$ such that $[a, b] \neq 0$. Since $\{C_L(l) : l \in A_{a,b}\}$ possesses $|F| + 2$ distinct centralizers, it follows that $\mathrm{Cent}(L) = \{C_L(l) : l \in A_{a,b}\}$. In particular,

$$L = C_L(a) \cup \bigcup_{\alpha \in F} C_L(\alpha a + b).$$

If $x \in L \backslash C_L(a)$ is any element, then $a + x \in L \backslash C_L(a)$ and hence there exist $\alpha, \beta \in F$ such that

$$x \in C_L(\alpha a + b) \qquad \text{and} \qquad a + x \in C_L(\beta a + b),$$

which imply that

$$\alpha[a,x]+[b,x]=0 \qquad \text{and} \qquad \beta[a,x]+[b,x]=[a,b].$$

Clearly, $\alpha \neq \beta$ and we have $[a,(\beta-\alpha)x-b]=0$, from which it follows that $x \in C_L(a)+Fb$. Therefore $L = C_L(a)+Fb$ and hence $\dim C_L(a) = \dim L - 1$. Similarly, $\dim C_L(b) = \dim L - 1$. On the other hand, $C_L(a) \cap C_L(b) \subseteq C_L(l)$ for all $l \in L$ so that $C_L(a) \cap C_L(b) \subseteq Z$. Now, we have

$$\begin{aligned}\dim \overline{L} &= \dim L - \dim Z \\ &\leq \dim L - \dim(C_L(a) \cap C_L(b)) \\ &\leq (\dim L - \dim C_L(a)) + (\dim L - \dim C_L(b)) = 2,\end{aligned}$$

which implies that $\dim \overline{L} = 2$, as required. □

For further analysis of centralizers, we need to introduce a particular class of Lie algebras.

Definition. A *CA-Lie algebra* is a Lie algebra all of whose proper centralizers are abelian.

We note that the elements of $\overline{\mathrm{Cent}}(L^*)$ in a CA-Lie algebra L give rise to a partition of $\overline{L}$ in the sense that no non-zero element of $\overline{L}$ belongs to two distinct elements of $\overline{\mathrm{Cent}}(L^*)$. The following result gives a lower bound for the number of centralizers of a CA-Lie algebra in terms of its bar-dimension.

Lemma 3.2. *If $\overline{\mathrm{Cent}}(L^*)$ partitions $\overline{L}$, then*

$$|F|^{\lceil \dim \overline{L}/2 \rceil} + 2 \leq |\mathrm{Cent}(L)|.$$

Proof. Let $\dim \overline{L} = n$. If $\dim \overline{C} = k > n/2$ for some $\overline{C} \in \overline{\mathrm{Cent}}(L^*)$, then since $\overline{L} = \bigcup_{\overline{X} \in \overline{\mathrm{Cent}}(L^*)} \overline{X}$ and $\dim \overline{X} \leq n-k$ for all $\overline{X} \in \overline{\mathrm{Cent}}(L^*) \setminus \{\overline{C}\}$, it follows that

$$|\overline{L}| \leq |\overline{C}| + (|\mathrm{Cent}(L)| - 2)(|\overline{F}|^{n-k} - 1).$$

Hence

$$|F|^{\lceil \frac{n}{2} \rceil} + 2 \leq |F|^k + 2 \leq |\mathrm{Cent}(L)|,$$

and we are done. Finally, suppose that $\dim \overline{X} \leq n/2$, for all $\overline{X} \in \overline{\mathrm{Cent}}(L^*)$. Then

$$|F|^{\lceil \frac{n}{2} \rceil} + 1 \leq \frac{|F|^n - 1}{|F|^{\lfloor \frac{n}{2} \rfloor} - 1} \leq |\mathrm{Cent}(L)| - 1,$$

as required. □

In what follows, $\mathrm{Cent}_L(S)$ (resp. $\overline{\mathrm{Cent}_L}(S)$) stands for the set of all $C_L(s)$ (resp. $\overline{C_L(s)}$) of elements $s \in S \subseteq L$, in which L is a given Lie algebra.

Lemma 3.3. *Let L be a Lie algebra of bar-dimension $n \geq 5$ such that $|\mathrm{Cent}(L)| \leq |F|^2 + |F| + 2$. For any commutative subspace $C \subseteq L$ of dimension 2 disjoint from $Z(L)$, there exists an element $l \in L$ such that $C_C(l) = 0$.*

Proof. Assume $C_C(l) \neq 0$ for all $l \in L$. Then $L = \bigcup_{l \in C \setminus \{0\}} C_L(l)$. There exists $\overline{X} \in \overline{\mathrm{Cent}_L}(C^*)$ such that $\dim \overline{X} = n-1$, for otherwise

$$|F|^n = |\overline{L}| \leq (|F|+1)|F|^{n-2} < |F|^n,$$

which is a contradiction. Let s be the number of those elements $\overline{Y}$ of $\overline{\mathrm{Cent}_L}(C^*)\backslash\{\overline{X}\}$ of dimension $n-1$ and $t = |F| - s$. Since $\dim(\overline{X} \cap \overline{Y}) = n-2$ for any such $\overline{Y}$, it follows that

$$|F|^n = |\overline{L}| \leq |\overline{X}| + s(|F|^{n-1} - |F|^{n-2}) + t(|F|^{n-2} - |F|^2),$$

which is impossible unless $s = |F|$ and $t = 0$. Thus $\dim \overline{C_L(l)} = n-1$ for all $l \in C\backslash\{0\}$. Assume $\dim \overline{C_L(x)} < n-1$ for some $x \in L$. Clearly, $x \in C_L(a)$ for some $a \in C\backslash\{0\}$. Let $D = \langle a, x\rangle$. Since D is commutative, the above argument grantee the existence of an element $l \in L$ such that $C_D(l) = 0$. Then $|\mathrm{Cent}_L(D+l)| = |F|^2$ otherwise $C_L(d_1 + l) = C_L(d_2 + l)$ for distinct elements $d_1, d_2 \in D$, which implies that $[d_1 + l, d_2 + l] = 0$ or $[d_1 - d_2, l] = 0$, a contradiction. Clearly, $|\mathrm{Cent}_L(D^*)| = |F| + 1$ and that $\mathrm{Cent}_L(D^*)$ and $\mathrm{Cent}_L(D+l)$ are disjoint. Thus $|\mathrm{Cent}(L)| = |F|^2 + |F| + 2$. On the other hand, $|\mathrm{Cent}_L(C^*)| = |F|+1$ and $\mathrm{Cent}_L(C^*)$ and $\mathrm{Cent}_L(D+l)$ are also disjoint for every centralizer in $\mathrm{Cent}_L(C^*)$ contains a but no centralizer in $\mathrm{Cent}_L(D+l)$ contains a. Hence, we must have $\mathrm{Cent}_L(C^*) = \mathrm{Cent}_L(D^*)$, which implies that $C_L(x) = C_L(b)$ has bar dimension $n-1$ for some $b \in C\backslash\langle a\rangle$, which is a contradiction. Therefore, all nontrivial centralizers in L have bar dimension $n-1$. If $l \in L\backslash Z(L)$ and $l' \in L\backslash C_L(l)$, then $Z(C_L(l)) \cap C_L(l') = Z(L)$, which is possible only if $\overline{Z(C_L(l))} = \langle \overline{l}\rangle$. This shows that, the map $\langle \overline{l}\rangle \longmapsto C_L(l)$ is injective and subsequently $|\mathrm{Cent}(L)| = |F|^n > |F|^2 + |F| + 2$, the final contradiction. □

Theorem 3.4. *Let L be a Lie algebra such that $|\mathrm{Cent}(L)| > |F| + 2$. Then $|\mathrm{Cent}(L)| \geq |F|^2 + 2$ and the equality holds if and only if*
(1) $\dim \overline{L} = 3$ *and all but one of the proper centralizers of L have bar-dimension* 1; *or*
(2) $\dim \overline{L} = 4$ *and all of the proper centralizers of L have bar-dimension* 2.

Proof. Let $\dim \overline{L} = n$. By Theorems 2.1, 2.2 and 2.3, we can assume that $n \geq 5$. Moreover, we assume $|\mathrm{Cent}(L)| \leq |F|^2 + 2$. If all proper centralizers of L have bar-dimension 1, then

$$|\mathrm{Cent}(L)| = 1 + \frac{|\overline{L}| - 1}{|F| - 1} = |F|^{n-1} + |F|^{n-2} + \cdots + |F| + 2 > |F|^2 + 2$$

and we are done. Hence one can assume that there exist commuting elements $a, b \in L\backslash Z$ such that $\langle \overline{a}\rangle \neq \langle \overline{b}\rangle$. Let $C = \langle a, b\rangle$. By Lemma 3.3, there exists an element $l \in L$ such that $C_C(l) = 0$. Then $|\mathrm{Cent}_L(C+l)| = |F|^2$ and consequently $|\mathrm{Cent}(L)| \geq |F|^2 + 2$.

Finally, suppose that the equality holds. Then $C_L(c) = C_L(a) = C_L(b)$, for all $c \in C\backslash\{0\}$. Let $S := C_L(a)$. We show that S is abelian. To this end, let $s \in S\backslash Z$. If $C_L(s) = C_L(c + l)$ for some $c \in C$, then $[a, l] = 0$ as $a \in C_L(s)$, which is a contradiction. Thus $C_L(s) = S$, as required. Since, the elements a and b were arbitrary, it follows that all proper centralizers of L are abelian so that L is a CA-Lie algebra. But then Lemma 3.2 yields $|F|^{\lceil \frac{n}{2} \rceil} \leq |F|^2$. Hence $n \leq 4$, which is a contradiction. □

Example. The following Lie algebras satisfy parts (1) and (2) of the above theorem.
(1) Let $L_\lambda = \langle x, y, z : [x, y] = y, [x, z] = \lambda z\rangle$ be a 3-dimensional Lie algebra with $\dim(L') = 2$ and $Z(L) = 0$. Then $|\mathrm{Cent}(L)| = |F|^2 + 2$.
(2) Let $L = \langle x, y, z, w : [x, z] = w, [x, w] = [y, z] = z, [y, w] = \lambda w\rangle$ if $\mathrm{char}(F) > 2$ and $\lambda \in F\backslash\{x^2 : x \in F\}$ and $L = \langle x, y, z, w : [x, z] = w, [x, w] = [y, z] = z, [y, w] = \lambda z + \lambda w\rangle$ if $\mathrm{char}(F) = 2$ and $\lambda^{-1} \in F\backslash\{x(x-1) : x \in F\}$. Then L is a 4-dimensional Lie algebra with $\dim(L') = 2$ and $Z(L) = 0$. One can easily see that $|\mathrm{Cent}(L)| = |F|^2 + 2$.

In what follows, we present the third smallest possible size of the set $\mathrm{Cent}(L)$ of a non-abelian Lie algebra L .

Theorem 3.5. *Let L be a Lie algebra such that $|\mathrm{Cent}(L)| > |F|^2 + 2$. Then $|\mathrm{Cent}(L)| \geq |F|^2 + |F| + 2$ and the equality holds if and only if $\dim \overline{L} = 3$ and all proper centralizers of L have bar-dimension* 1.

Proof. Let $\dim \overline{L} = n$. First observe that if $n \leq 4$, then the result holds by Theorems 2.1, 2.2 and 2.3. Hence it is enough to consider the case where $n \geq 5$. Suppose that $|\mathrm{Cent}(L)| \leq |F|^2 + |F| + 2$. We proceed in some steps.

Step 1. If C is a commuting subspace of L of dimension 2 disjoint from Z, then there exist elements a, b such that $C = \langle a, b\rangle$ and $C_L(a) = C_L(b)$. First of all we note that such a C exists as in the first paragraph of Theorem 3.4. Also, by Lemma 3.3, there exists an element $l \in L$ such that $C_C(l) = 0$. If $|\mathrm{Cent}_L(C)| < |F| + 2$, then we are done. Hence assume that $|\mathrm{Cent}_L(C)| = |F| + 2$. We know that $|\mathrm{Cent}_L(C+l)| = |F|^2$. In addition, $\mathrm{Cent}_L(C) \cap \mathrm{Cent}_L(C+l) = \emptyset$ for otherwise l commutes with a non-zero element of C. Thus $\mathrm{Cent}(L) = \mathrm{Cent}_L(C) \cup \mathrm{Cent}_L(C+l)$. Now, if $C = \langle a, b\rangle$ and $x \in C_L(a)\backslash C_L(b)$, then we must have $C_L(x) = C_L(c)$ for some $c \in C\backslash\{0\}$, otherwise $C_L(x) = C_L(c+l)$ for some $c \in C$ so that a commutes with l contradicting the assumption on l. But then $b \in C_L(x)$, which is a contradiction. Thus $C_L(a) \subseteq C_L(b)$. Similarly $C_L(b) \subseteq C_L(a)$, which implies that $C_L(a) = C_L(b)$, contradicting the fact that $|\mathrm{Cent}_L(C)| = |F| + 2$.

Step 2. If C is a commuting subspace of L of dimension 3 disjoint from Z, then $C_C(l) \neq 0$, for all $l \in L$. Indeed, if $C_C(l) = 0$ for some $l \in L$, then $\mathrm{Cent}_L(C + l)$ contains $|F|^3$ distinct centralizers, which is impossible.

Step 3. There is no element of dimension $n - 1$ in $\overline{\mathrm{Cent}}(L)$. Suppose on the contrary that $\overline{C}$ is such an element. Clearly,

$$|\mathrm{Cent}(L)| \geq |\mathrm{Cent}(C)| + |\mathrm{Cent}_L(L\backslash C)|.$$

From step 2, we know that $\dim \overline{Z(C)} \leq 2$. If $\dim \overline{Z(C)} = 2$, then $\dim \overline{C_L(l)} \leq n-2$ for all $l \in L\backslash C$ so that $|\mathrm{Cent}_L(L\backslash C)| \geq |F|^2$. Thus $|\mathrm{Cent}(C)| < |F|+2$, which is possible only if C is abelian (see Theorem 3.1). But then, $\dim \overline{C} = \dim \overline{Z(C)} = 2$ and consequently $\dim \overline{L} = 3$, a contradiction. Therefore $\dim \overline{Z(C)} = 1$. Clearly, $|\mathrm{Cent}_L(L\backslash C)| \geq |F|$ so that $|\mathrm{Cent}(C)| < |F|^2 + 2$. Now, by Theorem 3.4, $\dim C/Z(C) \leq 2$, which implies that $\dim \overline{L} \leq 4$, a contradiction.

Step 4. L has no commuting subspaces of dimension 3 disjoint from Z. Suppose on the contrary that C is such a subspace. We show that $|\mathrm{Cent}_L(C^*)| \leq |F|^2$. First observe that $\mathrm{Cent}_L(C^*)$ has at most $|F|^2 + |F| + 1$ elements. If every element of $\mathrm{Cent}_L(C^*)$ appears twice, then $|\mathrm{Cent}_L(C^*)| \leq (|F|^2 + |F| + 1)/2 < |F|^2$ and we are done. Otherwise, there exists an element $x \in C\backslash\{0\}$ such that $C_L(x) \neq C_L(l)$, for all $l \in C\backslash\langle x\rangle$. But then, by step 1, among $|F|$ subspaces of $\langle x, l\rangle$ of dimension 1 different from $\langle x\rangle$, there are two of which, say $\langle l_a\rangle$ and $\langle l_b\rangle$, satisfying $C_L(l_a) = C_L(l_b)$. Since $\langle l_a\rangle$ and $\langle l_b\rangle$ are all distinct when $\langle x, l\rangle$ ranges over all subspaces of C of dimension 2 including x and there are $|F| + 1$ such subspaces, it follows that

$$|\mathrm{Cent}_L(C^*)| \leq (|F|^2 + |F| + 1) - (|F| + 1) = |F|^2,$$

as required. On the other hand, from step 3, we know that all elements of $\overline{\mathrm{Cent}_L}(C^*)$ have dimension at most $n - 2$. Also, step 2 indicates that the elements of $\overline{\mathrm{Cent}_L}(C^*)$ cover $\overline{L}$. Thus $|\overline{L}| < |\mathrm{Cent}_L(C^*)||F|^{n-2} = |F|^n$, which is a contradiction.

Step 5. There is an element of dimension $n - 2$ in $\overline{\mathrm{Cent}}(L)$. If m denotes the maximum dimension among elements of $\overline{\mathrm{Cent}}(L^*)$, then

$$|F|^n = |\overline{L}| < |\mathrm{Cent}(L)||F|^m < |F|^{m+3},$$

which implies that $m \geq n - 2$. Since $m \leq n - 2$ by step 3, we must have $m = n - 2$.

Now, assume that $\overline{C} = \overline{C_L(x)} \in \overline{\mathrm{Cent}}(L)$ is an element of dimension $n - 2$. For any $l \in L\backslash C$, we have $|\overline{C_L(l)}| = |F|^{s+t}$, where $|F|^t = |\overline{C_L(l)} \cap \overline{C}|$, in which $s = 1, 2$ and $t \leq n - 2 - s$. Hence $|\overline{C_L(l)}\backslash\overline{C}| \leq$

$|F|^{n-2} - |F|^{n-4}$ so that

$$|\mathrm{Cent}_L(L\backslash C)| \geq \frac{|F|^n - |F|^{n-2}}{|F|^{n-2} - |F|^{n-4}} = |F|^2.$$

On the other hand, by step 4, $|\overline{C_L(l)} \cap \overline{C}| = |F|^2$ for all $\bar{l} \in \overline{C}\backslash\langle\overline{x}\rangle$ so that

$$|\mathrm{Cent}_L(C^*)| \geq \frac{|F|^{n-2} - |F|}{|F|^2 - |F|} = |F|^{n-4} + \cdots + |F| + 1.$$

Therefore $|\mathrm{Cent}(L)| = 1 + |\mathrm{Cent}_L(C^*)| + |\mathrm{Cent}_L(L\backslash C)| \geq |F|^2 + |F|^{n-4} + \cdots + |F| + 2$, which is possible only if $n = 5$ and $|\mathrm{Cent}(L)| = |F|^2 + |F| + 2$. Then $|\overline{C_L(l)}| = |F|^{n-2} = |F|^3$ and $|\overline{C_L(l)} \cap \overline{C}| = |F|^{n-4} = |F|$, for all $l \in L\backslash C$. Also, $C_L(a) = C_L(b)$, for all $\overline{a}, \overline{b} \in \langle\overline{x}, \bar{l}\rangle\backslash\langle\overline{x}\rangle$ and $\bar{l} \in \overline{C}\backslash\langle\overline{x}\rangle$. However, if $y \in L\backslash C$ and $\overline{C_L(x)} \cap \overline{C_L(y)} = \langle\overline{z}\rangle$, then $y \in C_L(z) = C_L(x+z)$ from which it follows that $y \in C_L(x)$, a contradiction. The proof is complete. □

Example. Let $L = \mathfrak{sl}_3(F) = \langle x, y, z : [x, y] = z, [x, z] = -2x, [y, z] = 2y\rangle$. Then $|\mathrm{Cent}(L)| = |F|^2 + |F| + 2$ and L satisfies the above theorem.

Acknowledgment

The authors would like to thank the referee for some helpful corrections.

References

[1] Abdollahi, A., Jafarian Amiri, S. M., Mohammadi Hassanabadi, A. (2007). Groups with specific number of centralizers. *Houston J. Math.* 33:43–57.
[2] Ashrafi, A. R. (2000). Counting the centralizers of some finite groups. *Korean J. Comput. Appl. Math.* 7:115–124.
[3] Ashrafi, A. R. (2000). On finite groups with a given number of centralizers. *Algebra Colloq.* 7:139–146.
[4] Ashrafi, A. R., Taeri, B. (2005). On finite groups with a certain number of centralizers. *J. Appl. Math. Comput.* 17:217–227.
[5] Barnea, Y., Isaacs, I. M. (2003). Lie algebras with few centralizer dimensions. *J. Algebra* 259:284–299.
[6] Belcastro, S. M., Sherman, G. J. (1994). Counting centralizers in finite groups. *Math. Mag.* 5:111–114.
[7] Jaikin-Zapirain, A. (2005). Centralizer sizes and nilpotency class in Lie algebras and finite p-groups. *Proc. Am. Math. Soc.* 133(10):2817–2820.
[8] Mann, A. (2006). Elements of minimal breadth in finite p-groups and Lie algebras. *J. Aust. Math. Soc.* 81:209–214.
[9] Zarrin, M. (2009). On element-centralizers in finite groups. *Arch. Math.* 93:497–503.