The Extended Abstracts of
The 3rd Seminar on Operator Theory and its Applications
8-9th March 2017, Ferdowsi University of Mashhad, Iran

Oral Presentation

MIXED C-COSINE FAMILIES ON BANACH SPACES

MASOUD MOSALLANEZHAD[1]* AND MOHAMMAD JANFADA[2]

[1] *Department of Pure Mathematics, Ferdowsi University of Mashhad, International Campus, Iran.*
masvame63@gmail.com

[2] *Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran.*
janfada@um.ac.ir

Abstract. In this paper a H generalized Cauchy equation

$$[S(t+s)+S(t-s)]C = H(S(s),S(t))$$

will be considered, where $\{S(t)\}$ is a one parameter family of bounded operators and $H : B(X)\times B(X) \to B(X)$ is a function. For the special case, when $H(S(s),S(t)) = 2S(s)S(t)+2D(S(s)-T(s))(S(t)-T(t))$ with $D \in B(X)$, solutions of H generalized Cauchy equation will be studied, where $\{T(t)\}$ is a C-cosine family of operators.

1. Introduction and preliminaries

Suppose that X is a Banach space, $\{S(t); t \in \mathbb{R}\}$ is a strongly continuous C-cosine operator function on X if it is a family of operators in $B(X)$ satisfying
(a) $S(0) = C$;

2010 *Mathematics Subject Classification.* Primary 47D60; Secondary 34G10, 47D03.
Key words and phrases. C-cosine family, Cauchy equation, sine operator.
* Speaker.

(b) $[S(t+s)+S(t-s)]C = 2S(t)S(s))$ for $t, s \in \mathbb{R}$;
(c) the function $S(.)x$ is continuous on $\mathbb{R}$ for every $x \in X$.
(see [1])

The associated sine operator function $\mathbb{S}(.)$ is defined by the formula $\mathbb{S}(t) = \int_0^t S(s)ds, t \in \mathbb{R}$. The second infinitesimal generator(or simply the generator) A of $S(.)$ is defined as $Ax = C^{-1}\lim_{t\to 0}\frac{2}{t^2}(S(t)-C)x$, with natural domain. A C-cosine operator function gives the solution of a well-posed Cauchy problem

$$CP(A;x;y)\begin{cases} \frac{d^2u(t)}{dt^2} = Au(t), & t \in \mathbb{R}, \\ u(0) = x, u^{'}(0) = y \end{cases}$$

For the theory of cosine operator function we refer to [1, 2, 4, 3].

2. Main results

Let X be a Banach space and C be an injective operator in $B(X)$. A family $\{S(t)\}_{t\in\Re} \subseteq B(X)$ is said to satisfy a H-C-cosine Cauchy equation if

$$[S(s+t)+S(s-t)]C = H(S(s), S(t)), \tag{2.1}$$

where $H : B(X) \times B(X) \to B(X)$ is a function. If $H(S(s), S(t)) = 2S(s)S(t)$, then $\{S(t)\}_{t\in\Re}$ satisfy in the first condition of C-cosine family of operators.

In this section we consider a special case when

$$H(S(s), S(t)) = 2S(s)S(t) + 2D(S(s)-T(s))(S(t)-T(t)), \tag{2.2}$$

where $\{T(t)\}_{t\in\Re}$ is a C-cosine family of operators and $D \in B(X)$. Trivially $D = 0$ is a the C-cosine condition.

Now consider the equation (2.1) with H as in (2.2). Let $A : D(A) \subseteq X \to X$ be defined as $A(x) = C^{-1}\lim_{h\to 0}\frac{2}{h^2}[S(h)x - Cx]$, where $D(A) = \{x \in X : \lim_{h\to 0}\frac{2}{h^2}[S(s)x - Cx]$ exists in the range of C$\}$. We shall think of A as the infinitesimal generator of $\{S(t)\}_{t\in\Re}$.

Lemma 2.1. *Let $\{S(t)\} \subseteq B(X)$ be a strongly continuous family, with $S(0) = C$ and it satisfies (2.1) with H as (2.2). Then*

(1) *$S(s) = S(-s)$ and $S(s)C = CS(s)$ for all s.*
(2) *If $I + D$ is injective and for any s, t, $T(s)S(t) = S(t)T(s)$, then $S(s)S(t) = S(t)S(s)$ for all s, t.*
(3) *If D is injective and $S(s)S(t) = S(t)S(s)$ for all s, t, then for any s, t, $T(s)S(t) = S(t)T(s)$.*
(4) *If $\{S(t)\}_{t\in\Re}$ is a commuting family and $x \in D(A)$ then for any t, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.*

Theorem 2.2. *Suppose that $\{S(t)\} \subseteq B(X)$ is a commuting strongly continuous family with $S(0) = C$ and it satisfies (2.1) with H as (2.2). Let $T_1(t)x := (1 + D)S(t)x - DT(t)x$, $x \in X$, then $\{T_1(t)\}$ is a C-cosine family. Furthermore if A_0 is the infinitesimal generator of the C-cosine family $\{T(t)\}$, then an extension of $A_1 := (1 + D)A - DA_0$ is the generator of $\{T_1(t)\}$.*

The previous theorem implies that if $(I + D)$ is invertible then the solution $S(t)$ of (2.1) with H as (2.2) is of the form

$$S(t)x = D(D + I)^{-1}T(t)x + (1 + D)^{-1}T_1(t)x, \quad x \in X.$$

For $D = -I$, the equation (2.1) reduces to

$$[S(t+s)+S(t-s)]C = 2S(t)S(s)+2(T(t)-S(t))(S(s)-T(s)). \quad (2.3)$$

Also in this case with A_0 and A as before, so we have for $x \in X$, $T^{'}(t)x = A_0 \int_0^t T(s)xds$ and,

Theorem 2.3. *Let $\{S(t)\}$ be a strongly continuous family satisfying (2.3). Then*

1. *For any $x \in D(A)$, $\frac{d^2}{dt^2}S(t)x = A_0S(t)x + (A - A_0)T(t)x$.*
2. *The solution of (2.3) in the uniform operator topology is of the form $S^{"}(t) = A_0S(t)x + (A - A_0)T(t)x$.*

The associated sine operator function $\mathbb{S}(.)$ is defined by the formula $\mathbb{S}(t) := \int_0^t S(s)ds, t \in \mathbb{R}$. We consider sine of C-cosine family $\{T(t)\}$ by $\Im(t) = \int_0^t T(m)dm$.

Proposition 2.4. *Suppose that $\{S(t)\} \subseteq B(X)$ is a commuting strongly continuous family with $S(0) = C$ and it satisfies (2.1) with H as (2.2). The following asertions hold*

$$S(t) = S(-t) \ for \ all \ t \in \Re; \quad (2.4)$$

$$\mathbb{S}(-t) = -\mathbb{S}(t) \ for \ all \ t \in \Re. \quad (2.5)$$

$$S(s), \mathbb{S}(s), S(t), \ and \ \mathbb{S}(t) \ commute \ for \ all \ t, s \in \Re. \quad (2.6)$$

$$[\mathbb{S}(s+t)+\mathbb{S}(s-t)]C = 2S(t)\mathbb{S}(s)+2D(S(t)-T(t))(\mathbb{S}(s)-\Im(s)). \quad (2.7)$$

$$\mathbb{S}(t + s)C = S(t)\mathbb{S}(s) + S(s)\mathbb{S}(t)$$
$$+D[(S(t) - T(t))(\mathbb{S}(s) - \Im(s)) + (S(s) - T(s))(\mathbb{S}(t) - \Im(t))]. \quad (2.8)$$

References

1. CC. Kuo and SY. Shaw, *C-cosine functions and the abstract Cauchy problem*, I$^*.J.Math.Anal.Appl\mathbf{210}(1997), 632-646$.
2. H. Singh and AB. Buche, *On some analytic properties of the modified exponential-cosine operator*, Indian J. pure Appl. Math. **10** (1979), no. 8, 913-919.
3. T. Takenaka and S. Piskarev, *Local C-cosine families and N-times integrated local cosine families*, Taiwan Journal of mathematics **8** (2004), no. 3, 515-545.
4. CC. Kuo, *Note on local integrated C-cosine function and abstract Cauchy problems*, Taiwan Journal of mathematics **17** (2013), no. 3, 957-980.