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FRAME FOR OPERATORS IN FINITE DIMENTIONAL HILBERT SPACE

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Abstract

In this paper, we study frames for operators (K-frames) in finite dimentional Hilbert spaces and express the dual of K-frames. Some properties of K-dual frames are investigated. Furthermore, the notion of their oblique K-duals and some properties are presented.

1. Introduction

Let $K \in B(\mathcal{H})$, the space of all bounded linear operators on a Hilbert space $\mathcal{H}$. A sequence $\{\varphi_j\}_{j\in\mathbb{J}}$ is said to be a K-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \le \sum_{j\in\mathbb{J}} |\langle x, \varphi_j \rangle|^2 \le B\|x\|^2, \quad (x \in \mathcal{H}). \tag{1}$$

We call A, B the lower and the upper K-frame bounds for $\{\varphi_j\}_{j\in\mathbb{J}}$, respectively. If $K = I_{\mathcal{H}}$, then $\{\varphi_j\}_{j\in\mathbb{J}}$ is the ordinary frame. If only the right inequalitiy holds, then $\{\varphi_j\}_{j\in\mathbb{J}}$ is called a Bessel sequence. Suppose that $\Phi := \{\varphi_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$. The operator $T_\Phi : \mathcal{H} \to \ell^2(\mathbb{J})$ defined by $T_\Phi(x) = \{\langle x, \varphi_j \rangle\}_{j\in\mathbb{J}}$ is called the analysis operator. T_Φ is bounded and $T_\Phi^* : \ell^2(\mathbb{J}) \to \mathcal{H}$ is given by $T_\Phi^*(\{c_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} c_j \varphi_j$. T_Φ^* is called the pre-frame or synthesis operator. The operator $S_\Phi : \mathcal{H} \to \mathcal{H}$ defined by $S_\Phi(x) = T_\Phi^* T_\Phi(x) = \sum_{j\in\mathbb{J}} \langle x, \varphi_j \rangle \varphi_j$ is called the frame operator of Φ. Note that, frame operator of a K-frame is not invertible on $\mathcal{H}$ in general, but it is invertible on the subspace $R(K) \subset \mathcal{H}$, that $R(K)$ is the range of K.

Given a positive integer N. Throughout this paper, we suppose that $\mathcal{H}^N$ is a real or complex N-dimensional Hilbert space. By $\langle \cdot, \cdot \rangle$ and $\|.\|$ we denote the inner product on $\mathcal{H}^N$ and its corresponding norm, respectively. Denote by P_W the orthogonal projection of $\mathcal{H}$ onto a closed subspace $W \subseteq \mathcal{H}$.

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2. Finite K-frames

In this section, we present K-frame theory in finite-dimensional Hilbert spaces.

Let $K \in B(\mathcal{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ be a family of vectors in $\mathcal{H}^N$. If $A\|K^*x\|^2 = \sum_{j=1}^M |\langle x, \varphi_j \rangle|^2$, then Φ is called an A-tight K-frame and if $\|K^*x\|^2 = \sum_{j=1}^M |\langle x, \varphi_j \rangle|^2$, then Φ is called a tight K-frame. If $\|\varphi_j\| = 1$ for all $j = 1, 2, ..., M$, this is an unit norm K-frame.

For an arbitrary K-frame, we obtain the optimal lower and upper K-frame bounds by eigenvalues of its frame operator.

Proposition 2.1. *Let $0 \neq K \in B(\mathcal{H}^N)$. Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $R(K)$ with K-frame operator S_Φ with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N > 0$. Then λ_1 is the optimal upper K-frame bound and if $\lambda_N \neq 0$ then $\frac{\lambda_N}{\|K\|^2}$ is the optimal lower K-frame bound.*

Now, we introduce a constructive method to extend a given frame to a tight K-frame.

Theorem 2.2. *Let $K \in B(\mathcal{H}^N)$. Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a frame for $\mathcal{H}^N$. Assume that the frame operator S_Φ has the eigenvalues $\{\lambda_j\}_{j=1}^N$, ordered as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N > 0$. Let $\{e_j\}_{j=1}^N$ be a corresponding eigenbasis. Then the collection $\{K\varphi_j\}_{j=1}^M \cup \{\sqrt{\lambda_1 - \lambda_j}\, Ke_j\}_{j=2}^N$ is a λ_1-tight K-frame for $\mathcal{H}^N$.*

In the following proposition, we express two inequality of A-tight K-frames.

Proposition 2.3. *(i) If $\Phi = \{\varphi_j\}_{j=1}^M$ is an A-tight K-frame for $\mathcal{H}^N$, then*

$$\max_{j=1,2,...,M} \|\varphi_j\|^2 \leq A\|K\|^2.$$

(ii) If $\Phi = \{\varphi_j\}_{j=1}^M$ is an unit norm A-tight K-frame for $\mathcal{H}^N$, then

$$A\|K\|^2 N \geq M.$$

In the last part of this section, we study conditions under which a linear combination of two K-frames is K-frame too.

Definition 2.4. *Let $K \in B(\mathcal{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ be K-frames for $\mathcal{H}^N$. Φ and Ψ are called strongly disjoint if $R(T_\Phi) \perp R(T_\Psi)$, where T_Φ and T_Ψ are the analysis operators of the sequences Φ and Ψ, respectively.*

Theorem 2.5. *Suppose that $K \in B(\mathcal{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ are strongly disjoint tight K-frames for $\mathcal{H}^N$. Also, assume that $A, B \in B(\mathcal{H}^N)$ are operators such that $AKK^*A^* + BKK^*B^* = I_{N \times N}$, then $\{A\Phi + B\Psi\}$ is a K-frame for $\mathcal{H}^N$. In particular, if $KK^* = \frac{1}{2(|\alpha|^2 + |\beta|^2)} I_{N \times N}$, then $\{\alpha\Phi + \beta\Psi\}$ is a K-frame for $\mathcal{H}^N$.*

3. Dual of K-frame

In this section, we introduce the concept of K-dual of K-frames in $\mathcal{H}^N$ and its properties are discussed. Also, the oblique K-dual is investigated.

Definition 3.1. *If $\Phi = \{\varphi_j\}_{j=1}^M$ is a K-frame for $\mathcal{H}^N$, a sequence $\Psi = \{\psi_j\}_{j=1}^M$ is called a K-dual frame for Φ if*

$$Kx = \sum_{j=1}^M \langle x, \psi_j \rangle \varphi_j, \ \ (x \in \mathcal{H}^N). \tag{2}$$

The systems $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ are referred to as a K-dual frame pair.

Proposition 3.2. *Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a tight K-frame for $\mathcal{H}^N$. Then $Tr(K) = \sum_{j=1}^M \langle \varphi_j, \psi_j \rangle$, where $\Psi = \{\psi_j\}_{j=1}^M$ is a K-dual of $\Phi = \{\varphi_j\}_{j=1}^M$.*

In the following theorem, we characterize the scalar sequences $v = \{v_j\}_{j=1}^M$ for which there exists a K-dual pair of frames $\{\varphi_j\}_{j=1}^M$ and $\{\psi_j\}_{j=1}^M$ such that $v_j = \langle \varphi_j, \psi_j \rangle$ for all $j = 1, 2, ..., M$.

Theorem 3.3. *Let $K \in B(\mathcal{H}^N)$ and $v = \{v_j\}_{j=1}^M \subset \mathbb{C}$ with $M > dim(R(K)) = rank(K)$ be given. Suppose that there exist K-dual frame pairs $\{\varphi_j\}_{j=1}^M$ and $\{\psi_j\}_{j=1}^M$ for $\mathcal{H}^N$ such that $v_j = \langle \varphi_j, \psi_j \rangle$ for all $j = 1, 2, ..., M$. Then there exists a tight K^*-frame $\{\theta_j\}_{j=1}^M$ and a corresponding dual frame $\Gamma = \{\gamma_j\}_{j=1}^M$ for $\mathcal{H}^N$ such that $v_j = \langle \theta_j, \gamma_j \rangle$ for all $j = 1, 2, ..., M$. Furthermore $Tr(K) = \sum_{j=1}^M v_j$.*

In the following result we characterize K-duals of a K-frame.

Proposition 3.4. *Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $\mathcal{H}^N$. Then $\Psi = \{\psi_j\}_{j=1}^M$ is a K-dual for Φ if and only if $R(T_\Phi) \perp R(T_\Theta)$, where T_Θ is the analysis operator of the sequence $\Theta = \{\theta_j\}_{j=1}^M = \{\psi_j - K^* S_\Phi^{-1} P_{S_\Phi(R(K))} \varphi_j\}_{j=1}^M$.*

Oblique dual frames in finite dimentional Hilbert space were studied in [5]. In the last part of this section, we study this notion for K-frames.

Definition 3.5. *Let $\mathcal{U}$ and $\mathcal{W}$ be two subspaces of $\mathcal{H}^N$ and suppose that $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ are in $\mathcal{H}^N$ and $\mathcal{W} = span\{\varphi_j : j = 1, 2, ..., M\}$, $\mathcal{U} = span\{\psi_j : j = 1, 2, ..., M\}$. The sequence $\Psi = \{\psi_j\}_{j=1}^M$ is an oblique K-dual frame of the K-frame $\Phi = \{\varphi_j\}_{j=1}^M$ on $\mathcal{W}$ if $Kx = \sum_{j=1}^M \langle x, \psi_j \rangle \varphi_j$, for all $x \in \mathcal{W}$.*

In the following two propositions a characterization of the oblique K-dual frames pair.

Proposition 3.6. *Suppose that $\mathcal{W}$ is a subspace of $\mathcal{H}^N$ and sequences $\Phi = \{\varphi_j\}_{j=1}^M$, $\Psi = \{\psi_j\}_{j=1}^L$ and $\Gamma = \{\gamma_j\}_{j=1}^L$ in $\mathcal{H}^N$ satisfy that $span(\Phi \cup \Gamma) = \mathcal{W}$. Then the following statements are equivalent:*

(i) $\Phi \cup \Psi$ is an oblique K-dual frame of $\Phi \cup \Gamma$ on $\mathcal{W}$.
(ii) For any $x \in \mathcal{W}$, $(K - S_\Phi)x = \sum_{j=1}^L \langle x, \psi_j \rangle \gamma_j$.

Proposition 3.7. *If* $\Psi = \{\psi_j\}_{j=1}^M$ *is an oblique K-dual frame of* $\Phi = \{\varphi_j\}_{j=1}^M$ *on* $\mathcal{W}$ *and* Φ *is K-minimal, then the oblique K-dual frame of* Φ *on* $\mathcal{W}$ *is unique in the sense that if* $\Gamma = \{\gamma_j\}_{j=1}^M$ *is another oblique K-dual frame of* Φ, *then* $\psi_j = \gamma_j$, $j = 1, ..., M$, *where* Ψ, Γ *are restricted in* $\mathcal{W}$.

Here, we state that if Φ is a K-frame for $R(K)$, then we can make an oblique K-dual frame of algebraic multiplicity of $\{\varphi_j\}_{j=1}^M \cup \{e_j\}_{j \neq j_0}$ where $\{e_j\}_{j=1}^d$ is an orthonormal eigenbasis of the frame operator S_Φ with associated eigenvalues $\{\lambda_j\}_{j=1}^d$.

Theorem 3.8. *Let* $K \in B(\mathcal{H}^N)$ *and* $\Phi = \{\varphi_j\}_{j=1}^M$ *be a K-frame for* $\mathcal{W} = R(K)$ *with* $dim \mathcal{W} = d$. *Also, let* $\{e_j\}_{j=1}^d$ *be an orthonormal eigenbasis of the frame operator* S_Φ *with associated eigenvalues* $\{\lambda_j\}_{j=1}^d$. *Then for any eigenvalue* $0 \neq \lambda_{j_0}$, *the sequence* $\{\frac{1}{\sqrt{\lambda_{j_0}}} K^* \varphi_j\}_{j=1}^M \cup \{\frac{(\lambda_{j_0} - \lambda_j)^{\frac{1}{3}}}{\sqrt{\lambda_{j_0}}} K^* e_j + K^* \gamma_j\}_{j: j \neq j_0}$, *is an oblique K-dual frame of* $\{\frac{1}{\sqrt{\lambda_{j_0}}} \varphi_j\}_{j=1}^M \cup \{\frac{(\lambda_{j_0} - \lambda_j)^{\frac{2}{3}}}{\sqrt{\lambda_{j_0}}} e_j\}_{j: j \neq j_0}$ *on* $\mathcal{W}$, *where* $\{\gamma_j\}_{j_0 \neq j=1}^d \subset \mathcal{H}^N$ *satisfies*

$$\sum_{j_0 \neq j=1}^d \langle x, K^* \gamma_j \rangle e_j = 0, \ (x \in \mathcal{W}).$$

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