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Some properties of central kernel and central autocommutator subgroups

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A classical result of Schur states that if the central quotient $G/Z(G)$ of a group G is finite, then the commutator subgroup G' is also finite. In this paper we introduce the notion of central autocommutator subgroup of a given group G. We study this concept and give some new results concerning the central kernel subgroup of G, which was first introduced by F. Haimo in 1955. More precisely, the analogue of Schur's result is proved. We also construct some upper bounds for the order of central kernel and central autocommutator subgroups of G in terms of the order of central kernel quotient of G.

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1. Introduction

Let G be a finite group then the *autocommutator* of the element $g \in G$ and the automorphism α in $\mathrm{Aut}(G)$ is defined to be

$$[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g).$$

Using this definition, the subgroup

$$K(G) = \langle [x, \alpha] : x \in G, \alpha \in \mathrm{Aut}(G) \rangle$$

is called the *autocommutator subgroup* of G. The concept of autocommutator subgroup has been already studied in [4]. Also

$$L(G) = \{x \in G : [x, \alpha] = 1,\ \forall \alpha \in \mathrm{Aut}(G)\},$$

is called the *autocenter* of G. Clearly if α runs over the inner automorphisms of G, then $K(G)$ and $L(G)$ will be the commutator subgroup, G', and the center, $Z(G)$, of G, respectively. One notes that, $K(G)$ and $L(G)$ are characteristic subgroups of G (see [3, 4] for more information).

In 1904, Schur [6] proved that for any group G, if $G/Z(G)$ is finite then so is G'. In 1994, Hegarty [3] showed that if $G/L(G)$ is finite, then $K(G)$ and the automorphism group $\mathrm{Aut}(G)$ are both finite.

In 1955, Haimo [2] introduced the following subgroup of a given group G, which we denote it by $L_c(G)$,

$$L_c(G) = \{x \in G : [x, \alpha] = 1, \forall \alpha \in \mathrm{Aut}_c(G)\},$$

where $\mathrm{Aut}_c(G)$ is the central automorphism group of G, and it is the set of all automorphisms α in $\mathrm{Aut}(G)$ for which $[x, \alpha] \in Z(G)$, for all $x \in G$.

We call $L_c(G)$ the *central kernel* of G and clearly it is a characteristic subgroup of G and contains $L(G)$. We may also define the *central autocommutator* subgroup of G as follows:

$$K_c(G) = \langle [x, \alpha] : x \in G, \alpha \in \mathrm{Aut}_c(G)\rangle.$$

It is easy to check that $K_c(G)$ is a central characteristic subgroup of G, which is contained in $K(G)$. Moreover, put $\langle 1\rangle = L_{c_0}(G)$ and $L_c(G) = L_{c_1}(G)$ then for $n \geq 1$, we may define inductively

$$L_{c_n}(G) = \{x \in G : [x, \alpha_1, \ldots, \alpha_n] = 1, \forall \alpha_i \in \mathrm{Aut}_c(G)\}.$$

Clearly, $L_{c_n}(G)$ is a characteristic subgroup of G and one obtains the following ascending series of G,

$$\langle 1\rangle = L_{c_0}(G) \subseteq L_c(G) = L_{c_1}(G) \subseteq L_{c_2}(G) \subseteq \cdots \subseteq L_{c_n}(G) \subseteq \cdots.$$

One should note that the above series is slightly different and more general than the one given in [2]. Note that our construction does not give [2]. So we work with the above definition throughout the rest of the paper.

One of our goals in this paper is to prove the analogue of Schur's result. We also construct upper bounds for the orders of $K_c(G)$ and $\mathrm{Aut}_c(G)$ in terms of the order of $G/L_c(G)$.

2. Preliminaries

Having the definition of $L_c(G)$ of a given group G, we may define

$$\mathrm{Aut}_{L_c}(G) = \{\alpha \in \mathrm{Aut}_c(G) : [x, \alpha] \in L_c(G),\ \forall\, x \in G\}$$

which is a normal subgroup of $\mathrm{Aut}_c(G)$.

Lemma 2.1. *For any group G, if $G/L_c(G)$ is finite, then so is* $\mathrm{Aut}_c(G)/\mathrm{Aut}_{L_c}(G)$.

Proof. Clearly every central automorphism $\alpha \in \mathrm{Aut}_c(G)$ induces an automorphism $\overline{\alpha} : G/L_c(G) \to G/L_c(G)$ given by

$$\overline{\alpha}(xL_c(G)) = \alpha(x)L_c(G).$$

Now the correspondence $\alpha \mapsto \overline{\alpha}$ is a homomorphism from $\mathrm{Aut}_c(G)$ into $\mathrm{Aut}_c(G/L_c(G))$ and the kernel of this homomorphism is $\mathrm{Aut}_{L_c}(G)$, which gives the assertion. □

Here we state the following fact of [5], which is useful in proving our next result.

Dicman's Lemma. *Let $\{x_1, \ldots, x_n\}$ be a finite normal subset of a group G, where $|x_i|$ is finite for each $1 \leq i \leq n$. Then $X = \langle x_1, \ldots, x_n \rangle$ is a finite normal subgroup of G and $|X| \leq \prod_{i=1}^{n} |x_i|$.*

Proof. See [5, 14.5.7]. □

The following lemma is very useful in our further investigations.

Lemma 2.2. *Let the factor group $G/L_c(G)$ be finite. Then the subgroup $K_c(G)$ of the group G is finite if and only if $\mathrm{Aut}_c(G)$ is finite if and only if $\mathrm{Aut}_{L_c}(G)$ is finite.*

Proof. Suppose that $G/L_c(G)$ is finite, then $\mathrm{Aut}_c(G)/\mathrm{Aut}_{L_c}(G)$ is also finite. Thus it is enough to show that $K_c(G)$ is finite if and only if $\mathrm{Aut}_{L_c}(G)$ is finite. Let $K_c(G)$ be finite and $\alpha \in \mathrm{Aut}_{L_c}(G)$, define $\alpha^* : G/L_c(G) \to L_c(G)$ given by

$$\alpha^*(xL_c(G)) = [x, \alpha],$$

for all $x \in G$. It is clear that α^* is a homomorphism. The correspondence $\alpha \mapsto \alpha^*$ is an injection from $\mathrm{Aut}_{L_c}(G)$ into the set $\mathcal{A} = \{\alpha^* \mid \alpha \in \mathrm{Aut}_{L_c}(G)\}$. If $\mathrm{Aut}_{L_c}(G)$ is infinite then the set $\mathcal{A}$ is also infinite and we have infinite number of elements of the form $[x, \alpha]$, for each $\alpha \in \mathrm{Aut}_c(G)$ and $x \in G$. This implies that $K_c(G)$ is infinite, which is a contradiction.

Now let $\mathrm{Aut}_{L_c}(G)$ be finite, then one can easily see that $\mathrm{Aut}_c(G)$ and the set $\mathcal{B} = \{[x, \alpha] : x \in G, \alpha \in \mathrm{Aut}_c(G)\}$ are both finite. Moreover, if $|G/L_c(G)| = n$, then for every element $x \in G$ we have $x^n \in L_c(G)$. Thus for all $\alpha \in \mathrm{Aut}_c(G)$ we may write

$$[x, \alpha]^n = [x^n, \alpha] = 1.$$

Hence the set $\mathcal{B}$ consists of only elements of finite order and by Dicman's Lemma, $K_c(G)$ is also finite. □

3. Main Results

The purpose of this section is to prove the following analogue of Schur's theorem. Also we derive some upper bounds for $|\mathrm{Aut}_c(G)|$ and $|K_c(G)|$.

Theorem 3.1. *If $G/L_c(G)$ is finite, then so are $K_c(G)$ and $\mathrm{Aut}_c(G)$.*

Proof. By Lemma 2.2 it is sufficient to show that if $G/L_c(G)$ is finite then $\mathrm{Aut}_{L_c}(G)$ is finite. Assume the contrary and $\mathrm{Aut}_{L_c}(G)$ is infinite, then Lemma 2.1 implies that $G/L_c(G)$ is infinite which gives a contradiction. □

In the following we construct upper bounds for $|\mathrm{Aut}_c(G)|$ and $|K_c(G)|$ in terms of $|G/L_c(G)|$.

Theorem 3.2. *Let $|G/L_c(G)| = n$ then*

(i) $|\mathrm{Aut}_c(G)| \leq n!\ d^{[\log_2 n]}$,
(ii) $|K_c(G)| \leq n^{n([\log_2 n]\times n!\ d^{[\log_2 n]})}$, *where d is the minimum number of generators of $L_c(G)$.*

Proof. (i) If $|G/L_c(G)| = n$, then $\mathrm{Aut}_c(G)/\mathrm{Aut}_{L_c}(G)$ is isomorphic to a subgroup of the symmetric group S_n. Thus

$$|\mathrm{Aut}_c(G)| \leq n!\ |\mathrm{Aut}_{L_c}(G)|.$$

Now using the argument as in the proof of Lemma 2.2, it implies that $|\mathrm{Aut}_{L_c}(G)| \leq |\mathcal{A}|$. However, each homomorphism α^* in $\mathcal{A}$ must map a generator of $G/L_c(G)$ into a generator of $L_c(G)$. Also a group of order n has less than or equal to $[\log_2 n]$ minimum number of generators (see [7, p. 48]). Therefore $|\mathrm{Aut}_c(G)| \leq n!\ d^{[\log_2 n]}$, where d is the minimum number of generators of $L_c(G)$.

(ii) Let $\{x_1, x_2, \ldots, x_r\}$ be a generating set for $K_c(G)$. Note that every generator of $K_c(G)$ has the form $[g_j, \alpha_k]$, where $g_j \in G$ and $\alpha_k \in \mathrm{Aut}_c(G)$. On the other hand, $[g_j, \alpha_k]^n = [g_j^n, \alpha_k] = 1$. Thus the order of each generator x_i dividing n for all $1 \leq i \leq r$. Since every element of $K_c(G)$ has order dividing n, Dicman's Lemma implies that $|K_c(G)| \leq n^{nr}$.

Note that for every $\alpha \in \mathrm{Aut}_c(G)$, the set of autocommutators $\{x^{-1}x^{\alpha} = [x, \alpha] : x \in G\}$ has at most n elements and $\langle [x, \alpha] : x \in G\rangle$ can be generated by at most the minimum number of generators of $G/L_c(G)$. As $d(G/L_c(G)) = [\log_2 n]$, we have

$$r \leq [\log_2 n] \times |\mathrm{Aut}_c(G)|.$$

Hence, the inequality (i) implies that $r \leq [\log_2 n] \times n!\, d^{[\log_2 n]}$, where d is the minimum number of generators of $L_c(G)$. Thus

$$|K_c(G)| \leq n^{n([\log_2 n]\times n!\, d^{[\log_2 n]})}.$$ □

Now the question arises that whether the converse of Theorem 3.1 is also true?

In the following we give a complete answer to this question.

Theorem 3.3. *If $K_c(G)$ and $\mathrm{Aut}_c(G)$ are both finite, then so is $G/L_c(G)$.*

Proof. For any automorphism $\alpha \in \mathrm{Aut}_c(G)$, let $C_G(\alpha) = \{g \in G : [g, \alpha] = 1\}$ be the centralizer of α in the group G. Assume $K_c(G)$ and $\mathrm{Aut}_c(G)$ are both finite, then the index $[G : C_G(\alpha)]$ is finite for each $\alpha \in \mathrm{Aut}_c(G)$, as $K_c(G)$ is finite. On the other hand, $L_c(G) = \bigcap C_G(\alpha)$ for every $\alpha \in \mathrm{Aut}_c(G)$, which is a finite intersection. Hence $G/L_c(G)$ is finite. □

Remark 3.4. Fournelle in [1] showed that for any prime number p, there is an infinite non-abelian group G such that $\mathrm{Aut}(G)$ is an uncountable elementary abelian p-group. Thus there exists a group G for which $K_c(G)$ is finite, while $G/L_c(G)$ is infinite (see [1, Examples 1–3]). So we cannot remove the finiteness hypothesis of $\mathrm{Aut}_c(G)$ in the above theorem.

Theorem 3.5. *Let G be any group with $|K_c(G)| = n$ and $|\mathrm{Aut}_c(G)| = m$. Then*

$$\left|\frac{G}{L_c(G)}\right| \leq n^m.$$

Proof. For all $g \in G$ and $\alpha \in \mathrm{Aut}_c(G)$

$$g^\alpha = g[g, \alpha] \in gK_c(G).$$

Thus for each centralizer of α in G, the size of the factor group $G/C_G(\alpha)$ has cardinality no larger than $|K_c(G)|$. It follows that

$$\left|\frac{G}{C_G(\alpha)}\right| \leq |K_c(G)|.$$

On the other hand, $L_c(G) = \bigcap_{\alpha \in \mathrm{Aut}_c(G)} C_G(\alpha)$ and hence Poincarés lemma implies that

$$\begin{aligned}\left|\frac{G}{L_c(G)}\right| &\leq \left|\frac{G}{C_G(\alpha_1)}\right| \times \cdots \times \left|\frac{G}{C_G(\alpha_m)}\right| \\ &\leq |K_c(G)| \times \cdots \times |K_c(G)| = |K_c(G)|^m = n^m.\end{aligned}$$ □

One can easily check that the central automorphisms of a group G fix the commutator subgroup G' elementwise, and hence $G' \subseteq L_c(G)$. This shows that $G/L_c(G)$ is abelian and if $L_c(G)$ is trivial, then G is abelian.

Lemma 3.6. *$G = L_{c_2}(G)$ if and only if $G/L_c(G) \cong \mathbb{Z}_2$.*

Proof. Let $G = L_{c_2}(G)$ and $x \in G \setminus L_c(G)$, then $[x, \alpha, \beta] = 1$ for all $\alpha, \beta \in \mathrm{Aut}_c(G)$. We remind that β fixes $[x, \alpha]$ and on the other hand every abelian group has a nontrivial automorphism which sends every element to its inverse.

Also $[xL_c(G), \bar{\alpha}, \bar{\beta}] = L_c(G)$, for all $\bar{\alpha}, \bar{\beta} \in \mathrm{Aut}_c(G/L_c(G))$. Thus, $\bar{\beta}([xL_c(G), \bar{\alpha}]) = [xL_c(G), \bar{\alpha}]^{-1} = [xL_c(G), \bar{\alpha}]$, which implies that $\bar{\alpha}(x^2 L_c(G)) = x^2 L_c(G)$, and so all central automorphisms of $G/L_c(G)$ are trivial. As $G/L_c(G)$ is abelian, it has a central automorphism θ, such that $\theta(xL_c(G)) = x^{-1}L_c(G) = xL_c(G)$ and so $x^2 \in L_c(G)$. Hence $G/L_c(G)$ is an elementary abelian 2-group with $\exp(G/L_c(G)) = 2$. If $G/L_c(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there is a nontrivial central automorphism and hence $G/L_c(G) \cong \mathbb{Z}_2$. The proof of the converse is obvious. □

Theorem 3.7. *Let $G = L_{c_n}(G)$ and $x \in G$, then $x^{2^{n-1}} \in L_c(G)$, $n \geq 2$.*

Proof. If $n = 2$, Lemma 3.6 implies that $x^2 \in L_c(G)$. Proceed by induction on n and assume the result holds for $n = k$. Now let $x \in L_{c_{k+1}}(G)$, then $[x, \alpha] \in L_{c_k}(G)$, for every $\alpha \in \mathrm{Aut}_c(G)$. Then the induction hypothesis implies that $[x, \alpha]^{2^{k-1}} \in L_c(G)$. Thus for each $\beta \in \mathrm{Aut}_c(G)$,

$$[[x, \alpha]^{2^{k-1}}, \beta] = 1 \Rightarrow [x^{2^{k-1}}, \alpha, \beta] = 1.$$

Hence $x^{2^{k-1}} \in L_{c_2}(G)$ and so $x^{2^k} \in L_c(G)$. □

Corollary 3.8. *Let $G = L_{c_n}(G)$, then $G/L_{c_{n-1}}(G) \cong \mathbb{Z}_2$ and $\exp(L_{c_n}(G)/L_c(G))$ divides 2^{n-1}.*

Theorem 3.9. *Let $G = L_{c_2}(G)$, then $\mathrm{Aut}_c(G)$ is an elementary abelian 2-group and $\mathrm{Aut}_c(G) = \mathrm{Aut}_{L_c}(G)$.*

Proof. Let $G = L_{c_2}(G)$, then the definition of $\mathrm{Aut}_{L_c}(G)$ implies that $\mathrm{Aut}_c(G) = \mathrm{Aut}_{L_c}(G)$. Also $G = L_c(G) \cup lL_c(G)$ and for every $x \in G \setminus L_c(G)$ there exists $l_x \in L_{c_2}(G)$ such that $x = ll_x$. For every $\alpha \in \mathrm{Aut}_c(G)$,

$$\alpha(x) = \alpha(ll_x) = \alpha(l)\alpha(l_x) = \alpha(l)l^{-1}x = [\alpha, l]x.$$

On the other hand, $l^2 \in L_c(G)$ and $[\alpha, l] \in Z(G) \cap L_c(G)$. Thus

$$\alpha^2(x) = \alpha(\alpha(l)l_x) = \alpha^2(l)l_x = [\alpha^2, l]x = [\alpha, l^2]x = x.$$

Moreover, for each $\alpha, \beta \in \mathrm{Aut}_c(G)$ and $x \in G$,

$$[x, \alpha\beta] = [x, \beta][x, \alpha] = [x, \alpha][x, \beta] = [x, \beta\alpha] \Rightarrow \alpha\beta(x) = \beta\alpha(x).$$

Hence $\mathrm{Aut}_c(G)$ is an elementary abelian 2-group. □

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