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NON-ABELIAN TENSOR ANALOGUES OF 2-AUTO ENGEL GROUPS

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Abstract. The concept of tensor analogues of right 2-Engel elements in groups were defined and studied by Biddle and Kappe [1] and Moravec [9]. Using the automorphisms of a given group G, we introduce the notion of tensor analogue of 2-auto Engel elements in G and investigate their properties. Also the concept of $2_{\otimes}$-auto Engel groups is introduced and we prove that if G is a $2_{\otimes}$-auto Engel group, then $G \otimes \mathrm{Aut}(G)$ is abelian.

Finally, we construct a non-abelian 2-auto-Engel group G so that its non-abelian tensor product by $\mathrm{Aut}(G)$ is abelian.

1. Introduction

Let G and H be groups equipped with the actions of G on H and H on G (both from the right), written as h^g and g^h for all $g \in G$ and $h \in H$, respectively. It is always understood that a group acts on itself by conjugation. As in [2, 3], all these actions must be *compatible* in the sense that

$$g'^{(h^g)} = ((g'^{g^{-1}})^h)^g, \qquad h'^{(g^h)} = ((h'^{h^{-1}})^g)^h,$$

for all $g, g' \in G$ and $h, h' \in H$.

Considering the above compatibilities of groups actions, the *non-abelian tensor product* $G \otimes H$ is the group generated by the symbols $g \otimes h$, satisfying the following relations:

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h),$$
$$g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'}),$$

for all $g, g' \in G$ and $h, h' \in H$. The concept of non-abelian tensor product of groups was introduced by Brown and Loday in [3]. Brown, Johnson and Robertson in [2] started the investigation of non-abelian tensor product as a group theoretical object. As a special case, $G \otimes G$ is said to be the tensor square of a group G.

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Recall that the set of all right n-Engel elements of a group G is defined by $R_n(G) = \{g \in G \mid [g,{}_n h] = 1, \forall h \in G\}$. Here $[g,h] = g^{-1}h^{-1}gh$, $[g_1,\dots,g_n,g_{n+1}] = [[g_1,\dots,g_n],g_{n+1}]$ and $[g,{}_n h] = [g,\underbrace{h,\dots,h}_{n}]$. It is shown in [6] that $R_2(G)$ is a subgroup of G.

Using the non-abelian tensor square, the set of all right $n_\otimes$-Engel elements of a group G is defined as follows:

$$R_n^{\otimes}(G) = \{g \in G \mid [g,{}_{n-1}h] \otimes h = 1_\otimes,\ \forall h \in G\},$$

see also [1, 9]. The set of right $n_\otimes$-Engel elements has been studied by several authors (see [1, 9, 11, 12]). Biddle and Kappe [1] proved that $R_2^{\otimes}(G)$ is always a characteristic subgroup of G contained in $R_2(G)$. In [9], Moravec determined some further information on $R_2^{\otimes}(G)$ and defined the concept of $2_\otimes$-Engel groups, which reads as G is a $2_\otimes$-Engel group when $[g,h]\otimes h = 1_\otimes$ for all $g,h \in G$. He also showed that if G is a $2_\otimes$-Engel group, then the non-abelian tensor square $G \otimes G$ is abelian.

The automorphisms group of a given group G is denoted by $\mathrm{Aut}(G)$ and inner automorphisms by $\mathrm{Inn}(G)$. In the present article, using the non-abelian tensor product $G \otimes \mathrm{Aut}(G)$, we introduce the concept of tensor analogue of 2-auto Engel elements in a group G and denote it by $AR_2^{\otimes}(G)$.

Definition. Let G be any group. Then

$$AR_2^{\otimes}(G) = \{g \in G \mid [g,\alpha] \otimes \alpha = 1_\otimes,\ \ \forall \alpha \in \mathrm{Aut}(G)\}$$

is the set of $2_\otimes$-auto Engel elements of G.

In Section 3, we show that $AR_2^{\otimes}(G)$ is a characteristic subgroup of G contained in $R_2^{\otimes}(G)$ and give some of its properties. Also we introduce the notion of $2_\otimes$-auto Engel groups and prove that if G is a $2_\otimes$-auto Engel group, then $G \otimes \mathrm{Aut}(G)$ is abelian.

2. Preliminary results

In this section we summarize some of the basic facts, which are needed for the proofs of our main results.

The following propositions of [2] are needed:

Proposition 2.1 ([2, Proposition 2]). *Let G and H be groups equipped with compatible actions on each other. Then*

(i) *the groups G and H act on $G \otimes H$ so that*

$$(g' \otimes h)^g = g'^g \otimes h^g, \qquad (g \otimes h')^h = g^h \otimes h'^h,$$

for all $g, g' \in G$, $h, h' \in H$.

(ii) *There are group homomorphisms $\lambda : G \otimes H \to G$ and $\lambda' : G \otimes H \to H$ such that*

$$(g \otimes h)\lambda = g^{-1}g^h, \qquad (g \otimes h)\lambda' = h^{-1^g}h,$$

for all $g \in G$ and $h \in H$.

Proposition 2.2 ([2, Proposition 3])**.** *Let G and H be groups equipped with compatible actions on each other. Then for all $g, g' \in G$ and $h, h' \in H$, the following identities are satisfied:*

(i) $(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$;
(ii) $(g' \otimes h')^{g \otimes h} = (g' \otimes h')^{[g,h]}$;
(iii) $g^{-1}g^h \otimes h' = (g \otimes h)^{-1}(g \otimes h)^{h'}$;
(iv) $g' \otimes h^{-1 g}h = (g \otimes h)^{-g'}(g \otimes h)$;
(v) $g^{-1}g^h \otimes h'^{-1 g'}h' = [g \otimes h, g' \otimes h']$.

For a given group G, we define the action of G on $\mathrm{Aut}(G)$ given by $\alpha^g = \alpha^{\varphi_g} = \varphi_g^{-1} \circ \alpha \circ \varphi_g$ and the action of $\mathrm{Aut}(G)$ on G given by $g^\alpha = (g)\alpha$ for all $g \in G$, $\alpha \in \mathrm{Aut}(G)$ and $\varphi_g \in \mathrm{Inn}(G)$.

In the following lemma, we show that the above actions are compatible.

Lemma 2.3. *The above actions are well-defined and compatible.*

Proof. Clearly the actions of G on $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ on G are well-defined. Also for all $\alpha, \beta \in \mathrm{Aut}(G)$, $g, h \in G$ and $\varphi_g \in \mathrm{Inn}(G)$, we have

$$\begin{aligned}(h)\beta^{(g^\alpha)} = (h)\beta^{\varphi_{(g)\alpha}} &= (h)(\varphi_{(g)\alpha}^{-1} \circ \beta \circ \varphi_{(g)\alpha}) \\ &= (g^{-1})\alpha((g)\alpha h(g^{-1})\alpha)\beta(g)\alpha \\ &= (g^{-1}(g)\alpha\beta\alpha^{-1}(h)\beta\alpha^{-1}(g^{-1})\alpha\beta\alpha^{-1}g)\alpha \\ &= ((g)\alpha\beta(h)\beta(g^{-1})\alpha\beta)\alpha^{-1} \circ \varphi_g \circ \alpha \\ &= (h)(\varphi_g^{-1} \circ (\alpha \circ \beta \circ \alpha^{-1}) \circ \varphi_g)^\alpha \\ &= (h)((\beta^{\alpha^{-1}})^g)^\alpha.\end{aligned}$$

Thus $\beta^{(g^\alpha)} = ((\beta^{\alpha^{-1}})^g)^\alpha$. Also,

$$\begin{aligned}h^{(\alpha^g)} = h^{(\alpha^{\varphi_g})} &= (h)(\varphi_g^{-1} \circ \alpha \circ \varphi_g) \\ &= g^{-1}(g)\alpha(h)\alpha(g^{-1})\alpha g \\ &= ((ghg^{-1})^\alpha)^g \\ &= ((h^{g^{-1}})^\alpha)^g.\end{aligned}$$

This completes the proof. □

Remark 1. One notes that we may use all the identities for the non-abelian tensor product $G \otimes \mathrm{Aut}(G)$.

Lemma 2.4 ([7])**.** *Let g and h be elements of a group G and $\alpha, \beta \in \mathrm{Aut}(G)$. Then the followings are held:*

(i) $[gh, \alpha] = [g, \alpha]^h[h, \alpha]$;
(ii) $[g, \alpha^{-1}] = ([g, \alpha]^{-1})^{\alpha^{-1}}$;

(iii) $[g^{-1},\alpha] = ([g,\alpha]^{-1})^{g^{-1}}$;
(iv) $[g,\alpha\beta] = [g,\beta][g,\alpha]^{\beta}$;
(v) $[g,\alpha]^{\beta} = [g^{\beta},\alpha^{\beta}]$.

Following [7], the set of all right 2-auto Engel elements in a given group G is defined as follows:

$$AR_2(G) = \{g \in G \mid [g,_2 \alpha] = [g,\alpha,\alpha] = 1,\ \forall \alpha \in \mathrm{Aut}(G)\}.$$

Proposition 2.5 ([7, Lemma 3.2])**.** *Let g be a right 2-auto Engel element and α, β and γ be arbitrary automorphisms of a group G. Then*

(i) $g^{\mathrm{Aut}(G)} = \langle g^{\alpha} \mid \alpha \in \mathrm{Aut}(G)\rangle$ *is abelian and its elements are right 2-auto Engel elements*;
(ii) $[g,\alpha,\beta] = [g,\beta,\alpha]^{-1}$;
(iii) $[g,[\alpha,\beta]] = [g,\alpha,\beta]^2$;
(iv) $[g,\alpha,\beta,\gamma]^2 = 1$;
(v) $[g,[\alpha,\beta],\gamma] = 1$.

The following corollary is an immediate consequence of Proposition 2.5(i).

Corollary 2.6. *Let G be a group, $g \in AR_2(G)$ and $\alpha,\beta \in \mathrm{Aut}(G)$. Then $[[g,\alpha],[g,\beta]] = 1$.*

Theorem 2.7 ([7, Theorem 3.2])**.** *The set of all right 2-auto Engel elements of a given group forms a characteristic subgroup.*

3. Main results

The main goal of this section is to study the tensor analogues of right 2-auto Engel elements of a given group G. So we define the set of all $2_{\otimes}$-auto Engel elements as follows:

$$AR_2^{\otimes}(G) = \{g \in G \mid [g,\alpha] \otimes \alpha = 1_{\otimes},\ \ \forall \alpha \in \mathrm{Aut}(G)\}.$$

We remind that $[g,\alpha] = g^{-1}(g)\alpha = g^{-1}g^{\alpha}$ is the *autocommutator* elements of $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, see also [5]. The subgroups $K(G) = \langle [g,\alpha] \mid g \in G,\ \alpha \in \mathrm{Aut}(G)\rangle$ and $L(G) = \{g \in G \mid [g,\alpha] = 1,\ \forall \alpha \in \mathrm{Aut}(G)\}$ are the *autocommutator subgroup* and the *absolute centre* of G, respectively. (see [5, 8, 10], for more information.)

The following properties of $AR_2^{\otimes}(G)$ are useful for our further investigations.

Lemma 3.1. *The set of all $2_{\otimes}$-auto Engel elements of a given group G contains $L(G)$ and is contained in the set of 2-auto Engel elements of G.*

Proof. It is obvious that $L(G)$ is a subset of $AR_2^{\otimes}(G)$. Now, by Proposition 2.1(ii), there exists a homomorphism $\theta : G \otimes \mathrm{Aut}(G) \to K(G)$ given by $(g \otimes \alpha)\theta = [g,\alpha]$, which gives the claim. □

Lemma 3.2. *Let G be a group. Then*

(i) $(g \otimes \alpha)^{-1} = g \otimes \alpha^{-1}$;

(ii) $([g,\beta]\otimes\alpha)([g,\alpha]\otimes\beta)=1_{\otimes}$;
(iii) $[g,\alpha]\otimes\varphi_g=1_{\otimes}$,

for all $g\in AR_2^{\otimes}(G)$ *and* $\alpha,\beta\in \mathrm{Aut}(G)$.

Proof. (i) Proposition 2.2(iii) implies that $1_{\otimes}=[g,\alpha]\otimes\alpha=(g\otimes\alpha)^{-1}(g\otimes\alpha)^{\alpha}$. Thus $(g\otimes\alpha)^{-1}=g\otimes\alpha^{-1}$, by Proposition 2.2(i).

(ii) Using part (i), we have $g\otimes\beta^{-1}\alpha=(g\otimes(\beta^{-1}\alpha)^{-1})^{-1}$. By a simple calculation we obtain

$$((g\otimes\beta)^{-1}(g\otimes\beta^{-1})^{-\alpha})^{-1}=(g\otimes\alpha)^{-1}(g\otimes\alpha^{-1})^{-\beta}.$$

Hence, by Proposition 2.2(iii), $([g,\beta]\otimes\alpha)^{-1}=([g,\alpha]\otimes\beta)$.

(iii) Clearly $1_{\otimes}=[g,\varphi_g\alpha]\otimes\varphi_g\alpha$, which gives $[g,\alpha]\otimes\varphi_g=1_{\otimes}$. □

Lemma 3.3. *Let* $g\in AR_2^{\otimes}(G)$ *and* $\alpha,\beta\in\mathrm{Aut}(G)$. *Then* $[g,\alpha]^{\beta}\otimes\alpha=1_{\otimes}$.

Proof. Using Lemma 2.4(iv), we have

$$1_{\otimes}=([g,\beta]\otimes\alpha\beta)^{[g,\alpha]^{\beta}}([g,\alpha]^{\beta}\otimes\alpha\beta),$$

or

$$1_{\otimes}=([g,\beta]\otimes\alpha)^{[g,\alpha]\beta}([g,\alpha]^{\beta}\otimes\beta)([g,\alpha]^{\beta}\otimes\alpha)^{\beta}.$$

As $g\in AR_2^{\otimes}(G)$, we observe that $[g,\alpha]$ acts trivially on α. Also using Lemma 3.1 and Corollary 2.6, one notes that $[g,\alpha]$ acts trivially on $[g,\beta]$. Hence we obtain

$$1_{\otimes}=([g,\beta]\otimes\alpha)^{\beta}([g,\alpha]^{\beta}\otimes\beta)([g,\alpha]^{\beta}\otimes\alpha)^{\beta}.$$

Finally, by Lemma 3.2(ii), the above equation reduces to $1_{\otimes}=[g,\alpha]^{\beta}\otimes\alpha$, which proves the claim. □

Biddle and Kappe in [1] proved that $R_2^{\otimes}(G)$ is a characteristic subgroup of G. Now we are in a position to show that $AR_2^{\otimes}(G)$ is also a characteristic subgroup of the group G.

Theorem 3.4. *For a given group* G*, the set of all* $2_{\otimes}$*-auto Engel elements is a characteristic subgroup of* G.

Proof. Clearly $AR_2^{\otimes}(G)$ is a characteristic set. We show that $AR_2^{\otimes}(G)$ is closed under the inverse and product. Assume g is any element of $AR_2^{\otimes}(G)$, which is also in $AR_2(G)$. Hence $1=[g,\varphi_g\alpha,\varphi_g\alpha]$, for all $\alpha\in\mathrm{Aut}(G)$ and $\varphi_g\in\mathrm{Inn}(G)$. This implies that $[g,\alpha,\varphi_g]=1$, using Lemma 2.4(iv). Now, by Lemma 2.4(iii) and using the previous identity, we obtain $[g^{-1},\alpha]=[g,\alpha]^{-1}$. So by Proposition 2.2(i), we have

$$[g^{-1},\alpha]\otimes\alpha=[g,\alpha]^{-1}\otimes\alpha=([g,\alpha]\otimes\alpha)^{-[g,\alpha]^{-1}}=1_{\otimes},$$

which implies that $g^{-1}\in AR_2^{\otimes}(G)$. By Lemma 2.4(i) and using the rules of non-abelian tensor product, we obtain

$$[gh,\alpha]\otimes\alpha=([g,\alpha]^{\varphi_h}\otimes\alpha)^{[h,\alpha]},$$

for all $g,h\in AR_2^{\otimes}(G)$ and $\alpha\in\mathrm{Aut}(G)$. Hence, Lemma 3.3 gives the proof. □

If A is a subset of $\mathrm{Aut}(G)$, we may define the *auto-tensor centralizer* of A in G as follows:

$$C_G^{\otimes}(A) = \{g \in G : g \otimes \alpha = 1_{\otimes}, \ \ \forall \alpha \in A\}.$$

It is easy to check that $C_G^{\otimes}(A)$ is a subgroup of G. The following proposition gives some useful properties of $AR_2^{\otimes}(G)$, which are needed in proving Theorem 3.8.

Proposition 3.5. *Let G be a group. Then for all $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$, $g \in AR_2^{\otimes}(G)$ and $n \in \mathbb{Z}$,*

(i) $[g, \alpha] \in C_G^{\otimes}(\alpha^{\mathrm{Aut}(G)})$;
(ii) $g^{-1} \otimes \alpha = (g \otimes \alpha)^{-1}$;
(iii) $[g, \alpha]^n \otimes \beta = ([g, \alpha] \otimes \beta)^n$;
(iv) $g \otimes \alpha^n = (g \otimes \alpha)^n$;
(v) $[g, \alpha] \otimes [\beta, \gamma] = 1_{\otimes}$;
(vi) $g \otimes [\alpha, \beta] = ([g, \alpha] \otimes \beta)^2$.

Proof. (i) It is obvious, by using Lemma 3.3.

(ii) By Proposition 2.2(i, iii) and Lemma 3.2(i, iii), we have $g^{-1} \otimes \alpha = (g \otimes \alpha^{-1})^{\varphi_g^{-1}}$ and hence we conclude that $g^{-1} \otimes \alpha = (g \otimes \alpha)^{-1}$.

(iii) It suffices to assume that $n > 0$. By Lemma 3.2(ii) we observe that $[g, \alpha]^n \otimes \beta = ([g, \alpha] \otimes \beta)([g, \alpha]^{n-1} \otimes \beta)$. Hence the claim follows by induction on n.

(iv) Using Proposition 2.2(iii), we have

$$g \otimes \alpha\beta = (g \otimes \beta)(g \otimes \alpha)([g, \alpha] \otimes \beta).$$

Now, $g \otimes \alpha^n = (g \otimes \alpha)^n$ is obtained by induction on n.

(v) Note that Lemma 3.2(ii) implies that the elements of the form $a \otimes \alpha$ commute, for all $a \in g^{\mathrm{Aut}(G)}$ and $\alpha \in \mathrm{Aut}(G)$. By considering the identity $[g, \alpha] \otimes \beta\gamma = ([g, \beta\gamma] \otimes \alpha)^{-1}$, we obtain

$$([g, \alpha] \otimes \gamma)([g, \alpha] \otimes \beta)^{\gamma} = ([g, \gamma] \otimes \alpha)^{-[g,\beta]^{\gamma}}([g, \beta] \otimes [\gamma^{-1}, \alpha^{-1}]\alpha)^{-\gamma}.$$

Finally, Proposition 2.2(iii, v) imply that $[g, \alpha] \otimes [\beta, \gamma] = 1_{\otimes}$.

(vi) Using parts (i) and (ii), Proposition 2.2(iii), Lemma 3.2(i) and the identity $g \otimes \alpha\beta = (g \otimes \beta)(g \otimes \alpha)([g, \alpha] \otimes \beta)$ give the claim as follows:

$$\begin{aligned} g \otimes [\alpha, \beta] &= (g \otimes \alpha\beta)(g \otimes \alpha^{-1}\beta^{-1}) \\ &= ([g, \alpha] \otimes \beta)([g, \alpha^{-1}] \otimes \beta^{-1}) \\ &= ([g, \alpha] \otimes \beta)^2. \end{aligned}$$ □

An immediate consequence of Proposition 3.5 is the following characterization of $AR_2^{\otimes}(G)$.

Corollary 3.6. *For any group G,*

$$AR_2^{\otimes}(G) = \{g \in G : [g, \alpha] \in C_G^{\otimes}(\alpha^{\mathrm{Aut}(G)}), \ \ \textit{for all } \alpha \in \mathrm{Aut}(G)\}.$$

It is known that g^G is abelian, for each element $g \in R_2(G)$. Also Moravec in [9], proves that if $g \in R_2^{\otimes}(G)$, then the normal closure $(g \otimes h)^{G\otimes G}$ is an abelian group for all $h \in G$.

The following corollary gives a similar result for $2_{\otimes}$-auto Engel elements.

Corollary 3.7. *Let $g \in AR_2^{\otimes}(G)$. Then the normal closure $(g \otimes \alpha)^{G\otimes \mathrm{Aut}(G)}$ is an abelian group for every $\alpha \in \mathrm{Aut}(G)$.*

Proof. Let $g \in AR_2^{\otimes}(G)$. We know that there exists a homomorphism $\varphi : G \otimes \mathrm{Aut}(G) \to K(G)$ given by $(g' \otimes \alpha') \mapsto [g', \alpha']$. Hence by Propositions 2.2(ii, v) and 3.5(v),

$$\begin{aligned}[(g \otimes \alpha), (g \otimes \alpha)^{(g'\otimes\alpha')}] &= [(g \otimes \alpha), (g \otimes \alpha)^{\varphi(g'\otimes\alpha')}] \\ &= [(g \otimes \alpha), (g^{[g',\alpha']} \otimes \alpha^{[g',\alpha']})] = 1_{\otimes}.\end{aligned}$$

This proves the result. □

Moravec in [9] shows that if G is a $2_{\otimes}$-Engel group, then the non-abelian tensor product $G \otimes G$ is abelian and $C_G^{\otimes}(g)$ is a normal subgroup of G, for each $g \in G$. We say a group G is $2_{\otimes}$-auto Engel group if $[g, \alpha] \otimes \alpha = 1_{\otimes}$, for all $g \in G$ and $\alpha \in \mathrm{Aut}(G)$.

Using [7], we conclude that the cyclic groups of orders 2 and 4 are the only non-trivial abelian 2-auto-Engel groups. Hence one can easily calculate that they are the only abelian $2_{\otimes}$-auto Engel groups.

Now we are in a position to prove the following:

Theorem 3.8. *Let G be a $2_{\otimes}$-auto Engel group. Then*

(i) *$G \otimes \mathrm{Aut}(G)$ is abelian;*
(ii) *$C_G^{\otimes}(\alpha)$ is a characteristic subgroup of G.*

Proof. (i) Using Propositions 2.2(v) and 3.5(v), we have

$$[g \otimes \alpha, h \otimes \beta] = [g, \alpha] \otimes [\varphi_h, \beta] = 1_{\otimes},$$

for all $g, h \in G$ and $\alpha, \beta \in \mathrm{Aut}(G)$.

(ii) It is clear that $C_G^{\otimes}(\alpha)$ is a subgroup of G. Then by Lemma 3.2(ii), for each $g \in C_G^{\otimes}(\alpha)$ and $\alpha, \beta \in \mathrm{Aut}(G)$,

$$\begin{aligned}g^{\beta} \otimes \alpha = g[g, \beta] \otimes \alpha &= (g \otimes \alpha)^{[g,\beta]}([g, \beta] \otimes \alpha) \\ &= ([g, \alpha] \otimes \beta)^{-1} \\ &= 1_{\otimes}.\end{aligned}$$

Hence $C_G^{\otimes}(\alpha)$ is a characteristic subgroup of G. □

In the following example we give a non-abelian 2-auto-Engel group G, for which $G \otimes \mathrm{Aut}(G)$ is abelian.

Example. Let $G = (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$ be the semidirect products of cyclic group of order 4, which is the group $(64, 68)$ in the GAP small groups library [4]. Using GAP, one can check that the centre and the group of automorphisms of G are both elementary abelian 2-groups. Hence, by Proposition 3.1 in [7], the group G is a purely non-abelian 2-auto-Engel group. Now by Corollary 3.7 in [7], $K(G) \leq Z(G)$ and $\mathrm{Aut}(G) = \mathrm{Aut_c}(G)$, where $\mathrm{Aut_c}(G)$ is the group of all central automorphisms of G. These automorphisms fix the central factor group of G element-wise, equivalently they commute with inner automorphisms of G. Hence, by Propositions 2.1(i) and 2.2(ii),

$$\begin{aligned}(g \otimes \alpha)^{-1}(g' \otimes \alpha')(g \otimes \alpha) &= (g' \otimes \alpha')^{[g,\alpha]} \\ &= g'^{[g,\alpha]} \otimes \alpha'^{[g,\alpha]} \\ &= g' \otimes \alpha',\end{aligned}$$

for all $g, g' \in G$ and $\alpha, \alpha' \in \mathrm{Aut}(G)$. Thus the non-abelian tensor product $G \otimes \mathrm{Aut}(G)$ is abelian.

Finally, using GAP one can check that $G = (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$ and $H = (\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ are the smallest non-abelian 2-auto-Engel groups of order 64. For more descriptions of these groups see [7]. We remark that by Lemma 3.1, $AR_2^{\otimes}(G)$ is contained in $AR_2(G)$ and using Theorem 3.8 if the group G is a $2_{\otimes}$-auto Engel, then $G \otimes \mathrm{Aut}(G)$ is abelian. Hence we strongly believe that G and H are both $2_{\otimes}$-auto Engel groups.

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