Asian-European Journal of Mathematics
Vol. 9, No. 2 (2016) 1650056 (7 pages)

DOI: 10.1142/S179355711650056X

Autonilpotent groups and their properties

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Communicated by K. P. Shum
Received August 12, 2015
Revised December 30, 2015
Published February 25, 2016

In 2010, the second author and coworker introduced and studied the concept of autonilpotent groups. In this paper, we investigate this concept from different point of view, and prove some new results. In fact, in this new notion some of our results do not hold in the nilpotent case.

Keywords: Absolute center; autocommutator subgroup; autosoluble; autonilpotent; A-nilpotent group.

AMS Subject Classification: 20D45, 20E36, 20E34, 20K10

1. Introduction and Preliminaries

For each element x of a given group G, and an automorphism α of $\mathrm{Aut}(G)$,

$$[x, \alpha] = x^{-1}x^{\alpha} = x^{-1}\alpha(x),$$

is called the *autocommutator* of x and α. For all $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(G)$, one may define the autocommutator $[x, \alpha_1, \alpha_2, \ldots, \alpha_n]$ inductively as follows:

$$[x, \alpha_1, \ldots, \alpha_n] = [[x, \alpha_1, \ldots, \alpha_{n-1}], \alpha_n], \quad n \geq 1.$$

Let G be any group, then

$$L(G) = \{x \in G : [x, \alpha] = 1, \forall\, \alpha \in \mathrm{Aut}(G)\}$$

and

$$K(G) = [G, \mathrm{Aut}(G)] = \langle [x, \alpha] : x \in G, \alpha \in \mathrm{Aut}(G) \rangle,$$

are called the *absolute center* and the *autocommutator subgroup* of G, respectively. The concepts of absolute center and autocommutator subgroup of a group ascend to the work of Baer [2]. Clearly, they are both characteristic subgroups and if the automorphism α runs over the inner automorphisms, then one gets the center, $Z(G)$, and the commutator subgroup, G', respectively. In 1994, Hegarty [6] proved if $G/L(G)$ is finite, then so is $K(G)$.

Put $G = K_0(G)$ and $K(G) = K_1(G)$, then for $n \geq 1$, we may define:

$$K_n(G) = [K_{n-1}(G), \mathrm{Aut}(G)] = \langle [x, \alpha_1, \ldots, \alpha_n] : \ x \in G, \alpha_i \in \mathrm{Aut}(G) \rangle,$$

which is called the *nth-autocommutator subgroup* of G (see also [12] for more information).

One can easily see that $\gamma_{n+1}(G) \leq K_n(G)$, $n \geq 1$ and $K_n(G)$ is the characteristic subgroup of G. Also, $K_n(G) = \gamma_{n+1}(G)$, when all the automorphisms α_i's run over the inner automorphisms of G. Hence we obtain the following descending series of G.

$$G = K_0(G) \supseteq K(G) = K_1(G) \supseteq K_2(G) \supseteq \cdots \supseteq K_n(G) \supseteq \cdots,$$

which we may call it the *lower autocentral series* of G. We also define

$$K^{(2)}(G) = K(K(G)) = [K(G), \mathrm{Aut}(K(G))]$$

and inductively,

$$K^{(n)}(G) = K(K^{(n-1)}(G)), \quad n \geq 2,$$

which is called the *nth-autoderived subgroup* of G (see also [10]). Clearly, if we consider the inner automorphisms of G, we obtain the nth-derived subgroup, $G^{(n)}$ of G and hence $G^{(n)}$ is contained in $K^{(n)}(G)$. It can be verified that for any natural number n,

$$G^{(n)} \leq \gamma_{n+1}(G) \leq K_n(G) \leq K^{(n)}(G). \tag{1.1}$$

Moreover, put $\langle 1 \rangle = L_0(G)$ and $L(G) = L_1(G)$ then for $n \geq 1$, we define inductively

$$L_n(G) = \{x \in G : \ [x, \alpha_1, \ldots, \alpha_n] = 1, \forall\, \alpha_i \in \mathrm{Aut}(G)\}.$$

Clearly, $L_n(G)$ is a characteristic subgroup and one obtains the following ascending series of G,

$$\langle 1 \rangle = L_0(G) \subseteq L(G) = L_1(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots.$$

By the above discussion, in 2013 (see [9]) we first introduced the notion of autonilpotent group, as follows.

Definition 1.1. A group G is said to be *autonilpotent* (henceforth we denoted by A-nilpotent) of class at most n if $L_n(G) = G$, for some natural number n.

One can easily see that $L_n(G) \leq Z_n(G)$ and so every A-nilpotent group is nilpotent. Moreover, $L_n(G) = G$ if and only if $K_n(G) = \langle 1 \rangle$. On the other hand, in 2010 (see [12]) Parvaneh and Moghaddam defined that if $K^{(n)}(G) = \langle 1 \rangle$ for some natural number n, then the group G is called *autosoluble*. According to (1.1), it is clear that the autosolubility of groups implies A-nilpotency, solubility and nilpotency, while the converses are not valid, in general. For example, consider the cyclic group $\mathbb{Z}_p$ of odd prime order p then $K(\mathbb{Z}_p) = \mathbb{Z}_p$. Also, the symmetric group S_3 is soluble, which is not autosoluble.

One should note that the above definition of A-nilpotency implies autonilpotency which has been defined in [12], while the converse does not hold. For example, one can easily check that the Dihedral group of order 8, D_8, is A-nilpotent as

$$L(D_8) = \{e, x^2\}, \quad L_2(D_8) = \{e, x, x^2, x^3\}, \quad L_3(D_8) = D_8.$$

On the other hand, the definition of autonilpotency in [12] implies that

$$L_2(D_8)/L(D_8) = L(D_8/L(D_8)) = L(\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle 1 \rangle.$$

Hence $L_2(D_8) = L(D_8)$, which implies that D_8 is not autonilpotent group.

So we work with the above definition throughout this paper. We remark that our definition of autonilpotency was given in [9] in 2013, and unfortunately the authors in [11] used our definition without giving any references.

The following example reveals some of the properties of this new notion.

Example 1.1. (i) Any non-trivial autoabelian group is an A-nilpotent group of class 1. It is emphasized that every non-trivial autoabelian group is isomorphic to $\mathbb{Z}_2$. The trivial group is A-nilpotent of class 0.

(ii) One can easily check that

$$L(\mathbb{Z}_2) = \mathbb{Z}_2; \quad L(\mathbb{Z}_3) = \langle 1 \rangle; \quad L_2(\mathbb{Z}_4) = \mathbb{Z}_4;$$
$$L(\mathbb{Z}_6) = \{e, x^3\} \quad \text{and} \quad L_2(\mathbb{Z}_6) = L(\mathbb{Z}_6).$$

Hence the cyclic groups of orders 2 and 4 are A-nilpotent and the ones of orders 3 and 6 are not, while they are nilpotent in the usual sense.

(iii) Clearly, the symmetric group S_3 is not nilpotent and it cannot be A-nilpotent either. Since it is easily checked that $L(S_3) = \langle 1 \rangle$.

(iv) It is easily checked that all cyclic groups of order 2^n, $n \geq 1$, are A-nilpotent, while arbitrary cyclic groups are not A-nilpotent.

2. Some Properties of A-Nilpotent Groups

In this section, it is shown that some of the known results of nilpotent groups can be carried over to A-nilpotent groups. We begin with some elementary facts about the A-nilpotent groups.

Lemma 2.1. *Let G be a non-trivial A-nilpotent group. Then its absolute center is non-trivial.*

Proof. Since G is A-nilpotent, we must have $L_n(G) = G$. Clearly, $n \geq 1$, otherwise G is trivial. Hence $L_1(G) = L(G) \neq \langle 1 \rangle$. $\square$

Remark 2.1. For each odd prime number p and $n \geq 1$, the cyclic group of order p^n, $\mathbb{Z}_{p^n}$, is not A-nilpotent since the absolute center of such a group is trivial.

Theorem 2.1. *Let G be an A-nilpotent group with a non-trivial characteristic subgroup N. Then $N \cap L(G)$ is non-trivial.*

Proof. By the assumption $G = L_n(G)$, for some non-negative integer n. So there exists a least positive integer i such that $N \cap L_i(G) \neq \langle 1 \rangle$. Now, $[N \cap L_i(G), \mathrm{Aut}(G)] \subseteq N \cap L_{i-1}(G) = \langle 1 \rangle$ and $N \cap L_i(G) \leq N \cap L(G)$. Hence $N \cap L(G) = N \cap L_i(G) \neq \langle 1 \rangle$. $\square$

Corollary 2.1. *A minimal characteristic subgroup of an A-nilpotent group is contained in the absolute center of the group.*

The automorphism groups of direct products of finite groups have been discussed in many articles (see [3, 5] for more details). Considering these ideas, we have the following results.

Theorem 2.2. *If H and K are finite groups with coprime orders, then*

$$\mathrm{Aut}(H \times K) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K).$$

Theorem 2.3. *Let $G_1, G_2, \ldots, G_n$ be finite A-nilpotent groups with coprime orders. Then $G_1 \times G_2 \times \cdots \times G_n$ is also A-nilpotent.*

Proof. The proof is obtained simply by using the above theorem and [10, Lemma 2.1]. $\square$

Clearly, the above results hold for the usual nilpotent groups. Now let $G = H \rtimes K$ be the semidirect product of a characteristic subgroup H by a subgroup K of G. Then Curran in [4] proved that $\mathrm{Aut}(H \rtimes K) \cong \mathcal{A}$, in which

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha \in \mathrm{Aut}(H), \gamma \in \mathrm{Aut}(K), \beta : K \to H \right\},$$

where α, β and γ satisfy $\beta(kk') = \beta(k)\beta(k')^{\gamma(k)}$ and $\alpha(h^k) = \alpha(h)^{\beta(k)\gamma(k)}$, for all $k, k' \in K$ and $h \in H$.

In fact, he showed that any automorphism θ of G may be defined by $\theta(hk) = \alpha(h)\beta(k)\gamma(k)$, and $\theta(h) = \alpha(h)$, for all $h \in H$ and $k \in K$.

Using the result of Curran, we show that the semidirect product of A-nilpotent groups is also A-nilpotent.

Theorem 2.4. *Let $G = H_1 \rtimes H_2$ be the semidirect product of a characteristic subgroup H_1 by the subgroup H_2. If H_1 and H_2 are both A-nilpotent, then so is G.*

Proof. Assume $K_m(H_1) = K_n(H_2) = \langle 1 \rangle$, for some natural numbers m and n. Then by the above discussion for every $h_1 \in H_1$, $h_2 \in H_2$ and $\theta \in \mathrm{Aut}(G)$, we have

$$\begin{aligned}
[h_1h_2, \theta] &= (h_1h_2)^{-1}\theta(h_1h_2) \\
&= {h_2}^{-1}{h_1}^{-1}\alpha(h_1)\beta(h_2)\gamma(h_2) \\
&= {h_2}^{-1}{h_1}^{-1}\alpha(h_1)h_2{h_2}^{-1}\beta(h_2)h_2{h_2}^{-1}\gamma(h_2) \\
&= [h_1, \alpha]^{h_2}\beta(h_2)^{h_2}[h_2, \gamma] \in K(H_1)H_1K(H_2) \\
&= H_1K(H_2).
\end{aligned}$$

Thus $K(G) \subseteq H_1K(H_2)$. By induction argument, we conclude that $K_n(G) \subseteq H_1K_n(H_2) = H_1$. Now, we have

$$K_{n+1}(G) = [K_n(G), \mathrm{Aut}(G)] \subseteq [H_1, \mathrm{Aut}(G)] \subseteq K(H_1).$$

Using induction argument, we have

$$K_{n+m}(G) \subseteq K_m(H_1) = \langle 1 \rangle.$$

Thus G is an A-nilpotent group of class at most $n + m$. □

Clearly, the Dihedral group $D_8 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ is an example for the above theorem.

3. Autocenter-by-Autosoluble Groups

In [1], the concept of Engel set is introduced and the nilpotency property of groups generated by a finite Engel set is studied. In this section, we concentrate on the A-nilpotency property of groups without using Engel set. Let $\mathcal{P}$ and $\mathcal{Q}$ be some group properties, then we remind that G is said to be $\mathcal{P}$-by-$\mathcal{Q}$ group, if there exists a normal subgroup N of G such that $N \in \mathcal{P}$ and $G/N \in \mathcal{Q}$.

Theorem 3.1. *An autocenter-by-autosoluble group G is A-nilpotent.*

Proof. Let N be an autocenter subgroup of G such that G/N is autosoluble. Then there exists a positive integer m such that $K^{(m)}(G/N) = 1_{G/N}$. Thus (1.1) implies that $K_m(G/N) = 1_{G/N}$.

On the other hand, $K(G)N/N \subseteq K(G/N)$. Therefore $K(G/N) = 1_{G/N}$ implies that $K(G) \subseteq N$. Clearly by induction on m we obtain $K_m(G)N/N \subseteq K_m(G/N) = 1_{G/N}$. Hence $K_m(G) \subseteq N$ and as $N \subseteq L(G)$,

$$K_{m+1}(G) = [K_m(G), \mathrm{Aut}(G)] \subseteq [N, \mathrm{Aut}(G)] = \langle 1 \rangle.$$

Therefore G is A-nilpotent of class at most $m + 1$. □

According to the inequalities (1.1) and the above theorem, we have the following corollary.

Corollary 3.1. (a) *Let G be an autocenter-by-autoabelian group. Then G is A-nilpotent of class* 2.

(b) *Let G be an autocenter-by-(A-nilpotent) group of class n. Then G is A-nilpotent of class at most $n+1$.*

(c) *Let G be an autocenter-by-autometabelian group. Then G is A-nilpotent of class 3.*

Note that subgroups and homomorphic images of A-nilpotent groups are not necessarily A-nilpotent. In general, there are no relations between the automorphisms of a subgroup of a given group and the automorphisms of the whole group. For example $S_3/A_3 \cong \mathbb{Z}_2$, thus S_3/A_3 is A-nilpotent, while S_3 is not A-nilpotent. Also the Dihedral group D_8 is A-nilpotent, but $\mathbb{Z}_2 \times \mathbb{Z}_2$ cannot be A-nilpotent as $L(\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle 1 \rangle$. However, in this context there are several articles in which they discussed the relation of automorphisms of groups and their subgroups and gave the connections under some conditions (see [8, 13, 15] for more information).

Finally, we state and prove some interesting results about the subgroups and homomorphic images of A-nilpotent groups.

Theorem 3.2. *For a characteristic subgroup N of a given group G, if N and G/N are both A-nilpotent, then so is G.*

Proof. Suppose that there exist positive integers r and s such that $K_r(G/N) = 1_{G/N}$ and $K_s(N) = \langle 1 \rangle$. Clearly, $K(G)N/N \subseteq K(G/N)$ and by induction on r one gets $K_r(G)N/N \subseteq K_r(G/N) = 1_{G/N}$ and hence $K_r(G) \subseteq N$. Let $[k, \alpha]$ be an arbitrary generator of $K_{r+1}(G) = [K_r(G), \mathrm{Aut}(G)]$. One can easily see that $[k, \alpha|_N] \in [N, \mathrm{Aut}(N)]$, so $K_{r+1}(G) \subseteq K(N)$. By induction on s we have $K_{r+s}(G) \subseteq K_s(N) = \langle 1 \rangle$. Thus G is an A-nilpotent of class at most $r+s$. □

Theorem 3.3. *Let N be a proper characteristic subgroup of a given group G with G/N is A-nilpotent of class r. If $N \cap K_r(G) = \langle 1 \rangle$, then G is A-nilpotent.*

Proof. Similar to the argument in the proof of the above theorem, one can easily see that $K_r(G)N/N \subseteq K_r(G/N)$. Now, as $N \cap K_r(G) = \langle 1 \rangle$, we have

$$\frac{K_r(G)}{N \cap K_r(G)} \cong \frac{K_r(G)N}{N} \subseteq K_r\left(\frac{G}{N}\right) = 1_{G/N}.$$

Hence $K_r(G) = \langle 1 \rangle$, which gives the A-nilpotency of G. □

Acknowledgments

The authors would like to thank the referee for his or her helpful suggestions, which made some improvements of the article.

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