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SOME PROPERTIES OF PERFECT LIE RINGS

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Abstract

In [2], Batten and Stitzinger introduced the notion of Schur multiplier and cover of Lie algebras, and gave some properties of finite dimensional perfect Lie algebras. In the present article, we study the concept of finite perfect Lie rings and prove some results similar to Lie algebras.

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1. Introduction

A Lie ring L is an additive abelian group with a (not necessarily associative) multiplication denoted by $[\,,\,]$ satisfying the following properties:

(i) $[x, x] = 0$ for all $x \in L$ (anti-commutativity).

(ii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in L$ (Jacobi identity).

(iii) $[x + y, z] = [x, z] + [y, z]$ and $[x, y + z] = [x, y] + [x, z]$ for all $x, y, z \in L$ (bilinearity).

The product $[x, y]$ is called the *commutator* of x and y. Subrings, ideals and homomorphisms of Lie rings are defined as usual, and we write $M \leq L$ and $N \trianglelefteq L$ if M is a subring and N is an ideal of L. The *centre* of L is an ideal defined as $Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$. If $L = Z(L)$, then L is called *abelian*.

Let M and N be the Lie subrings of a given Lie ring L. Then $[M, N]$ is the Lie subring of L generated by all commutators $[m, n]$, with $m \in M$ and $n \in N$. We denote the derived subring of L, by $L^2 = [L, L]$.

A finite Lie ring L is called *perfect* if $L = L^2$. This is analogous to the notion of finite dimensional perfect Lie algebras as introduced by Batten and Stitzinger in [2]. The purpose of this article is to show some properties of the Schur multipliers and covers of finite perfect Lie rings. Our results have analogues in the theory of groups and Lie algebras (see [5] for the case of groups). However, the structure of Lie rings is rather different in nature.

Throughout the paper, we consider all Lie rings to be finite and the notations are taken from [4].

Let L and A be arbitrary and abelian Lie rings, respectively. A *central extension* of L by A is a short exact sequence:

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} L \longrightarrow 0$$

of Lie rings such that $\alpha(A)$ is contained in the centre of B. A *section* of β is a map $\lambda : L \to B$ such that $\beta \circ \lambda = \mathrm{id}_L$. If a section of β exists and it is a homomorphism, then E is a *split* extension.

The central extensions E and E_1 of L by A are said to be *equivalent* if there exists a Lie ring homomorphism γ, which makes the following diagram commutative:

$$\begin{array}{ccccccccc} E: 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & L & \longrightarrow & 0 \\ & & \| & & \downarrow \gamma & & \| & & \\ E_1: 0 & \longrightarrow & A & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & L & \longrightarrow & 0. \end{array}$$

By the well-known Five lemma (whose standard proof in the category of groups is via diagram chasing transfers almost verbatim to the category of Lie rings), it follows that γ is an isomorphism.

One observes that any central extension is equivalent to the one in which α is simply the inclusion map.

Free Lie rings and (finite) presentation of a Lie ring are again defined as usual, see [6] for more details. Let L be a Lie ring with a free presentation $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$, where F is a free Lie ring and R is an ideal of F, and it is written $L \cong F/R$. Clearly, $R/[F, R]$ is a central ideal of $F/[F, R]$, and hence the short exact sequence:

$$0 \longrightarrow R/[F, R] \longrightarrow F/[F, R] \longrightarrow L \longrightarrow 0$$

is a central extension.

Using the above assumptions and notations, the following results from [4] are needed in our further investigation.

Lemma 1.1 ([4], Lemma 5.1)**.** *Suppose* $0 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} C \longrightarrow 0$ *is a central extension and* $\alpha : L \to C$ *is a homomorphism.*

Then there exists a homomorphism $\beta : F/[F, R] \to B$ *making the following diagram commutative*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/[F,R] & \longrightarrow & F/[F,R] & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \overset{\varphi}{\longrightarrow} & C & \longrightarrow & 0, \end{array}$$

where the map from $R/[F, R]$ *into A is the restriction of* β *to* $R/[F, R]$.

Let L be a Lie ring and A be a trivial L-module. The factor group $H^2(L, A)$ is called the *second cohomology* group of L with coefficients in A, and the elements of $H^2(L, A)$ are called *cohomology classes*, see [4] for more information.

The multiplier of a finite group is defined to be the second cohomology group $H^2(G, \mathbb{C}^*)$. In group theory, the multiplier of a group is unique, up to isomorphism, but the corresponding cover is not necessarily unique. In [2], multipliers and covers of Lie algebras are investigated. Also, it is shown that all the covers of a Lie algebra are isomorphic. This fact was shown earlier by Moneyhun [7]. We have similar results on Lie rings.

The Schur multiplier $M(L)$ of a Lie ring L is defined as $M(L) = H^2(L, \mathbb{C}^*)$, here $\mathbb{C}^*$ is trivial L-module. This notion has been studied in [1, 3].

Theorem 1.2 ([4], Theorem 6.5)**.** *Let* $L = F/R$ *be a finite Lie ring, where F is a free Lie ring and R is an ideal of F. Then* $H^2(L, \mathbb{C}^*) \cong (R \cap F^2)/[F, R]$ *and it is finite.*

For groups and Lie algebras, there is an extensive literature on their Schur multipliers and their Schur covers (see e.g., [5, 8, 10]). However, not much significant is available on Schur multipliers and Schur covers of Lie rings, see [3] for more details.

The following definition is essential in the next sections.

Definition 1.3. Let L be a Lie ring. A *cover* of L is a Lie ring C with a central ideal $K \leq C^2$ so that $C/K \cong L$. A *Schur cover* of L is a cover C which additionally satisfies that $K \cong M(L)$.

Schur proved that every finite group has a Schur cover and that every cover is a quotient of a Schur cover in [9]. In [3], Eick et al. proved that the Schur covers of a finite Lie ring L can be characterized as the covers of L of maximal size.

The following theorem from [3] is useful for more explanation.

Theorem 1.4 ([3], Theorem 10)**.** *Let L be a finite Lie ring, and let C be an arbitrary cover of L with $C/K \cong L$. Then there exists a Schur cover $\hat{C}$ of L so that C is isomorphic to a quotient of $\hat{C}$ and K is isomorphic to a quotient of $M(L)$.*

2. Preliminary Results

This section is devoted to some basic results, which are vital in our investigations.

The following definition is essential in our further study.

Definition 2.1. Let L be a fixed Lie ring, $E: 0 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} L \longrightarrow 0$ and $E_1: 0 \longrightarrow A_1 \longrightarrow B_1 \xrightarrow{\psi} L \longrightarrow 0$ be central extensions of L. Then E covers E_1 if there exists a homomorphism $\theta: B \to B_1$, so that the following diagram commutes:

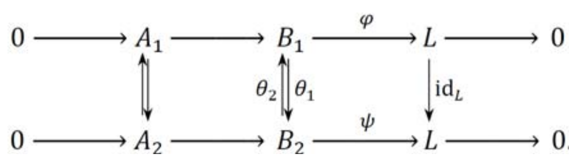
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\varphi} & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \theta & & \downarrow \mathrm{id}_L & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \xrightarrow{\psi} & L & \longrightarrow & 0, \end{array}$$

in which the vertical map from A to A_1, is the restriction of θ to A. If θ is unique, then we say that E *uniquely covers* E_1. The extension E is called *universal* if it covers any central extension of L uniquely.

The following lemmas and corollaries are needed for the proofs of our main results in the next section.

Lemma 2.2. *If* E_1 *and* E_2 *are universal central extensions of a finite Lie ring* L, *then there exists an isomorphism* $B_1 \to B_2$ *which carries* A_1 *onto* A_2.

Proof. Let $\theta_1 : B_1 \to B_2$ and $\theta_2 : B_2 \to B_1$ be the unique homomorphisms such that the following diagram commutes:



Hence $\psi \circ \theta_1 = \mathrm{id}_L \circ \varphi = \varphi$ and $\varphi \circ \theta_2 = \mathrm{id}_L \circ \psi = \psi$. Then the mapping $\theta_2 \circ \theta_1 : B_1 \to B_1$ gives $\varphi \circ \theta_2 \circ \theta_1 = \psi \circ \theta_1 = \varphi$. Since θ_1 is unique, $\theta_2 \circ \theta_1 = \mathrm{id}_{B_1}$ and likewise $\theta_1 \circ \theta_2 = \mathrm{id}_{B_2}$. Hence θ_1 is an isomorphism from B_1 onto B_2 and therefore when restricted to A_1, carries A_1 onto A_2.

Corollary 2.3. *If* E *is a universal central extension*, *then both* B *and* L *are perfect*.

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} L \longrightarrow 0$ be a universal central extension and consider the central extension:

$$0 \longrightarrow A \times (B/B^2) \longrightarrow B \times (B/B^2) \xrightarrow{\psi} L \longrightarrow 0,$$

where $\psi(b, \bar{b}') = \varphi(b)$ for all $b \in B$ and $\bar{b}' \in B/B^2$. Define the homomorphism $\theta_i : B \to B \times (B/B^2)$ for $i = 1, 2,$ given by

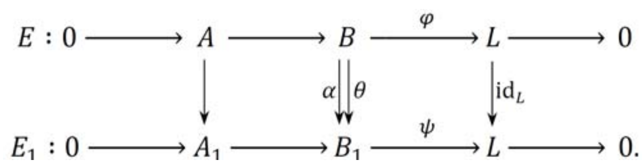
$$\theta_1(b) = (b, 0), \quad \theta_2(b) = (b, \bar{b}).$$

Then $\psi \circ \theta_1(b) = \psi(b, 0) = \varphi(b)$ and $\psi \circ \theta_2(b) = \psi(b, \bar{b}) = \varphi(b)$. Hence $\psi \circ \theta_i = \varphi$ for $i = 1, 2$. Since E is universal, $\theta_1 = \theta_2$ and so $B/B^2 = 0$ or $B = B^2$. Consequently, $L^2 = (B/A)^2 = (B^2 + A)/A = (B + A)/A = B/A = L$. Therefore, both B and L are perfect.

Lemma 2.4. *By the above assumption, if B is perfect, then E covers E_1 if and only if E uniquely covers E_1.*

Proof. Clearly, if E uniquely covers E_1, then E covers E_1.

Conversely, if E covers E_1, then there exists a homomorphism $\theta : B \to B_1$ such that the following diagram commutes:



Suppose that there exists another homomorphism $\alpha : B \to B_1$ such that the above diagram with α commutes. We need to show that $\alpha = \theta$. For all $x, y \in B$, we have

$$\psi(\alpha(x) - \theta(x)) = \psi(\alpha(x)) - \psi(\theta(x)) = \varphi(x) - \varphi(x) = 0.$$

Therefore, $\alpha(x) - \theta(x) \in \mathrm{Ker}\psi = A_1 \subseteq Z(B_1)$. Likewise, $\alpha(y) - \theta(y) \in \mathrm{Ker}\psi = A_1 \subseteq Z(B_1)$. So $\alpha(x) = \theta(x) + a$ and $\alpha(y) = \theta(y) + b$, for some $a, b \in Z(B_1)$,

$$\begin{aligned}\alpha([x, y]) = [\alpha(x), \alpha(y)] &= [\theta(x) + a, \theta(y) + b] \\ &= [\theta(x), \theta(y)] + [\theta(x), b] + [a, \theta(y)] + [a, b] \\ &= [\theta(x), \theta(y)] \text{ since } a, b \in Z(B_1) = \theta([x, y]).\end{aligned}$$

Hence α and θ coincide on B^2 and as B is perfect, they imply that $\alpha = \theta$.

Lemma 2.5. *Assume that L is a finite perfect Lie ring. The extension* $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ *is universal if and only if every central extension of L splits.*

Proof. Suppose that $E : 0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal, and $E_1 : 0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} L \longrightarrow 0$ is a central extension of L. Then there exists a unique homomorphism $\theta : L \to B$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
E: 0 & \longrightarrow & 0 & \longrightarrow & L & \xrightarrow{\mathrm{id}_L} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}_L} & & \\
E_1: 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\alpha} & L & \longrightarrow & 0.
\end{array}$$

Hence $\alpha \circ \theta = \mathrm{id}_L$ and the extension E_1 splits.

Conversely, suppose that every central extension of L splits. Let $0 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} L \longrightarrow 0$ be a central extension of L. Then there exists a homomorphism $\beta : L \to B$ such that $\varphi \circ \beta = \mathrm{id}_L$. Then the following diagram is commutative:

$$\begin{array}{ccccccccc}
E: 0 & \longrightarrow & 0 & \longrightarrow & L & \xrightarrow{\mathrm{id}_L} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{id}_L} & & \\
E_1: 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\varphi} & L & \longrightarrow & 0,
\end{array}$$

and therefore E covers E_1. Using Lemma 2.4 and the perfectness of L, E uniquely covers E_1. Hence E is universal.

Corollary 2.6. *Let* $E_1 : 0 \longrightarrow B \longrightarrow G \xrightarrow{\varphi} L \longrightarrow 0$, $E_2 : 0 \longrightarrow C \longrightarrow L \xrightarrow{\psi} H \longrightarrow 0$ *be the central extensions*, $\pi = \psi \circ \varphi$ *and* $K = \mathrm{Ker}\pi$. *If G is a perfect Lie ring, then* $E_3 : 0 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 0$ *is a central extension.*

Proof. If $k \in K = \operatorname{Ker}\pi$, then $\psi \circ \varphi(k) = \pi(k) = 0$. Hence $\varphi([k, x]) = [\varphi(k), \varphi(x)] = 0$. Assume that $\lambda_k : G \to G$ is a map given by $\lambda_k(x) = [k, x]$, for every $x \in G$. Since $\varphi(\lambda_k(x)) = 0$, it follows that $\lambda_k(x) \in \operatorname{Ker}\varphi \subseteq Z(G)$. Clearly, $\lambda_k([x, y]) = [k, [x, y]] = -[y[k, x]] - [x, [y, k]] = 0$ for all $x, y \in G$. Therefore, λ_k is the trivial map on $G^2 = G$ and so $k \in Z(G)$. Hence E_3 is a central extension.

Corollary 2.7. *Consider the extensions* E_1, E_2, E_3, *and the maps defined as in Corollary* 2.6. *If* E_1 *is universal, then so is* E_3.

Proof. Suppose that the extension $E_1 : 0 \longrightarrow B \longrightarrow G \xrightarrow{\varphi} L \longrightarrow 0$ is universal. Then by Corollary 2.3, both G and L are perfect. Since H is a homomorphic image of G, it implies that H is perfect.

Let $E_4 : 0 \longrightarrow D \longrightarrow S \xrightarrow{\mu} H \longrightarrow 0$ be another central extension of H and

$$T = \{(a, b) \in L \times S \mid \psi(a) = \mu(b)\}.$$

Define a multiplication in T by $[(a, b), (c, d)] = ([a, c], [b, d])$. Clearly, T is closed under the Lie bracket, as $\psi([a, c]) = [\psi(a), \psi(c)] = [\mu(b), \mu(d)] = \mu([b, d])$. Hence T is a subring of $L \times S$. Let λ be the projection of T onto L. Since E_1 is universal, there exists a unique homomorphism $\alpha : G \to T$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
E_1: 0 & \longrightarrow & B & \longrightarrow & G & \xrightarrow{\varphi} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow \mathrm{id}_L & & \\
0 & \longrightarrow & 0 \times D & \longrightarrow & T & \xrightarrow{\lambda} & L & \longrightarrow & 0.
\end{array}$$

Hence $\lambda \circ \alpha = \varphi$. Let $\gamma : T \to S$ be the natural projection given by $\gamma(a, b) = b$, and let $\beta = \gamma \circ \alpha$. For every element $g \in G$, set $\alpha(g) = (a, b)$. Then $\beta(g) = \gamma \circ \alpha(g) = \gamma(a, b) = b$ and $\lambda \circ \alpha(g) = \lambda(a, b) = a = \varphi(g)$.

Therefore, $(\mu \circ \beta)(g) = \mu(b) = \psi(a) = \psi \circ \varphi(g) = \pi(g)$ and hence the following diagram commutes. Hence E_3 covers E_4 and as G is perfect, E_3 uniquely covers E_4. Therefore, E_3 is universal.

$$\begin{array}{lccccccccc}
E_3: & 0 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & H & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle \mathrm{id}_H} & & \\
E_4: & 0 & \longrightarrow & D & \longrightarrow & S & \xrightarrow{\mu} & H & \longrightarrow & 0,
\end{array}$$

Lemma 2.8. *Assume that L is a finite perfect Lie ring. If* $0 \longrightarrow 0 \longrightarrow B \xrightarrow{\alpha} L \longrightarrow 0$ *is a universal extension, then so is* $0 \longrightarrow 0 \longrightarrow L \xrightarrow{\mathrm{id}_L} L \longrightarrow 0.$

Proof. Suppose $0 \longrightarrow 0 \longrightarrow B \xrightarrow{\alpha} L \longrightarrow 0$ is universal and L is perfect. Let $0 \longrightarrow A \longrightarrow L^* \xrightarrow{\psi} L \longrightarrow 0$ be any central extension of L. Then there exists a unique homomorphism $\theta : B \to L^*$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & B & \xrightarrow{\alpha} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle \mathrm{id}_L} & & \\
0 & \longrightarrow & A & \longrightarrow & L^* & \xrightarrow{\psi} & L & \longrightarrow & 0.
\end{array}$$

Clearly, α is an isomorphism and $\alpha = \psi \circ \theta$. Let $\beta = \theta \circ \alpha^{-1}$. Then β is a homomorphism from L into L^* such that $\psi \circ \beta = \psi \circ \theta \circ \alpha^{-1} = \alpha \circ \alpha^{-1} = \mathrm{id}_L$. Thus, the extension $0 \longrightarrow A \longrightarrow L^* \xrightarrow{\psi} L \longrightarrow 0$ splits. Since L is perfect and every central extension of L splits, by Lemma 2.5, $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal.

3. The Main Results

In this final section, using the discussion of the previous section, we give and prove our main results on finite perfect Lie rings.

Our results are somehow similar to those given in [2], in which the authors studied the notion of Lie algebras.

The following theorem gives the existence of a cover for a finite perfect Lie ring.

Theorem 3.1. *Let* $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ *be a free presentation of a finite perfect Lie ring L. Then*

(i) $F^2/[F, R]$ *is a cover of L and the central extension*

$$0 \longrightarrow (F^2 \cap R)/[F, R] \longrightarrow F^2/[F, R] \longrightarrow L \longrightarrow 0$$

is universal.

(ii) *If* $0 \longrightarrow A \longrightarrow L^* \longrightarrow L \longrightarrow 0$ *is a universal central extension and L is perfect, then* $A \cong M(L)$ *and* L^* *is a cover of L.*

Proof. (i) By definition, $M(L) \cong (F^2 \cap R)/[F, R]$ and $L \cong L^2 \cong (F/R)^2 \cong F^2/(F^2 \cap R) \cong (F^2/[F, R])/(F^2 \cap R)/[F, R]$. So we only need to verify $(F^2 \cap R)/[F, R] \subseteq Z(F^2/[F, R]) \cap (F^2/[F, R])^2$. Clearly, $(F^2 \cap R)/[F, R] \subseteq Z(F^2/[F, R])$ and $(F^2 \cap R)/[F, R] \subseteq F^2/[F, R]$. It suffices to show that $(F^2/[F, R])^2 = F^2/[F, R]$. $(F^2/[F, R])^2 = ([F^2, F^2] + [F, R])/[F, R]$ and $[F^2, F^2] + [F, R] \subseteq F^2$. Now, by the perfectness of L, we have $F/R = (F/R)^2 = (F^2 + R)/R$. So for every $x_i \in F$, $x_i = y_i + r_i$, for some $y_i \in F^2$ and $r_i \in R$. Hence

$$[x_1, x_2] = [y_1 + r_1, y_2 + r_2] = [y_1, y_2] + [y_1, r_2] + [r_1, y_2] + [r_1, r_2]$$
$$\in [F^2, F^2] + [F, R].$$

Therefore, $F^2 \subseteq [F^2, F^2] + [F, R]$. Hence $F^2/[F, R]$ is a cover L.

Now, let

$$0 \longrightarrow R/[F,\ R] \longrightarrow F/[F,\ R] \xrightarrow{\ \pi\ } L \longrightarrow 0$$

be the natural exact sequence. As $L^2 = L$, it follows that L is the image of

$$(F,\ [F,\ R])^2 \cong (F^2 + [F,\ R])/[F,\ R] \cong F^2/[F,\ R].$$

Therefore, the restriction of π to $F^2/[F,\ R]$ induces a central extension

$$0 \longrightarrow (F^2 \cap R)/[F,\ R] \longrightarrow F^2/[F,\ R] \longrightarrow L \longrightarrow 0.$$

Now suppose that $E_1 : 0 \longrightarrow A \longrightarrow B \longrightarrow L \longrightarrow 0$ is a central extension of L. Then by Lemma 1.1, it is covered by the natural exact sequence, E_2, rendering the following diagram commutative:

$$\begin{array}{lcccccccc}
E_2: & 0 & \longrightarrow & R/[F,R] & \longrightarrow & F/[F,R] & \longrightarrow & L & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow \beta & & \downarrow \mathrm{id}_L & \\
E_1: & 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\ \varphi\ } & L & \longrightarrow 0,
\end{array}$$

since the following diagram:

$$\begin{array}{lcccccccc}
E: & 0 & \longrightarrow & (F^2 \cap R)/[F,R] & \longrightarrow & F^2/[F,R] & \longrightarrow & L & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow \theta & & \downarrow \mathrm{id}_L & \\
E_1: & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & L & \longrightarrow 0
\end{array}$$

commutes, where θ is the restriction of β to $F^2/[F,\ R]$. Hence E_1 is covered by E. Since $F^2/[F,\ R]$ is perfect and covers E_1, the exact sequence E uniquely covers E_1 by Lemma 2.4, and therefore E is universal.

(ii) By Lemma 2.2, there exists an isomorphism $L^* \to F^2/[F,\ R]$ which carries A onto $(F^2 \cap R)/[F,\ R] \cong M(L)$. Hence $M(L) \cong A$ and L^* is a cover of L.

The following theorem gives a connection between the Schur multipliers of $M(L)$ and $M(L/Z)$, for any central ideal Z of a finite perfect Lie ring L.

Theorem 3.2. *Let the Schur multiplier of a finite perfect Lie ring L be trivial. Then*

(i) *For every abelian Lie ring A, $H^2(L, A) = 0$.*

(ii) *If Z is a central ideal of L, then $Z \cong M(L/Z)$ and L is a cover of L/Z.*

Proof. (i) By the assumption $M(L) = 0$, and using Theorem 3.1(i), the exact sequence $0 \longrightarrow 0 \longrightarrow F^2/[F, R] \longrightarrow L \longrightarrow 0$ is also universal. Then by Lemma 2.8, $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal. Therefore, by Lemma 2.5, every central extension of L splits and so $H^2(L, A) = 0$.

(ii) By part (i), $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal. Since L is perfect, Corollary 2.7 implies that $0 \longrightarrow Z \longrightarrow L \longrightarrow L/Z \longrightarrow 0$ is also universal. Therefore, by Theorem 3.1(ii), $M(L/Z) \cong Z$. Hence, L is a cover of L/Z.

We use some cohomological machinery in proving our final result.

Theorem 3.3. *If L is a finite perfect Lie ring and C is a cover of L, then C is perfect with trivial Schur multiplier.*

Proof. Put $M = M(L)$, hence $L \cong C/M$ and $M \subseteq Z(C) \cap C^2$. So $L/L^2 \cong (C/M)/(C/M)^2 \cong (C/M)/(C^2 + M)/M \cong C/(C^2 + M)$. By the assumption $L = L^2$, $C = C^2 + M = C^2$ and hence C is perfect. By Theorem 4.1 of [2] and Theorem 5.5 in [4], the following sequence is exact:

$$0 \longrightarrow \mathrm{Hom}(L, C^*) \xrightarrow{\inf_1} \mathrm{Hom}(C, C^*) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(M, C^*)$$

$$\xrightarrow{\mathrm{Tra}} M(L) \xrightarrow{\inf_2} M(C) \xrightarrow{f} C/C^2 \otimes M,$$

where C^* is the set of non-zero complex numbers. As C is perfect, we have $(C/C^2) \otimes M = 0$. Therefore, $M(C) = \text{Ker} f = \text{Im}(\text{inf}_2)$. Also, the perfectness of C implies that $\text{Hom}(C, C^*) = 0$. Then $0 = \text{Im}(\text{Res}) = \text{Ker}(\text{Tra})$, and so $\text{Im}(\text{Tra}) \cong \text{Hom}(M, C^*) \cong M \cong M(L)$. Therefore,

$$M(C) = \text{Im}(\text{inf}_2) \cong M(L)/\text{Ker}(\text{inf}_2) = M(L)/\text{Im}(\text{Tra}) = M(L)/M(L) = 0.$$

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