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Effect of the Casimir Force on Size-Dependent Dynamic Pull-In Instability in Micro-Bridge Gyroscopes with a Proof Mass

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Abstract

This paper concentrates on size-dependent dynamic pull-in instability of micro-bridge gyroscopes under the combined effects of instantaneous DC voltage and the Casimir attraction. Hamilton's principle is employed to derive the governing equations of motion based on the modified couple stress theory. Galerkin's reduced order model is then utilized to obtain the ordinary differential equations of motion. In order to solve the reduced equations, fourth order Runge-Kutta method is used. The accuracy of the present model is validated through comparison with available results in the literature. Afterward effects of system parameters on the dynamic pull-in behavior of the system are investigated. It is found that neglecting the Casimir effect can lead to false results especially for systems with small initial gaps.

Keywords: Instantaneous DC voltage; Micro-bridge gyroscope; Modified couple stress theory; Casimir force.

1. Introduction

Micro-electro-mechanical-systems (MEMS) cover a wide area of research interests nowadays. Small size, low power consumption and capability of mass production are the main factors that made analysis micro devices a desired topic among the researchers [1]. Based on these factors, micro devices have found their ways into many applications such as navigations [2].

As the analysis of micro system grows, the need for an accurate modelling of MEMS devices has arisen. Many researchers have employed micro-beams to model the behavior of different microsystems [3]. Ghommem et al. [4] investigated the behavior of an electrically actuated microgyroscope by modelling the system with a rotating micro-cantilever with a tip mass.

Convergence of gyroscope and MEMS technologies has led to the developments of different types of micro-gyroscopes. In general, a micro-gyroscope, which is a device employed to detect angular velocity, has found growing applications in many research fields such as anti-rollover systems and virtual reality [5]. Among all kinds of micro gyroscopes, Coriolis based ones are the most common type of these devices due to their simple manufacturing process. These micro gyroscopes are usually modelled with rotating beams, tuning forks and vibrating rings [6]. Rotating beams are mostly employed to trace the response of the Coriolis based micro-gyroscopes. In these devices, a vibrating beam is placed in a rotating system. Upon actuation of dynamic loads in a specific direction, which is called drive direction, the effect of Coriolis acceleration combined with dynamic loadings will cause the system to begin a vibratory motion around the static deflection in a direction perpendicular to the actuation direction called sense direction. By measuring the motion of the system in the sense direction, one can obtain the angular velocity of the system.

Electrical actuation is widely used by researchers for the actuation of micro-gyroscopes. As the applied voltage reaches a critical value, the elastic force will not be able to overcome the effect of electrical actuation and the vibratory micro-beam will hit the stationary substrate underneath it. This phenomenon is called pull-in instability and the corresponding voltage is pull-in voltage. Since the pull-in instability is the most common type of instability in such micro-systems, investigating the pull-in behavior of such systems have turned into a desirable topic. Mojahedi et al. [7] modelled a micro gyroscope as a cantilever beam with a tip mass subjected to the step-input DC voltage. They considered the effects of squeeze film damping, nonlinear curvatures and air pressure in their theoretical formulations. To obtain the pull-in voltage of the system, they employed Hamilton's principle and derived the equations of motion and solved them numerically. They observed that consideration of geometric nonlinearities would increase the stiffness of the beam that leads to higher values of pull-in voltage.

When at least one of the dimensions of a system is took place in the order of micron, it has been proved that the classical theories are incapable of predicting the true behavior of systems. Therefore, many researchers have tried to introduce new continuum theories to remove such incapabilities [8]. In this way, some continuum size-dependent theories, which usually contain some higher order material length scale parameters, have been introduced [8-10]. Modified couple stress theory (MCST) is one of these theories that has been proposed by Yang et al. [8] in which only one material length scale parameter is introduced. Recently, it has been proved that the modified couple stress theory is so accurate in problems with bending loadings only [11]. Hence, it has been widely employed to determine the response of micro-systems due to its accuracy and simplicity.

Although the size dependency of micro devices has been greatly investigated, the size effect on the behavior of micro-gyroscopes has received little attention. Ghayesh et al. [12] investigated the size effect on the dynamic behavior of a cantilever type micro gyroscope by considering the modified couple stress theory. They utilized Hamilton's principle to derive the governing equations of motion and applied the Galerkin's reduced order (ROM) method to obtain the reduced ordinary differential equations of motion. The reduced equations are then solved via a continuation method.

Another issue that the miniaturization process may face is the effect of Casimir force. The Casimir effect represents attractive force between two flat parallel plates of solids that arises from quantum fluctuations in the ground state of the electromagnetic field [13]. Although Casimir force seems to have an important role in determining the behavior of micro-gyroscope, only a few researchers have studied the effect of Casimir force on the micro-gyroscopes. Mojahedi et al. [14] investigated the effect of Casimir force on the static pull-in behavior of a micro-bridge gyroscope.

Although the static pull-in behaviour of the micro-bridge gyroscopes has been studied, size-dependent dynamic pull-in analysis of such systems has not been conducted yet. The main purpose of this paper is to investigate the combined effects of size and the Casimir force on the threshold of the dynamic pull-in instability in micro-bridge gyroscopes with a proof mass. To this end, Galerkin's weighted residual method is employed to obtain the reduced governing ordinary differential equations of motion. Afterward, the fourth order Runge-Kutta method is utilized to solve the resulting initial value problems. The results are compared and validated by those available in the previous studies and very good agreements between them are observed.

2. Theoretical formulations

Consider an electrically actuated micro-bridge of length *L*, thickness *b* in the sense direction, thickness *h* in the drive direction and proof mass *M* in the middle as illustrated in Figure 1. The initial gap in the sense and drive directions are considered to be d_w and d_v, respectively. Since the square cross-section is the most common cross-section of beams for micro gyroscopes, the beam is considered to have a square cross-section in this study and hence, *b* is equal to *h*. In addition, to achieve symmetric static deflections, d_w and d_v are also assumed to have the same values. It is worth noting that the system in Figure 1 rotates with a constant angular velocity of Ω along the *x*-axis.

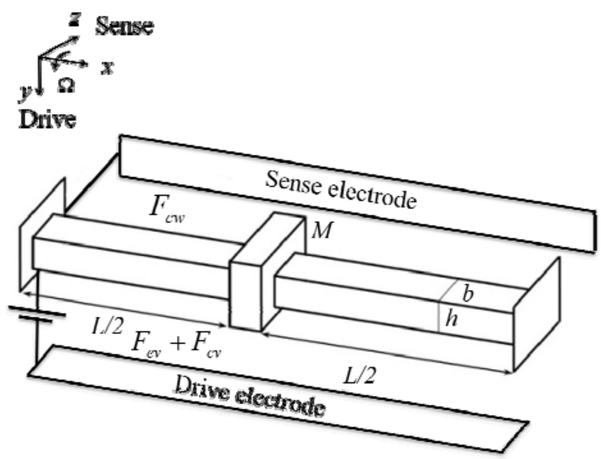


Figure 1. Schematic of an electrically actuated rotating micro-bridge.

2.1 Strain energy

To simplify the strain energy expression of the system, Euler-Bernoulli beam model is assumed. The displacement field correspond to the Euler-Bernoulli beam theory can be given as

$$u = u_0 - \frac{\partial v_0}{\partial x} y - \frac{\partial w_0}{\partial x} z \tag{1a}$$

$$v = v_0 \tag{1b}$$

$$w = w_0 \tag{1c}$$

where u_0, v_0 and w_0 are the displacements of a point on the neutral axis of the beam, respectively, in the *x*, *y* and *z* directions with respect to the rotating coordinate system. Considering the effect of mid-plane stretching, one can obtain the non-zero component of the strain tensor as [15]

$$\varepsilon_{xx} = u_0{}' - v_0{}'' y - w_0{}'' z + \frac{1}{2} v_0{}'^2 + \frac{1}{2} w_0{}'^2 \tag{2}$$

where prime denotes the derivatives with respect to *x*.

According to the MCST, non-zero components of symmetric curvature tensor can be obtained as [12]

$$\chi_{xy} = -\frac{1}{2}\frac{\partial^2 w_0}{\partial x^2}, \chi_{xz} = \frac{1}{2}\frac{\partial^2 v_0}{\partial x^2} \quad \chi_x = \chi_y = \chi_z = \chi_{yz} = 0 \tag{3}$$

Therefore, the strain energy expression of the system based on the MCST can be written as

$$U = \frac{1}{2}\int_0^L \left(EA(u_0' + \frac{v_0'^2}{2} + \frac{w_0'^2}{2})^2 + \left(EI_{zz} + \mu l^2 A\right) v_0''^2 + \left(EI_{yy} + \mu l^2 A\right) w_0''^2 \right) dx \tag{4}$$

where E is the Young modulus of the micro-beam and

$$I_{yy} = \int_A z^2 dA, \; I_{zz} = \int_A y^2 dA \tag{5}$$

Since the present investigation concentrates on the square cross-section, I_{yy} and I_{zz} are equal and will be denoted by I hereinafter.

2.2 Kinetic Energy

To obtain the kinetic energy of the system, at the first step, velocity of a point P on a cross-section of the beam at a distance x from one end of the beam is determined. Neglecting the second mass moment of inertia of the cross-sectional area and the time derivatives of displacement along the x axis, velocity of a point P can be written as

$$\mathbf{v_P} = \left(\dot{v}_0 - \Omega w_0\right)\hat{j} + \left(\dot{w}_0 + \Omega v_0\right)\hat{k} \tag{6}$$

where dot represents the derivatives with respect to time.

From Eq. (6), the kinetic energy of the beam can be obtained as

$$T = \frac{1}{2}\int_o^L \left(\rho A + M\delta\left(x - \frac{L}{2}\right)\right)\left((\dot{v}_0 - \Omega w_0)^2 + (\dot{w}_0 + \Omega v_0)^2\right) dx \tag{7}$$

2.3 Electrostatic force

Considering the first order fringing field effect and neglecting the area of the capacitance, the electrostatic excitation by instantaneous DC voltage in the drive direction can be expressed as [16]

$$F_{ev} = \frac{\varepsilon b\left(V_{DCv}\right)^2}{2\left(d_v - v_0\right)^2}\left(1 + 0.65\frac{d_v - v_0}{b}\right) \tag{8}$$

where ε is the dielectric constant of the medium between the electrodes.

2.4 Casimir force

The Casimir force per unit length of the beam can be given as [17]

$$F_{ci} = \frac{\pi^2 \hbar c b}{240\left(d - i\right)^4} \tag{9}$$

where $\hbar = 1.055\times10^{-34}$ J s is the Planck's constant divided by 2π, $c = 2.998\times10^8$ m/s is the speed of light and i can be either v or w.

2.5 Equations of motion

Employing the Hamilton's principle, the equations of motion can be obtained as

$$\frac{\partial}{\partial x}\left[EA\left(u_0' + \frac{1}{2}v_0'^2 + \frac{1}{2}w_0'^2 \right)\right] = 0 \tag{10a}$$

$$\left(\rho A + M\delta\left(x - \frac{L}{2}\right)\right)\frac{\partial}{\partial t}\left(\dot{v}_0 - \Omega w_0\right) - \left(\rho A + M\delta\left(x - \frac{L}{2}\right)\right)\left(\Omega^2 v_0 + \Omega \dot{w}_0\right) - EA\frac{\partial}{\partial x}\left(u_0' v_0' + \frac{1}{2}v_0' w_0'^2 + \frac{1}{2}v_0'^3\right) + \left(EI + \mu l^2 A\right) v_0'''' = F_{ev} + F_{cv} \tag{10b}$$

$$\left(\rho A+M\delta\left(x-\frac{L}{2}\right)\right)\frac{\partial}{\partial t}\left(\dot{w}_0+\Omega v_0\right)-\left(\rho A+M\delta\left(x-\frac{L}{2}\right)\right)\left(\Omega^2 w_0-\Omega\dot{v}_0\right)-EA\frac{\partial}{\partial x}\left(u_0'w_0' +\frac{1}{2}w_0'v_0'^2+\frac{1}{2}w_0'^3\right)+\left(EI+\mu l^2 A\right)w_0'''' = F_{cw} \tag{10c}$$

where δ is the Dirac's delta. The corresponding boundary conditions can be given as

$$u_0\left(0,t\right)=0,\, u_0\left(L,t\right)=0 \tag{11a}$$

$$k\left(0,t\right)=0,\, k\left(L,t\right)=0 \tag{11b}$$

$$k'\left(0,t\right)=0,\, k'\left(L,t\right)=0 \tag{11c}$$

where k can be either v_0 or w_0.

Eq. (10a) can be solved analytically for u_0 along with the boundary conditions in Eq. (11a). The solution of u_0 in terms of v_0 and w_0 can be obtained as

$$u_0=-\frac{1}{2}\int_0^x\left(v_0'^2+w_0'^2\right)dx+\frac{1}{2L}\left(\int_0^L\left(v_0'^2+w_0'^2\right)dx\right)x \tag{12}$$

To derive the non-dimensional equations of motion, following non-dimensional parameters are introduced

$$\hat{w}=\frac{w_0}{d_w},\ \hat{v}=\frac{v_0}{d_v},\ \hat{x}=\frac{x}{L},\ M_r=\frac{M}{\rho AL},\ \hat{t}=t\tilde{t}=t\sqrt{\frac{EI}{\rho A\left(L\right)^4}},\ \Omega_r=\frac{\Omega}{\tilde{t}},\ \alpha=6\left(\frac{d}{h}\right)^2$$
$$\beta_1=\frac{\varepsilon bV^2}{2d_v^2 e_1},\ k=\frac{d_w}{d_v},\ \ \beta_2=0.65\frac{d_v}{b},\ \eta=\frac{\mu l^2 A}{EI},\ \lambda_L=\frac{\pi^2\hbar cbL^4}{240d^5 EI} \tag{13}$$

Substitution of Eq. (12) and the dimensionless parameters in Eq. (13) into Eqs. (10b) and (10c), and dropping the hats, one would get

$$\left(1+M_r\delta\left(x-\frac{1}{2}\right)\right)\left(\ddot{v}_0-\Omega_r^2 v_0-2\Omega_r k\dot{w}_0\right)-\alpha\int_0^1\left(\left(v_0'\right)^2+\left(w_0'\right)^2\right)dx\, v_0''+\left(1+\eta\right)v'''' = \frac{\beta_1}{\left(1-v_0\right)^2}\left(1+\beta_2\left(1-v_0\right)\right)+\frac{\lambda_L}{\left(1-v_0\right)^4} \tag{14a}$$

$$\left(1+M_r\delta\left(x-\frac{1}{2}\right)\right)\left(\ddot{w}_0-\Omega_r^2 w_0+2\frac{\Omega_r}{k}\dot{v}_0\right)-\alpha\int_0^1\left(\left(v_0'\right)^2+\left(w_0'\right)^2\right)dx\, w_0''+\left(1+\eta\right)w_0'''' = \frac{\lambda_L}{\left(1-w_0\right)^4} \tag{14b}$$

2.6 Solution Procedure

To solve the equations of motion, a Galerkin-based ROM is employed to convert the partial differential equations in Eqs. (14) to some ODEs. To this end, a single-mode approximation has been employed and deflection of an arbitrary point on the beam can be expressed as

$$i=\psi\left(x\right)q_i \tag{15}$$

where i can be either v or w, $\psi\left(x\right)$ is the normalized dimensionless un-damped mode shape of a clamped-clamped beam and q_i is the proof-mass displacement in corresponding direction of i. Mode shape of clamped-clamped beam is utilized to obtain the ordinary differential equations of motion. Hence, the mode shape of the beam can be written as [18]

$$\psi(x) = \Lambda\left[\cosh(\gamma x) - \cos(\gamma x) - \sigma\left(\sinh(\gamma x) - \sin(\gamma x)\right)\right] \tag{16}$$

where

$$\Lambda = 0.6297, \gamma = 4.73, \sigma = 0.9825 \tag{17}$$

Substitution of Eq. (15) into Eqs. (14), multiplying the resultant by ψ and integrating over the non-dimensional domain will lead to

$$\begin{aligned}\sigma_1\ddot{q}_v + \sigma_2 q_v + \sigma_3\dot{q}_w + \sigma_4 q_v^3 + \sigma_5 q_v q_w^2 &= \int_0^1 \frac{\beta_1\psi}{(1-\psi q_v)^2}\left(1+\beta_2\left(1-\psi q_v\right)\right)dx \\ &+ \int_0^1 \frac{\lambda_L\psi}{(1-\psi q_v)^4}dx\end{aligned} \tag{18a}$$

$$\sigma_1'\ddot{q}_w + \sigma_2' q_w + \sigma_3'\dot{q}_v + \sigma_4' q_w^3 + \sigma_5' q_w q_v^2 = \int_0^1 \frac{\lambda_L\psi}{(1-\psi q_w)^4}dx \tag{18b}$$

where

$$\sigma_1 = \sigma_1' = \int_0^1\left(1 + M_r\delta\left(1-\frac{1}{2}\right)\right)\psi^2 dx \tag{19a}$$

$$\sigma_2 = \sigma_2' = (1+\eta)\int_0^1 \psi\psi'''' dx - \Omega_r^2\int_0^1\left(1 + M_r\delta\left(1-\frac{1}{2}\right)\right)\psi^2 dx \tag{19b}$$

$$\sigma_3 = -\sigma_3' = -2\Omega_r\int_0^1\left(1 + M_r\delta\left(1-\frac{1}{2}\right)\right)\psi^2 dx \tag{19c}$$

$$\sigma_4 = \sigma_5 = \sigma_4' = \sigma_5' = -\alpha\int_0^1 \psi'^2 dx\int_0^1 \psi\psi'' dx \tag{19d}$$

3. Results and discussions

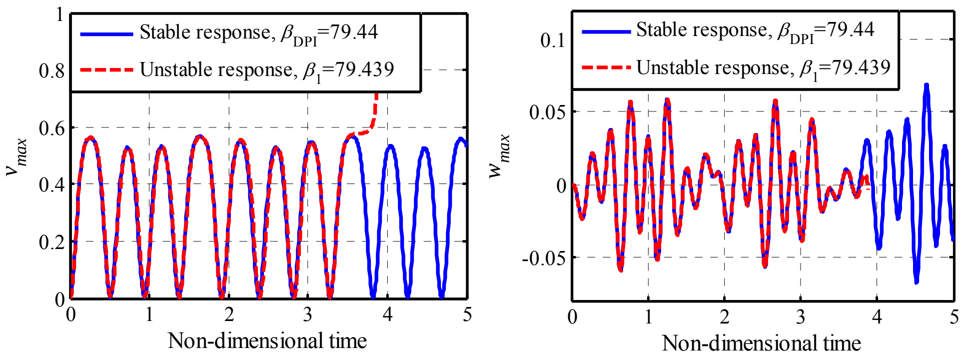
To obtain numerical results, the micro-beam is considered to be made of silicon with material properties given in Table 1.

Table 1. Material properties of micro-beam

E (GPa)	ν	ρ (kg/m^3)	l (μm)
169	0.33	2331	0.592

For the purpose of validation, a comparison has been made between the present findings and the results reported by Moghimi Zand et al. [19]. In this comparison, the parameters *b*, *h*, *L*, *d* and effective Young Modulus *E* are, respectively, set to 20 μm, 2 μm, 300 μm, 2 μm and 189 GPa. It should be noted that in this comparison, the cross-section of the beam is not a square since the system is not a gyroscope. Furthermore, the values of parameters Ω_r, λ_L, M_r and η are chosen to be equal to zero in order to perform this comparison. The dimensionless dynamic pull-in voltage for these set of parameters is calculated as $\beta_1 = 63.79$ which agrees excellently with $\beta_1 = 64.59$ reported by Moghimi Zand et al. [19]. It is worth mentioning that, for the purpose of simplicity, the dimensionless dynamic pull-in voltage is denoted by β_{DPI} hereinafter.

Time responses of the system in the drive and sense directions for both stable and unstable states are shown in Figure 2. To obtain this figures, the values of the parameters Ω_r, λ_L, M_r and η are set to 1, 0.5, 0.05 and 0.3953, respectively. It is noteworthy that β_{DPI} is calculated as 79.44 for this case.



(a) (b)

Figure 2. Time response of the micro gyroscope for stable and unstable conditions for (a) drive direction and (b) sense direction

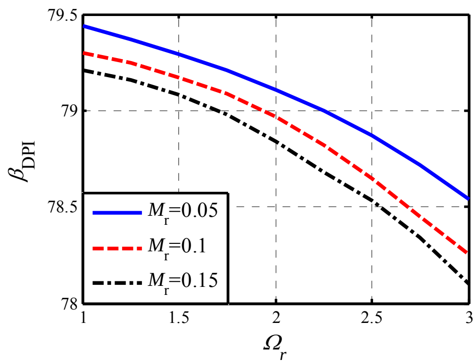


Figure 3. Non-dimensional dynamic pull-in voltage versus non-dimensional rotation frequency for two mass ratios 0.05 and 0.1.

Figure 3 represents the effects of these base rotation frequency and proof mass on the dynamic pull-in voltage of the micro gyroscope. It can be observed from Figure 3 that increasing of these two parameters leads to reduction of dynamic pull-in voltage. The reason is that increasing the Ω_r induces a higher value of centrifugal force to the system and M_r increases the system's inertia. Hence, dynamic pull-in happens at a lower voltage.

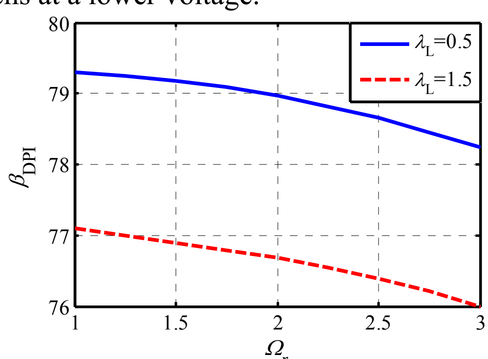


Figure 4. Non-dimensional dynamic pull-in voltage versus non-dimensional rotation frequency for two Casimir parameters λ_L (0.5 and 1.5).

Figure 4 represents the effect of the Casimir force on the dynamic pull-in behaviour of the system when M_r=0.1. It is obvious from Figure 4 that for higher values of the non-dimensional Casimir parameter λ_L, dynamic pull-in instability occurs at lower voltages. This is due to the fact that, beside the electrostatic force, the Casimir force also attracts the movable electrode toward the fixed substrate. Therefore, it is so essential to account for the effect of Casimir force especially for systems with lower initial gaps. Because, according to Eq. (14), the Casimir parameter λ_L increases with the decrease of the gap parameter.

4. Conclusion

This study aimed to investigate the effect of the Casimir force on the dynamic pull-in voltage of a micro-bridge gyroscope. Hamilton's principle was utilized to derive the size-dependent governing equations of motion based on the modified couple stress theory. Galerkin's weighted residual method was then employed to obtain the associated reduced ordinary differential equations of motion which solved through the fourth-order Runge-Kutta method. The accuracy of the present model was also validated by those available in the literature for simpler systems without the effect of the Casimir attraction. Finally, a parametric study was also conducted to account for the effects of different parameters on the dynamic pull-in behaviour of the system. It was found that increasing the angular velocity and the proof mass leads to the reduction of dynamic pull-in voltage of the system. Furthermore, the results revealed that considering the effect of the Casimir force had a noticeable influence on reducing the threshold of the dynamic pull-in instability of the system especially for those with lower initial gaps.

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