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On groups satisfying a symmetric Engel word

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Abstract We show that a finite group satisfying the law $[y,_n x] = [x,_n y]$ $(n > 1)$ is nilpotent and utilizing the results of Macdonalds on the structure of groups satisfying the law $[y, x] = [x, y]$, we investigate groups satisfying both of the laws $[y, x] = [x, y]$ and $[y,_n x] = [x,_n y]$ for small n. Our results can be applied to obtain special commutators, which can be expressed as the product of commutators squares.

Keywords Symmetry · Engel word · Engel group · Nilpotent group

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1 Introduction

Let G be a finite group. A word $w = w(x, y)$ is called *symmetric* on the group G if $w(g_1, g_2) = w(g_2, g_1)$, for all $g_1, g_2 \in G$. Now let $E_n = E_n(x, y) = [y,_n x]$ be the nth Engel word. Then G is said to be an E_n-symmetric group if E_n is symmetric on

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G. If G is finite and $E_n \equiv 1$, then it is known that G is nilpotent. In this paper, we shall generalize this result by showing that G is still nilpotent if E_n $(n \geq 2)$ is symmetric on G. See [4] and references therein for further results on equality of Engel words. If P is an elementary abelian 2-group and ϕ is a fixed-point-free automorphism of P of odd prime order p then $G = P \rtimes \langle \phi \rangle$ is clearly a finite E_1-symmetric group, which is not nilpotent. So E_1-symmetric groups are not necessarily nilpotent. Macdonald [3] have shown that if G is an E_1-symmetric group, then $G/Z(G)$ is metabelian and Neumann's construction of 3-metablelian non-metabelian groups indicates that E_1-symmetric groups are not necessarily metabelian. We show that an E_1-symmetric group is an extension of a 2-group by an abelian group of odd order. Also we will present some more results concerning groups, which are both E_1- and E_n-symmetric for small n and we give conditions under which some commutators of weight > 1 can be expressed as the product of commutators squares.

2 E_n-symmetric groups, $n \geq 2$

It is well-known that a finite Engel group is nilpotent (see [7, Theorem 12.3.4]). Now we generalize this result by showing that a finite group satisfying a symmetric n-Engel word $(n \geq 2)$ is also nilpotent.

Theorem 2.1 *If G is a finite E_n-symmetric group $(n \geq 2)$, then G is nilpotent.*

Proof First suppose that G is solvable. Clearly $[y, x] \in G^{(1)}$ and if $[y,_{1+kn} x] \in G^{(k+1)}$ then

$$[y,_{1+(k+1)n} x] = [y,_{1+kn} x,_n x] = \big[x,_n [y,_{1+kn} x]\big] \in G^{(k+2)}.$$

Hence we reach to $[y,_{1+(m-1)n} x] = 1$ by choosing m to be the solvability length of G that is G is an Engel group. Using [7, Theorem 12.3.4] we conclude that G is nilpotent. Now suppose that G is a finite E_n-symmetric group and the result holds for all groups of order less than $|G|$. Since the proper subgroups of G inherit the same property as G does, each of which should be nilpotent. Hence by [7, Theorem 9.1.9], G is solvable and consequently G is nilpotent. □

Theorem 2.1 can be generalized in the following form.

Corollary 2.2 *Let G be a finite group. If for each $x, y \in G$ there exist integers $m_{x,y}, n_{x,y} > 1$ such that $[y,_{m_{x,y}} x] = [x,_{n_{x,y}} y]$, then G is nilpotent.*

3 E_1-symmetric groups

E_1-symmetric groups are different from E_n-symmetric groups $(n \geq 2)$ as they are not nilpotent in general. Macdonald have shown in [3, Theorem 4] and its consequences that an E_1-symmetric group G satisfies the following properties:

(1) $[[x, y], [x, z]] = 1$;
(2) $[[x_1, x_2], [x_3, x_4]] = \big[[x_{\pi_1}, x_{\pi_2}], [x_{\pi_3}, x_{\pi_4}]\big]$;

(3) $[\gamma_2(G), \gamma_3(G)] = 1$;
(4) $[\gamma_2(G), \gamma_2(G), G] = 1$;
(5) $G'^4 = 1$,

for each $x, y, z, x_1, x_2, x_3, x_4 \in G$ and $\pi \in S_4$.

Also the results (4) and (5) are in a sense best possible due to the construction of 3-metabelian non-metabelian groups by Neumann [5], where a group is called 3-metabelian if all subgroups generated by three elements is metabelian. Such groups are E_1-symmetric and form the variety defined by (1).

It is investigated by several authors that, when a commutator (or an expression involving commutators) can be expressed as the product of special elements of the group, say squares, cubes and etc.? For example, it is proved that any commutator $[y, x]$ is the product of squares, $[y, x, x]$ is the product of cubes and the fifth Engel word $[y, x, x, x, x, x]$ is the product of forth powers (see [1,2]).

Utilizing (3) and (4), we observe that in an arbitrary group G the commutators of the form $[[a, b, c], [d, e]]$ and $[[a, b], [c, d], e]$ can be expressed as the product of commutators squares.

We shall give further properties and applications of E_1-symmetric groups.

The structure of finite E_1-symmetric groups can be described in an alternative way as follows.

Theorem 3.1 *If G is a finite E_1-symmetric group, then G is a semidirect product of a normal Sylow 2-subgroup P by an abelian subgroup H of odd order. In particular, $[P, H]$ is an elementary abelian 2-group.*

Proof Let $x \in G$ be a 2-element of order 2^n. Then, $[y,_{2^n} x] = [y, x^{2^n}] = 1$ for each $y \in G$ and consequently x is a right Engel element. By [6], the set of all right Engel elements of G coincides with the Fitting subgroup $F(G)$ of G. Thus, $F(G)$ possesses all Sylow 2-subgroups of G. Let P be a Sylow 2-subgroup of G [hence of $F(G)$]. As $F(G)$ is a characteristic nilpotent subgroup of G its Sylow 2-subgroup P is normal in G and hence by Schur-Zassenhaus theorem [7, Theorem 9.1.2], P has a complement H in G. Clearly, H is abelian. Finally, we know by (2) that $[P, H]' \subseteq [P', H'] = 1$. Hence $[P, H]$ is an elementary abelian 2-group and the proof is complete. □

Here is a simple example of an E_1-symmetric group.

Example Let F be a field of characteristic 2 and let $G = U(n, F)$ be the Unitriangular group of matrices of dimension $n \leq 4$ over F. Then G is an E_1-symmetric group.

To prove our results on E_1- and E_n-symmetric groups, we need some more properties of E_1-symmetric groups.

Lemma 3.2 *If G is an E_1-symmetric group, then*

(i) $[[x, y_1, \ldots, y_m], [x, z_1, \ldots, z_n]] = 1$;
(ii) $[[x, y], [z, w]] = [x, y, z, w][x, y, w, z]$,

where $x, y, z, w, y_1, \ldots, y_m, z_1, \ldots, z_n \in G$ and m, n are natural numbers.

Proof (i) The case $(m, n) = (1, 1)$ is given in (1). Now if the result is true for (m, n), then by expanding $\big[[x, y_1, \ldots, y_{m-1}, y_m y_{m+1}], [x, z_1, \ldots, z_n]\big] = 1$ we obtain the result for $(m + 1, n)$ and similarly for $(m, n + 1)$.

(ii) If $x, y, z, w \in G$, then

$$[x, y, zw] = [x, y, w][x, y, z][x, y, z, w].$$

On the other hand, by applying (i),

$$\begin{aligned} [x, y, zw] &= [x, y, wz[z, w]] \\ &= [[x, y], [z, w]]\,[x, y, wz]^{[z,w]} \\ &= [[x, y], [z, w]]\,[x, y, z]^{[z,w]}[x, y, w]^{[z,w]}[x, y, w, z]^{[z,w]} \\ &= [[x, y], [z, w]]\,[x, y, z][x, y, w][x, y, w, z]. \end{aligned}$$

From these two identities and applying (i) once more we obtain the result. □

4 E_1- and E_n-symmetric groups, $n \geq 2$

In this section, we investigate groups satisfying both E_1- and E_n-symmetric properties for small n. We will show that in an E_1-symmetric group both E_2- and E_3-symmetric properties are equivalent to the 2- and 3-Engel properties, respectively.

Lemma 4.1 *If G is an E_1- and E_n-symmetric group ($n \geq 2$), then G is an $(n + 1)$-Engel group.*

Proof Let $x, y \in G$. Then, by Lemma 3.2(i)

$$[y,_{n+1} x] = [[y, x],_n x] = [x,_n [y, x]] = \big[[y, x, x], [y, x],_{n-2} [y, x]\big] = 1,$$

as required. □

Lemma 4.2 *If G is an E_1-symmetric group, then $[y, x, x, y] = [x, y, y, x]$, for all $x, y, \in G$.*

Proof If $x, y \in G$, then

$$[y, x, x, y] = [y, x^2, y] = [x^2, y, y] = [x^2, y^2]$$

and

$$[x, y, y, x] = [x, y^2, x] = [y^2, x, x] = [y^2, x^2],$$

from which the result follows. □

Theorem 4.3 *If G is both E_1- and E_2-symmetric group, then G is nilponent of class at most* 2.

Proof By the assumption, $[y, x, x] = [x, y, y]$ holds for all $x, y \in G$ and hence expanding $[xy, x, x] = [x, xy, xy]$ we obtain $[y, x, x] = 1$, that is G is a 2-Engel group. Now let $x, y, z \in G$. Then

$$\begin{aligned}[x, y, z] &= [z, [x, y]] = [z, u^2v^2w^2] \\ &= [z, w^2][z, v^2]^{w^2}[z, u^2]^{v^2w^2} = [z, w, w][z, v, v]^{w^2}[z, u, u]^{v^2w^2} = 1,\end{aligned}$$

where $u = x^{-1}$, $v = xy^{-1}$ and $w = y$. Hence G is nilpotent of class at most 2. □

Theorem 4.3 asserts that in an arbitrary group G and elements $x, y, z \in G$, there always exist elements x_i, y_i and z_i, w_i such that

$$[x, y, z] = \prod [x_i, y_i]^2 \prod [z_i, w_i, w_i][w_i, z_i, z_i]^{-1}.$$

Thus the commutators $[x, y, z]$ can be expressed as the product of commutators squares if and only if the elements $[y, x, x][x, y, y]^{-1}$ have the same property.

Theorem 4.4 *If G is both E_1- and E_3-symmetric group, then G is a 3-Engel group.*

Proof Let $x, y \in G$. Expanding the identity $[y, yx, yx, yx] = [yx, y, y, y]$ and utilizing Lemmas 4.1 and 4.2, we obtain

$$[y, x, x, x] = [y, x, y, x, y][y, x, y, x, y, x] = [y, x, y, x, y]^x,$$

which implies that

$$[y, x, x, x] = [y, x, y, x, y].$$

By replacing y by y^2 in the last identity we get

$$\begin{aligned}[x, y, x, y, x] &= [y, x, x, y, x] = [x, y, y, x, x] = [x, y^2, x, x] = [y^2, x, x, x] \\ &= [y^2, x, y^2, x, y^2] = [x, y^2, y^2, x, y^2] = [x, y, y, y, y, x, y^2] = 1.\end{aligned}$$

Therefore $[y, x, y, x, y] = 1$ and consequently $[y, x, x, x] = 1$, as required. □

Theorem 4.5 *If G is both E_1- and E_3-symmetric group, then G is metabelian.*

Proof Expanding the identity $[z, xy, xy, xy] = 1$ in conjunction with the fact that, by Lemma 3.2(i, ii), $[a, b, c, d] = [a, b, d, c]$ when $a, b, c, d \in \{x, y, z\}$, we obtain $[z, x, y, y] = [z, y, x, x]$. Moreover

$$[z, y, y, x, x] = [z, y^2, x, x] = [z, x, y^2, y^2] = [z, x, y, y, y, y] = 1,$$

from which we deduce that $[w, z, y, x, x] = 1$, for all $x, y, z, w \in G$.

Now expanding $[w, z, xy, xy] = [w, xy, z, z]$ in conjunction with the previous identities we reach to $[w, z, x, y] = [w, z, y, x]$. Therefore $G'' = \langle 1 \rangle$. □

From Theorem 4.5, we conclude that each commutator $[[x, y], [z, w]]$ can be expressed as

$$[[x, y], [z, w]] = \prod [x_i, y_i]^2 \prod [z_i, w_i, w_i, w_i][w_i, z_i, z_i, z_i]^{-1}.$$

Hence in an arbitrary group G, the commutators $[[x, y], [z, w]]$ can be expressed as the product of commutators squares if the elements $[y, x, x, x][x, y, y, y]^{-1}$ have the same property.

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