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An extension of the Gegenbauer pseudospectral method for the time fractional Fokker-Planck equation

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The time fractional Fokker-Planck equation has been used in many physical transport problems which take place under the influence of an external force field. In this paper we examine pseudospectral method based on Gegenbauer polynomials and Chebyshev spectral differentiation matrix to solve numerically a class of initial-boundary value problems of the time fractional Fokker-Planck equation on a finite domain. The presented method reduces the main problem to a generalized Sylvester matrix equation, which can be solved by the global generalized minimal residual method. Some numerical experiments are considered to demonstrate the accuracy and the efficiency of the proposed computational procedure.

KEYWORDS

anomalous diffusion, fractional Fokker-Planck equations, Gegenbauer polynomials, generalized Sylvester matrix equation, global GMRES, pseudospectral methods, Riemman-Liouville fractional derivative

1 | INTRODUCTION

Over the last few years, ordinary and partial differential equations of fractional order have been focused on many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering.[1] Consequently, considerable attention has been given to the solutions of fractional ordinary differential, integral, and fractional partial differential equations of physical interest.[2,3] In most fractional partial differential equations, it is not possible to find the exact analytic solutions, so approximation and numerical techniques should be used.[4-10]

Recently, the phenomena of anomalous diffusion have been observed in many physical systems, eg, pollutant transport through porous media, electron transfer in semiconductor and nuclear proliferation.[1,2] The study of anomalous diffusion is also of special significance in chemistry, biology, environmental science, and even in economics.[11,12] The fractional Fokker-Planck equations (FPEs) have been recently treated by a number of authors and are found to be a useful tool for the description of transport dynamics in complex systems that are governed by anomalous diffusion and non-Markovian processes.[13,14] Fractional derivatives play a key role in modelling particle transport in anomalous diffusion.[15,16] For the description of anomalous transport in the presence of an external field, Metzler and Klafter[13] introduced a time-fractional FPE (TFFPE) as an extension of the FPE.[17,18] As a model for subdiffusion in an external potential field $v(x)$, the TFFPE

$$\frac{\partial w(x,t)}{\partial t} = D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{v'(x)}{m\eta_\alpha} + K_\alpha\frac{\partial^2}{\partial x^2}\right]w(x,t) \tag{1}$$

has been suggested.[14,15] Here, $K_\alpha > 0$ denotes the generalized diffusion coefficient with dimension $[K_\alpha] = cm^2 s^{-\alpha}$, and η_α is the generalized friction coefficient with dimension $[\eta_\alpha] = s^{\alpha-1}$. Equation 1 uses the Riemann-Liouville fractional derivative of order $1-\alpha$, defined by[1,2]

$$D_t^{1-\alpha} w(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{w(x,s)}{(t-s)^{1-\alpha}}\, ds, \tag{2}$$

where $0 \le \alpha < 1$[1,2,17,19] and $\Gamma(x)$ is the Euler gamma function. For convenience, let $f(x) = \frac{v'(x)}{m\eta_\alpha}$, the TFFPE can be written as

$$\frac{\partial w(x,t)}{\partial t} = D_t^{1-\alpha} \left[\frac{\partial}{\partial x} f(x) + K_\alpha \frac{\partial^2}{\partial x^2} \right] w(x,t). \tag{3}$$

There have been some attempts in deriving numerical methods and analysis techniques for the fractional FPEs. Ma and Liu[20] have considered a one-dimensional generalized fractional nonlinear FPE and extract the exact solution expressed by q-exponetial function. Zhuang et al[21] proposed an implicit numerical method and 2 solution techniques for improving its order of convergence for this equation. Liu et al[5,6] presented practical numerical methods to solve fractional FPEs. Pseudo-spectral methods on Gauss or Gauss-Lobatto nodes are high-order methods. A Legendre pseudospectral method has been developed for the determination of the control parameter in a 3-dimensional diffusion equation in Shamsi and Dehghan.[22] In Izadkhah and Saberi Nadjafi,[23] for solving the time fractional convection-diffusion equations with variable coefficients, Gegenbauer spectral method have been proposed. The aim of this work is to present an effective numerical method for the TFFPE (3) along with supplementary conditions by considering the Gegenbauer pseudospectral (GPS) method for the fractional derivative in time, while the spatial derivatives is approximated by pseudospectral method based on Chebyshev-Gauss-Lobatto (CGL) nodes. Finally, we obtain a generalized Sylvester matrix equation. There are different schemes for solving a generalized Sylvester matrix equation. Among these approaches, we consider the global GMRES(k) suggested by Jbilou et al,[24] which is also used in Tohidi and Toutounian[25] to numerically solve a class of 1-dimensional parabolic partial differential equations.[26,27]

Let us turn the attention towards the issue that the advantage of the pseudospectral method based on Gegenbauer polynomials compared with pseudospectral methods based on Legendre polynomials is the parameter λ in the construction of Gegenbauer polynomials. Through which, one can exploit appropriate parameter λ related to the order of fractional derivative in pseudospectral method. This approach can improve the pseudospectral method compared with other ones. In the current paper, the temporal derivative is considered in fractional form, and due to this, we approximate the solution of the time fractional Fokker-Planck by Gegenbauer polynomials in time.

The outline of this paper is considered as follows. In the next Section, we introduce some preliminaries of fractional calculus and present practical properties of Gegenbauer polynomials. For the sake of simply application of the proposed method, the problem is reformulated in Section 3. In Section 4, we introduce the GPS method for solving (3) with the appropriate initial-boundary conditions. Section 5 is devoted to report 2 numerical experiments which demonstrate the accuracy of the proposed numerical scheme for solving (3) compare with those of the methods presented in Chen et al and Deng.[4,28] And Section 6 includes some concluding remarks.

2 | PRELIMINARIES

2.1 | Fractional calculus

In this subsection, we give some basic definitions and properties of the fractional calculus theory which will be used further in this paper. For more details see Podlubny.[2] For the finite interval $[t_0, T]$, we define the Riemann-Liouville fractional integrals and derivatives.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, of a function $w(x,t)$ with respect to time is defined as

$${}_{t_0}J_t^\alpha w(x,t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-\zeta)^{\alpha-1} w(x,\zeta)\, d\zeta, \quad t > t_0, \quad \alpha > 0, \tag{4}$$

$${}_{t_0}J_t^0 w(x,t) = w(x,t). \tag{5}$$

Definition 2.2. The fractional partial derivative of $w(x,t)$ of order α, with respect to time, in the Riemann-Liouville sense defined is as

$$ {}_{t_0}D_t^{\alpha} w(x,t) = \frac{\partial}{\partial t}\left({}_{t_0}J_t^{1-\alpha} w(x,t)\right), \tag{6}$$

for $0 \le \alpha < 1$, $t > t_0$.

For ease of use, we will drop the low terminal t_0 in definitions (4) and (6) whenever $t_0 = 0$. For $0 < \alpha < 1$, the following properties hold[2]

$$ J_t^{\alpha}(D_t^{\alpha} w(x,t)) = w(x,t) - [J_t^{1-\alpha} w(x,t)]_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{7}$$

$$ D_t^{\alpha}(J_t^{\alpha} w(x,t)) = w(x,t). \tag{8}$$

Also, according to equation 2.66 given in Podlubny,[2] we have

$$ J_t^{1-\alpha}\left(\frac{\partial w(x,t)}{\partial t}\right) = D_t^{\alpha} w(x,t) - \frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{9}$$

Also, we have an important property of Riemman-Liouville fractional derivative of order $\alpha > 0$ corresponding (6) when $t_0 = -1$ and $w(x,t) \equiv (1+t)^r$, $r \in \mathbb{N}_0$ as following

$$ {}_{-1}D_t^{\alpha}(1+t)^r = \frac{r!}{\Gamma(r-\alpha+1)}(1+t)^{r-\alpha}, \quad r \in \mathbb{N}_0, \tag{10}$$

where $\mathbb{N}_0$ stands for the set $\{0, 1, 2, \dots\}$.

The following lemma gives us an equivalent form of the Equation 1.

Lemma 2.1. *If $w(x,t) \in C_{x,t}^{2,1}([a,b] \times [0,T])$, then we can rewrite*

$$ \frac{\partial w(x,t)}{\partial t} = D_t^{1-\alpha}\left[\frac{\partial}{\partial x} f(x) + K_{\alpha}\frac{\partial^2}{\partial x^2}\right] w(x,t) \tag{11}$$

in the following equivalent form

$$ D_t^{\alpha} w(x,t) - \frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)} = \left[\frac{\partial}{\partial x} f(x) + K_{\alpha}\frac{\partial^2}{\partial x^2}\right] w(x,t). \tag{12}$$

Proof. See Chen et al.[4] □

2.2 | Gegenbauer polynomials

Spectral methods typically use special cases of Jacobi polynomials, which are the eigenfunctions of the singular Strüm-Liouville problem. The Gegenbauer polynomials $C_n^{(\lambda)}(x)$ of order n associated with the real parameter λ, $\lambda > -\frac{1}{2}$, $\lambda \neq 0$, appear as the eigensolutions to the following singular Strüm-Liouville problem in the finite domain $[-1, 1]$,

$$ -\frac{d}{dx}\left((1-x^2)^{\lambda+\frac{1}{2}}\frac{dy(x)}{dx}\right) = (1-x^2)^{\lambda-\frac{1}{2}}\varrho y(x),$$

and the corresponding eigenvalues are

$$ \varrho_n^{\lambda} = n(n+2\lambda).$$

With the first 2 polynomials

$$ C_0^{(\lambda)}(x) = 1, \quad C_1^{(\lambda)}(x) = 2\lambda x,$$

the remaining polynomials of the subsequent order are given through the following recurrence formula

$$ C_{n+1}^{(\lambda)}(x) = \frac{2(\lambda+n)}{n+1} x C_n^{(\lambda)}(x) - \frac{2\lambda+n-1}{n+1} C_{n-1}^{(\lambda)}(x), \quad n \ge 1.$$

The special cases $\lambda = 0, 1$ and $\frac{1}{2}$ correspond to the Chebyshev polynomials of first kind, second kind, and Legendre polynomials, respectively. Actually, for Chebyshev polynomials of first kind, we have[29]

$$T_n(x) = \frac{n}{2}\lim_{\lambda\to 0}\frac{C_n^{(\lambda)}(x)}{\lambda}, \quad n \geq 1.$$

The weight function for the Gegenbauer polynomials is $w^\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}$, while they satisfy the weighted orthogonality relation[29]

$$\int_{-1}^{1} C_m^{(\lambda)}(x)C_n^{(\lambda)}(x)w^\lambda(x)\,dx = \gamma_n^\lambda \delta_{mn},$$

where

$$\gamma_n^\lambda = \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma^2(\lambda)},$$

and δ_{mn} is the Kronecker delta function. As mentioned in Szegö,[29] the sequence of Gegenbauer polynomials $\{C_n^{(\lambda)}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{w^\lambda}[-1,1]$-orthogonal system, where $L^2_{w^\lambda}[-1,1]$ is the weighted space defined by

$$L^2_{w^\lambda}[-1,1] := \{u | u \text{ is measurable and } \|u\|_{w^\lambda} < \infty\},$$

equipped with the following norm

$$\|u\|_{w^\lambda} = \left(\int_{-1}^{1} |u(x)|^2 w^\lambda(x)\,dx\right)^{\frac{1}{2}},$$

and the following inner product

$$(u,v)_{w^\lambda} = \int_{-1}^{1} u(x)v(x)w^\lambda(x)\,dx, \quad \forall u,v \in L^2_{w^\lambda}[-1,1].$$

The analytic form of the Gegenbauer polynomials $C_n^\lambda(t)$ of degree n, associated with the parameter λ is given by[30]

$$C_n^{(\lambda)}(t) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)}\sum_{r=0}^{n}\frac{(-1)^{n-r}\Gamma(n+r+2\lambda)}{2^r\Gamma(\lambda+r+\frac{1}{2})(n-r)!r!}(t+1)^r. \tag{13}$$

We prove the following theorem, which is needed in the sequel.

Theorem 1. *Let $C_n^{(\lambda)}(t), t \in [-1,1]$ denotes the Gegenbauer polynomial of degree n, associated with the parameter λ, and suppose $\alpha > 0$. Then the derivative of order α in the Riemman-Liouville sense for $C_n^{(\lambda)}(t)$ is*

$${}_{-1}D_t^\alpha(C_n^{(\lambda)}(t)) = \sum_{r=0}^{n} b_{n,r}^{(\lambda,\alpha)}(t+1)^{r-\alpha},$$

where

$$b_{n,r}^{(\lambda,\alpha)} = \frac{(-1)^{n-r}(2\lambda)_{n+r}}{2^r(n-r)!(\lambda+\frac{1}{2})_r\Gamma(r+1-\alpha)},$$

and the notation $(\beta)_k$ stands for Pochhammer symbol which defined by $(\beta)_0 = 1$ and $(\beta)_k = \beta(\beta+1)\cdots(\beta+k-1)$.

Proof. Taking the Riemman-Liouville fractional derivative ${}_{-1}D_t^\alpha$ of $C_n^{(\lambda)}(t)$ in finite series representation (13), by using (10), we have

$$\begin{aligned}
{}_{-1}D_t^\alpha(C_n^{(\lambda)}(t)) &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)}\sum_{r=0}^{n}\frac{(-1)^{n-r}\Gamma(n+r+2\lambda)}{2^r\Gamma(\lambda+r+\frac{1}{2})(n-r)!r!}\ {}_{-1}D_t^\alpha((t+1)^r)\\
&= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)}\sum_{r=0}^{n}\frac{(-1)^{n-r}\Gamma(n+r+2\lambda)}{2^r\Gamma(\lambda+r+\frac{1}{2})(n-r)!r!}\frac{r!}{\Gamma(r-\alpha+1)}\ (t+1)^{r-\alpha}\\
&= \sum_{r=0}^{n}\frac{(-1)^{n-r}(2\lambda)_{n+r}}{2^r(n-r)!(\lambda+\frac{1}{2})_r\Gamma(r+1-\alpha)}\ (t+1)^{r-\alpha}.
\end{aligned} \tag{14}$$

The last expression in (14), is obtained by using Pochhammer notation. Thus, the proof is completed. □

For computing the coefficients $b_{n,r}^{(\lambda,\alpha)}$, we use the following recursive formula:

$$b_{n,r+1}^{(\lambda,\alpha)} = \frac{-(2\lambda+n+r)(n-r)}{2(\lambda+\frac{1}{2}+r)(r+1-\alpha)} b_{n,r}^{(\lambda,\alpha)}, \qquad r = 0, 1, \dots, n,$$

with

$$b_{n,0}^{(\lambda,\alpha)} = \frac{(-1)^n (2\lambda)_n}{n!\Gamma(1-\alpha)},$$

for any integer $n \in \mathbb{N}_0$.

2.3 | Spectral differentiation matrix

For applying the pseudospectral method, we use the CGL nodes $z_{N,0}, \dots, z_{N,N}$[31] defined by

$$z_{N,j} = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, 1, \dots, N. \tag{15}$$

Let $l_{N,j}(x), j = 0, 1, \dots, N$, be the Lagrange polynomials based on CGL nodes, that are expressed as

$$l_{N,j}(x) = \prod_{\substack{i=0 \\ i\neq j}}^{N} \frac{x - z_{N,i}}{z_{N,j} - z_{N,i}}, \quad j = 0, \dots, N,$$

with Kronecker property

$$l_{N,j}(z_{N,k}) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

In pseudospectral methods, it is crucial that to express the derivatives $l_{N,j}^{(m)}(x)$ in terms of $l_{N,j}(x)$, ie,

$$l_{N,j}^{(m)}(x) = \sum_{k=0}^{N} l_{N,j}^{(m)}(z_{N,k}) l_{N,k}(x), \quad j = 0, \dots, N. \tag{16}$$

Let $\boldsymbol{\phi}_N(x) = [l_{N,0}(x), l_{N,1}(x), \dots, l_{N,N}(x)]$, and $\tilde{\boldsymbol{\phi}}_N(\xi)$ denotes the $(N-1)$-dimensional row vector obtained from $\boldsymbol{\phi}_N(\xi)$ by removing the first and last components. Then, from (16) we have

$$\boldsymbol{\phi}_N^{(m)}(x) = \boldsymbol{\phi}_N(x) \mathbf{D}_{N+1}^{(m)}, \tag{17}$$

and

$$\tilde{\boldsymbol{\phi}}_N^{(m)}(\xi) = \boldsymbol{\phi}_N(\xi) \tilde{\mathbf{D}}_{N+1}^{(m)}, \tag{18}$$

where $\mathbf{D}_{N+1}^{(m)}$ is the differentiation matrix of order m with the following entries:

$$\left[\mathbf{D}_{N+1}^{(m)}\right]_{i+1,j+1} = l_{N,j}^{(m)}(z_{N,i}), \quad i,j = 0, \dots, N,$$

and $\tilde{\mathbf{D}}_{N+1}^{(m)}$ denotes the matrix obtained from $\mathbf{D}_{N+1}^{(m)}$ by removing the first and last columns. Note that the subscript $N+1$ in $\mathbf{D}_{N+1}^{(m)}$ stands for dimension.

More computationally practical methods for deriving these entries, in accurate and stable manner, can be found in Canuto et al, Baltensperger and Trummer, Costa and Don, and Weideman and Reddy.[31-34] For 2 special cases $m = 1$ and $m = 2$, $\mathbf{D}_{N+1}^{(m)}$ has the following explicit formula in terms of CGL nodes[31]:

$$\left[\mathbf{D}_{N+1}^{(1)}\right]_{p+1,l+1} = \begin{cases} \frac{\bar{c}_p}{\bar{c}_l} \frac{(-1)^{p+l}}{z_{N,p} - z_{N,l}}, & p \neq l, \\ -\frac{z_{N,l}}{2(1-z_{N,l}^2)}, & 1 \leq p = l \leq N-1, \\ \frac{2N^2+1}{6}, & p = l = 0, \\ -\frac{2N^2+1}{6}, & p = l = N, \end{cases} \tag{19}$$

and

$$\left[\mathbf{D}_{N+1}^{(2)}\right]_{p+1,l+1}=\begin{cases}\frac{(-1)^{p+l}}{\bar{c}_l}\frac{z_{N,p}^2+z_{N,p}z_{N,l}-2}{\left(1-z_{N,p}^2\right)\left(z_{N,p}-z_{N,l}\right)^2}, & \begin{matrix}1\le p\le N-1\\ 0\le l\le N, p\ne l\end{matrix},\\ -\frac{(N^2-1)\left(1-z_{N,p}^2\right)+3}{3\left(1-z_{N,p}^2\right)^2}, & 1\le p=l\le N-1,\\ \frac{2}{3}\frac{(-1)^l}{\bar{c}_l}\frac{(2N^2+1)(1-z_{N,l})-6}{(1-z_{N,l})^2}, & p=0,\quad 1\le l\le N,\\ \frac{2}{3}\frac{(-1)^{l+N}}{\bar{c}_l}\frac{(2N^2+1)(1+z_{N,l})-6}{(1+z_{N,l})^2}, & p=N,\quad 0\le l\le N-1,\\ \frac{N^4-1}{15}, & p=l=0,\quad p=l=N,\end{cases} \tag{20}$$

where

$$\bar{c}_j=\begin{cases}2,\ j=0,N,\\ 1,\ j=1,\ \dots\ ,N-1.\end{cases} \tag{21}$$

3 | PROBLEM REFORMULATION

In this section, we reformulate the TFFPE:

$$\frac{\partial w(x,t)}{\partial t}=D_t^{1-\alpha}\left[\frac{\partial}{\partial x}f(x)+K_\alpha\frac{\partial^2}{\partial x^2}\right]w(x,t),\quad a\le x\le b,\quad 0<t\le T \tag{22}$$

subject to the initial condition

$$w(x,0)=\varphi(x),\quad a\le x\le b, \tag{23}$$

and the boundary conditions

$$w(a,t)=g_1(t),\quad w(b,t)=g_2(t),\quad 0<t\le T. \tag{24}$$

We assume that the problem (Equations 22-24) have a unique solution $w(x,t)\in C_{x,t}^{2,1}([a,b]\times[0,T])$. By using Lemma 2.1, Equation 22 can be considered as follows:

$$D_t^\alpha w(x,t)-\frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)}=\frac{\partial}{\partial x}(f(x)w(x,t))+K_\alpha\frac{\partial^2}{\partial x^2}w(x,t). \tag{25}$$

To simplify our method, we reformulate the problem (Equations 22-24), by applying the transformation

$$v(x,t)=w(x,t)+\mu(x,t),$$

with

$$\mu(x,t)=\frac{x-b}{b-a}g_1(t)+\frac{a-x}{b-a}g_2, \tag{26}$$

which transforms the boundary conditions (24) to the homogeneous boundary conditions. So, we have the following problem,

$$D_t^\alpha v(x,t)-\frac{v(x,0)t^{-\alpha}}{\Gamma(1-\alpha)}=\frac{\partial}{\partial x}(f(x)v(x,t))+K_\alpha\frac{\partial^2}{\partial x^2}v(x,t)+\psi(x,t),\qquad 0<t\le T,\quad a\le x\le b, \tag{27}$$

with the initial condition

$$v(x,0)=\tilde{\varphi}(x),\quad a\le x\le b, \tag{28}$$

and the homogeneous boundary conditions

$$v(a,t)=v(b,t)=0,\quad 0<t\le T, \tag{29}$$

where

$$\psi(x,t) = D_t^{\alpha}\mu(x,t) - \frac{\mu(x,0)t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{\partial}{\partial x}(f(x)\mu(x,t)), \tag{30}$$

$$\tilde{\varphi}(x) = \varphi(x) + \mu(x,0). \tag{31}$$

We note that $\frac{\partial^2}{\partial x^2}\mu(x,t) \equiv 0$. In addition, for considering the problem (Equations 27-29) on the domain $\xi \in \Lambda = [-1,1]$ in space and $\tau \in I = [-1,1]$ in time, we also use the following change of variables:

$$t = \frac{T(\tau+1)}{2}, \quad \tau \in I,$$

and

$$x = \frac{1}{2}((b-a)\xi + (b+a)), \quad \xi \in \Lambda.$$

So, the problem (Equations 27-29) can be written as follows:

$$\begin{aligned}(T/2)^{-\alpha}\left[{}_{-1}D_{\tau}^{\alpha}u(\xi,\tau) - \frac{u(\xi,-1)(\tau+1)^{-\alpha}}{\Gamma(1-\alpha)}\right] &= \left(\frac{2}{b-a}\right)\frac{\partial}{\partial\xi}(F(\xi)u(\xi,\tau)) + K_{\alpha}\left(\frac{2}{b-a}\right)^2\frac{\partial^2}{\partial\xi^2}u(\xi,\tau)\\ &\quad + \Psi(\xi,\tau), \quad (\xi,\tau) \in \Lambda \times I,\end{aligned} \tag{32}$$

$$u(\xi,-1) = \bar{\varphi}(\xi), \quad \xi \in \Lambda, \tag{33}$$

$$u(-1,\tau) = u(1,\tau) = 0, \quad \tau \in I, \tag{34}$$

where

$$u(\xi,\tau) := v\left(\frac{1}{2}((b-a)\xi + (b+a)), \frac{T(\tau+1)}{2}\right), \quad F(\xi) := f\left(\frac{1}{2}((b-a)\xi + (b+a))\right),$$

$$\Psi(\xi,\tau) := \psi\left(\frac{1}{2}((b-a)\xi + (b+a)), \frac{T(\tau+1)}{2}\right), \quad \bar{\varphi}(\xi) := \tilde{\varphi}\left(\frac{1}{2}((b-a)\xi + (b+a))\right).$$

Moreover, by the Riemman-Liouville definition of fractional derivative (6) for $t_0 = -1$, we have

$${}_{-1}D_{\tau}^{\alpha}u(\xi,\tau) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_{-1}^{\tau}(\tau-s)^{-\alpha}u(\xi,s)\,ds, \quad 0 \le \alpha < 1. \tag{35}$$

4 | DESCRIPTION OF THE METHOD

According to the idea of spectral method, we consider the approximate solution of the problem (Equations 32-34) in the following form

$$U(\xi,\tau) = \sum_{j=1}^{N-1}\sum_{k=0}^{M}u_{j,k}l_{N,j}(\xi)C_k^{(\lambda)}(\tau), \tag{36}$$

where $u_{j,k}, j = 1, \ldots, N-1, k = 0, \ldots, M$ are the unknown coefficients to be determined. We mention that, in (36), due to the homogeneous boundary conditions, in (36) j varies from 1 to $N-1$.

By defining the unknown vector function

$$\Theta(\tau) = [\theta_1(\tau), \theta_2(\tau), \ldots, \theta_{N-1}(\tau)]^T, \tag{37}$$

with

$$\theta_j(\tau) = \sum_{k=0}^{M}u_{j,k}C_k^{(\lambda)}(\tau), \tag{38}$$

we have

$$U(\xi,\tau) = \sum_{j=1}^{N-1}\theta_j(\tau)l_{N,j}(\xi) = \tilde{\boldsymbol{\phi}}_N(\xi)\Theta(\tau), \tag{39}$$

and

$$ {}_{-1}D_{\tau}^{\alpha}U(\xi,\tau) = \tilde{\boldsymbol{\phi}}_N(\xi)\ {}_{-1}D_{\tau}^{\alpha}\Theta(\tau). \tag{40}$$

In addition, we consider the spectral approximation

$$ F_N(\xi) = \boldsymbol{\phi}_N(\xi)\mathbf{F} \tag{41}$$

for $F(\xi)$, where $\mathbf{F} = [F(z_{N,0}), F(z_{N,1}), \ \dots\ , F(z_{N,N})]^T$.

By using (39), (41), (17), and (18), we have

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)\frac{\partial}{\partial\xi}(F(\xi)U(\xi,\tau)) + \left(\frac{2}{b-a}\right)^2 K_\alpha \frac{\partial^2}{\partial\xi^2}U(\xi,\tau)\\
&\quad\simeq \left(\frac{2}{b-a}\right)\frac{\partial}{\partial\xi}(F_N(\xi)U(\xi,\tau)) + \left(\frac{2}{b-a}\right)^2 K_\alpha \frac{\partial^2}{\partial\xi^2}U(\xi,\tau)\\
&\quad= \left[\left(\frac{2}{b-a}\right)\frac{d}{d\xi}\left(F_N(\xi)\tilde{\boldsymbol{\phi}}_N(\xi)\right) + \left(\frac{2}{b-a}\right)^2 K_\alpha \frac{d^2}{d\xi^2}\tilde{\boldsymbol{\phi}}_N(\xi)\right]\Theta(\tau),\\
&\quad= \left[\left(\frac{2}{b-a}\right)\left(\boldsymbol{\phi}_N(\xi)\mathbf{D}_{N+1}^{(1)}\mathbf{F}\tilde{\boldsymbol{\phi}}_N(\xi) + F_N(\xi)\boldsymbol{\phi}_N(\xi)\tilde{\mathbf{D}}_{N+1}^{(1)}\right) + \left(\frac{2}{b-a}\right)^2 K_\alpha \boldsymbol{\phi}_N(\xi)\tilde{\mathbf{D}}_{N+1}^{(2)}\right]\Theta(\tau).
\end{aligned} \tag{42}$$

By using the notation $[\mathbf{A}]_{(p)}$ for the pth row of the matrix $\mathbf{A}$, we note that

$$ \boldsymbol{\phi}_N(\xi)\tilde{\mathbf{D}}_{N+1}^{(1)} = \tilde{\boldsymbol{\phi}}_N(\xi)\hat{\tilde{\mathbf{D}}}_{N+1}^{(1)} + \mathbf{r}^{(1)}(\xi), \tag{43}$$

$$ \boldsymbol{\phi}_N(\xi)\tilde{\mathbf{D}}_{N+1}^{(2)} = \tilde{\boldsymbol{\phi}}_N(\xi)\hat{\tilde{\mathbf{D}}}_{N+1}^{(2)} + \mathbf{r}^{(2)}(\xi), \tag{44}$$

$$ \boldsymbol{\phi}_N(\xi)\mathbf{D}_{N+1}^{(1)} = \tilde{\boldsymbol{\phi}}_N(\xi)\hat{\mathbf{D}}_{N+1}^{(1)} + \tilde{\mathbf{r}}^{(1)}(\xi), \tag{45}$$

where

$$\begin{aligned}
\mathbf{r}^{(1)}(\xi) &= \left[\tilde{\mathbf{D}}_{N+1}^{(1)}\right]_{(1)} l_{N,0}(\xi) + \left[\tilde{\mathbf{D}}_{N+1}^{(1)}\right]_{(N+1)} l_{N,N}(\xi),\\
\mathbf{r}^{(2)}(\xi) &= \left[\tilde{\mathbf{D}}_{N+1}^{(2)}\right]_{(1)} l_{N,0}(\xi) + \left[\tilde{\mathbf{D}}_{N+1}^{(2)}\right]_{(N+1)} l_{N,N}(\xi),\\
\tilde{\mathbf{r}}^{(1)}(\xi) &= \left[\mathbf{D}_{N+1}^{(1)}\right]_{(1)} l_{N,0}(\xi) + \left[\mathbf{D}_{N+1}^{(1)}\right]_{(N+1)} l_{N,N}(\xi),
\end{aligned}$$

and $\hat{\tilde{\mathbf{D}}}_{N+1}^{(1)}$, $\hat{\tilde{\mathbf{D}}}_{N+1}^{(2)}$, and $\hat{\mathbf{D}}_{N+1}^{(1)}$ are the matrices obtained from $\tilde{\mathbf{D}}_{N+1}^{(1)}$, $\tilde{\mathbf{D}}_{N+1}^{(2)}$, and $\mathbf{D}_{N+1}^{(1)}$, respectively, by removing the first and the last rows. As we will see, there is no need to compute the 3 terms $\mathbf{r}^{(1)}(\xi)$, $\mathbf{r}^{(2)}(\xi)$, and $\tilde{\mathbf{r}}^{(1)}(\xi)$ in (43) to (45), because these are eliminated during the collocation procedure with CGL points.

By substituting (43) to (45) in (42), we have

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)\frac{\partial}{\partial\xi}(F_N(\xi)U(\xi,\tau)) + \left(\frac{2}{b-a}\right)^2 K_\alpha \frac{\partial^2}{\partial\xi^2}U(\xi,\tau)\\
&\quad= \left[\left(\frac{2}{b-a}\right)\left(\tilde{\boldsymbol{\phi}}_N(\xi)\hat{\mathbf{D}}_{N+1}^{(1)}\mathbf{F}\tilde{\boldsymbol{\phi}}_N(\xi) + F_N(\xi)\tilde{\boldsymbol{\phi}}_N(\xi)\hat{\tilde{\mathbf{D}}}_{N+1}^{(1)}\right) + \left(\frac{2}{b-a}\right)^2 K_\alpha \tilde{\boldsymbol{\phi}}_N(\xi)\hat{\tilde{\mathbf{D}}}_{N+1}^{(2)} + \mathbf{r}(\xi)\right]\Theta(\tau),
\end{aligned} \tag{46}$$

where

$$ \mathbf{r}(\xi) = \left(\frac{2}{b-a}\right)\left[\tilde{\mathbf{r}}^{(1)}(\xi)\mathbf{F}\tilde{\boldsymbol{\phi}}_N(\xi) + F_N(\xi)\mathbf{r}^{(1)}(\xi) + \left(\frac{2}{b-a}\right)K_\alpha \mathbf{r}^{(2)}(\xi)\right].$$

It is important to note that $\mathbf{r}(\xi)$ in the internal CGL points (15) is a zero vector of dimension $N-1$, because

$$ \tilde{\mathbf{r}}^{(1)}(z_{i,N}) = \mathbf{0}, \quad \mathbf{r}^{(1)}(z_{i,N}) = \mathbf{0}, \quad \mathbf{r}^{(2)}(z_{i,N}) = \mathbf{0}, \quad i = 1, 2, \ \dots\ , N-1. \tag{47}$$

Now, we consider (32) to (34) for the approximate solution $U(\xi,\tau)$, given in (36), instead of the exact solution $u(\xi,\tau)$. We substitute (40) and (46) in (32) and collocate the resulting equation in the internal grid points $z_{i,N}, i = 1, 2, \dots, N-1$. This and the fact that the $\tilde{\boldsymbol{\phi}}_N(z_{N,p})$ is the pth row of the identity matrix I_{N-1}, imply that

$$\begin{aligned} {}_{-1}D_\tau^\alpha \theta_p(\tau) = \left(\frac{T}{2}\right)^\alpha \Bigg[\left(\frac{2}{b-a}\right) \left([\hat{\mathbf{D}}_{N+1}^{(1)}]_{(p)} \mathbf{F} \theta_p(\tau) + F(z_{N,p}) [\hat{\tilde{\mathbf{D}}}_{N+1}^{(1)}]_{(p)} \Theta(\tau) \right) \\ + \left(\frac{2}{b-a}\right)^2 K_\alpha [\hat{\tilde{\mathbf{D}}}_{N+1}^{(2)}]_{(p)} \Theta(\tau) + \Psi(z_{N,p},\tau) \Bigg] + \bar{\varphi}(z_{N,p}) \frac{(\tau+1)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} \tag{48}$$

Based on the above relation, we can establish the following matrix form:

$${}_{-1}D_\tau^\alpha \Theta(\tau) = \left(\frac{T}{2}\right)^\alpha \left[\left(\frac{2}{b-a}\right) \left(\mathrm{diag}(\hat{\mathbf{D}}_{N+1}^{(1)} \mathbf{F}) + \mathrm{diag}(\tilde{\mathbf{F}}) \hat{\tilde{\mathbf{D}}}_{N+1}^{(1)} \right) + \left(\frac{2}{b-a}\right)^2 K_\alpha \hat{\tilde{\mathbf{D}}}_{N+1}^{(2)} \right] \Theta(\tau) + \left(\frac{T}{2}\right)^\alpha \Omega(\tau) + \Phi(\tau), \tag{49}$$

where $\tilde{\mathbf{F}}$ is the vector obtained from $\mathbf{F}$ by removing the first and last components, and

$$\begin{aligned} \Phi(\tau) &= \frac{(\tau+1)^{-\alpha}}{\Gamma(1-\alpha)} [\bar{\varphi}(z_{1,N}), \bar{\varphi}(z_{2,N}), \dots, \bar{\varphi}(z_{N-1,N})]^T, \\ \Omega(\tau) &= [\Psi(z_{1,N},\tau), \Psi(z_{2,N},\tau), \dots, \Psi(z_{N-1,N},\tau)]^T. \end{aligned}$$

Here, we use notation "diag($\mathbf{v}$)" for a vector $\mathbf{v}$, which means a diagonal matrix that puts the vector $\mathbf{v}$ on the main diagonal.

On the other hand, from (37) and (38), the vector function $\Theta(\tau)$ can be written as follows:

$$\Theta(\tau) = \Lambda_u \Upsilon_M^\lambda(\tau), \tag{50}$$

where Λ_u is a $(N-1)\times(M+1)$ matrix with unknown entries $(u_{j,k})$, $1 \le j \le N-1, 0 \le k \le M$, and

$$\Upsilon_M^\lambda(\tau) = \left[C_0^{(\lambda)}(\tau), C_1^{(\lambda)}(\tau), \dots, C_M^{(\lambda)}(\tau) \right]^T.$$

By using (50), the approximate solution $U(\xi,\tau)$, given in (39), can be written as follows:

$$U(\xi,\tau) = \tilde{\boldsymbol{\phi}}_N(\xi) \Lambda_u \Upsilon_M^\lambda(\tau). \tag{51}$$

By Theorem 1, we have

$${}_{-1}D_\tau^\alpha \Theta(\tau) = \Lambda_u \Upsilon_M^{\lambda,\alpha}(\tau), \tag{52}$$

with

$$\Upsilon_M^{\lambda,\alpha}(\tau) = \left[b_{0,0}^{(\lambda,\alpha)}(1+\tau)^{-\alpha}, b_{1,0}^{(\lambda,\alpha)}(1+\tau)^{-\alpha} + b_{1,1}^{(\lambda,\alpha)}(1+\tau)^{1-\alpha}, \dots, \sum_{r=0}^{M} b_{M,r}^{(\lambda,\alpha)}(1+\tau)^{r-\alpha} \right]^T.$$

By substituting (51) and (52) in (49), we get

$$\begin{aligned} \Lambda_u \Upsilon_M^{\lambda,\alpha}(\tau) = \left(\left(\frac{T}{2}\right)^\alpha \left[\left(\frac{2}{b-a}\right) \left(\mathrm{diag}(\hat{\mathbf{D}}_{N+1}^{(1)} \mathbf{F}) + \mathrm{diag}(\tilde{\mathbf{F}}) \hat{\tilde{\mathbf{D}}}_{N+1}^{(1)} \right) + \left(\frac{2}{b-a}\right)^2 K_\alpha \hat{\tilde{\mathbf{D}}}_{N+1}^{(2)} \right] \right) \Lambda_u \Upsilon_M^\lambda(\tau) \\ + \left(\frac{T}{2}\right)^\alpha \Omega(\tau) + \Phi(\tau). \end{aligned} \tag{53}$$

For determining the matrix of unknowns Λ_u, we collocate (53) in $\tau_{M,j} j = 0, 1, \dots, M$, where $\tau_{M,i}, i = 0, 1, \dots M$, are the real simple roots of the Gegenbauer polynomial of degree $M+1$ associated with the parameter λ, and we obtain

$$\Lambda_u \Upsilon_M^{\lambda,\alpha}(\tau_{M,j}) + \mathbf{M} \Lambda_u \Upsilon_M^\lambda(\tau_{M,j}) = \left(\frac{T}{2}\right)^\alpha \Omega(\tau_{M,j}) + \Phi(\tau_{M,j}), \quad j = 0, 1, \dots, M, \tag{54}$$

where

$$\mathbf{M} = -\left(\frac{T}{2}\right)^\alpha \left[\left(\frac{2}{b-a}\right) \left(\mathrm{diag}(\hat{\mathbf{D}}_{N+1}^{(1)} \mathbf{F}) + \mathrm{diag}(\tilde{\mathbf{F}}) \hat{\tilde{\mathbf{D}}}_{N+1}^{(1)} \right) + \left(\frac{2}{b-a}\right)^2 K_\alpha \hat{\tilde{\mathbf{D}}}_{N+1}^{(2)} \right].$$

Finally, Equation 54 can be summarized in the following generalized Sylvester matrix equation

$$\Lambda_u \Upsilon_M^{\lambda,\alpha} + \mathbf{M}\Lambda_u \Upsilon_M^{\lambda} = \mathbf{C}, \tag{55}$$

where $\mathbf{C} = (\frac{T}{2})^{\alpha}\Omega + \Phi$ is a $(N-1)\times(M+1)$ matrix and the matrices Ω, Φ, Υ_M^{λ} and $\Upsilon_M^{\lambda,\alpha}$ are defined as follows:

$$\begin{aligned}
\Omega &= [\Omega(\tau_{M,0}), \Omega(\tau_{M,1}), \dots, \Omega(\tau_{M,M})],\\
\Phi &= [\Phi(\tau_{M,0}), \Phi(\tau_{M,1}), \dots, \Phi(\tau_{M,M})],\\
\Upsilon_M^{\lambda} &= \left[\Upsilon_M^{\lambda}(\tau_{M,0}), \Upsilon_M^{\lambda}(\tau_{M,1}), \dots, \Upsilon_M^{\lambda}(\tau_{M,M})\right],
\end{aligned}$$

and

$$\Upsilon_M^{\lambda,\alpha} = \left[\Upsilon_M^{\lambda,\alpha}(\tau_{M,0}), \Upsilon_M^{\lambda,\alpha}(\tau_{M,1}), \dots, \Upsilon_M^{\lambda,\alpha}(\tau_{M,M})\right].$$

We note that the generalized Sylvester matrix Equation 55 has a unique solution if the matrix $\left((\Upsilon_M^{\lambda,\alpha})^T \otimes \mathbf{I}_{N-1} + (\Upsilon_M^{\lambda})^T \otimes \mathbf{M}\right)$ is nonsingular, which throughout this paper, we assume that this condition is verified. As Jbilou et al[24] and Bouhamidi and Jbilou,[35] for solving the generalized Sylvester matrix Equation 55, the restarted global generalized minimal residual (GMRES) algorithm, denoted by GlGMRES(k), can be used. We exploit the modified global Arnoldi algorithm described as Algorithm 1 to construct an F-orthonormal basis $V_1, V_2, \dots, V_k$ for the corresponding matrix Krylov subspace $\mathcal{K}_k(A, V_1)$, associated with the matrix Equation 55. In Algorithm 1, $\|A\|_F$ is the Ferobenius norm of the matrix A, defined by $\|A\|_F = \sqrt{\text{tr}(A^T A)}$, where tr($B$) denotes the trace of the matrix B. As mentioned in Bouhamidi and Jbilou,[35] to save memory and CPU-time requirements, the global GMRES method should be used in a restarted mode. The restarted global GMRES algorithm for solving generalized Sylvester Equation 55 is summarized as Algorithm 2. We note that γ_{k+1} is the last component of the vector $g_k = \|R_0\|_F \tilde{Q}_k e_1$.

Algorithm 1 Modified global Arnoldi algorithm for the matrix Equation 55

1. Choose an $N \times (M+1)$ matrix V_1 such that $\|V_1\|_F = 1$ ($V_1 = V/\|V\|_F$)
2. For $j = 1, 2, \dots, k$ Do

$\tilde{V} = V_j \mathbf{\Upsilon}_M^{\lambda,\alpha} + \mathbf{M} V_j \mathbf{\Upsilon}_M^{\lambda}$.
For $i = 1, 2, \dots, j$ Do

$h_{i,j} = \langle \tilde{V}, V_i \rangle_F$,
$\tilde{V} = \tilde{V} - h_{i,j} V_i$,

End Do
$h_{j+1,j} = \|\tilde{V}\|_F$. If $h_{j+1,j} = 0$ then Stop
$V_{j+1} = \tilde{V}/h_{j+1,j}$

End Do

Algorithm 2 Global GMRES(k) algorithm for the matrix Equation (55)

1. Choose X_0, a tolerance ϵ and set $iter = 0$.

Compute $R_0 = \mathbf{C} - X_0 \mathbf{\Upsilon}_M^{\lambda,\alpha} - \mathbf{M} X_0 \mathbf{\Upsilon}_M^{\lambda}$, $\beta = \|R_0\|_F$, and $V_1 = R_0/\beta$

2. Construct the F-orthonormal basis $V_1, V_2, \dots, V_k$ by Algorithm 1
3. Determine y_k as solution of the least square problem:

$$\min_{y \in \mathbb{R}^k} \| \|R_0\|_F e_1 - \tilde{H}_k y \|_2$$

4. Compute $X_k = X_0 + \mathcal{V}_k(y_k \otimes \mathbf{I}_{M+1})$
5. Compute the residual R_k and $\|R_k\|_F$
6. If $\|R_k\|_F < \epsilon$ Then Stop
7. else $X_0 = X_k$, $R_0 = R_k$, $\beta = \|R_0\|_F$, $V_1 = R_0/\beta$ and $iter = iter + 1$, Go to 2

Further remarks. The functions μ and Ψ are used during discretization. These functions are defined in Equations 26 and 30, in a complex form. However, it is not necessary to obtain μ and Ψ in closed form. Because we just need the values of μ and Ψ in collocation points $(z_{N,p}, \tau_{M,j})$ to construct the final generalized Silvester matrix Equation 55. By using (26) and (30), we obtain the values of μ and Ψ in the collocation points. Also, we mention that the partial derivative can be approximated by using differentiation matrix and GPS approximation.

5 | NUMERICAL ILLUSTRATIONS

This section is devoted to the numerical experiments, for demonstrating the effectiveness of the GPS method to solve numerically the TFFPE (Equations 22-24). We implemented the GPS method with MATLAB 8.5 software in a PC laptop COREi3 with 2.13 GHz of CPU and 4 GB of RAM.

We used the global GMRES(20) Algorithm 2, for solving the associated generalized Sylvester matrix Equation 55, with the stopping criterion $\|R\|_F < 10^{-8}$, where $R = \mathbf{C} - \tilde{\Lambda}_u \Upsilon_M^{\lambda,\alpha} - \mathbf{M}\tilde{\Lambda}_u \Upsilon_M^{\lambda}$ for $\tilde{\Lambda}_u$ as an approximation of Λ_u. We calculated the computational order of the method presented in this article with

$$\text{C-order} = \log_2 \left(\frac{E_\infty(M_1, N_1)}{E_\infty(M_2, N_2)} \right),$$

where $E_\infty(M, N) = \max |w(x_i, T) - W(x_i, T)|$ for approximate solution $W(x, t)$ of exact solution $w(x, t)$.

Example 5.1. Consider the following TFFPE:

$$\frac{\partial w(x,t)}{\partial t} = D_t^{1-\alpha} \left[\frac{\partial}{\partial x}(-1) + \frac{\partial^2}{\partial x^2} \right] w(x,t), \quad 0 \le x \le 1, \quad 0 < t \le T, \quad 0 \le \alpha < 1,$$

subject to the initial condition

$$w(x, 0) = x(1 - x), \quad 0 \le x \le 1,$$

and the boundary conditions

$$w(0,t) = -\frac{3t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad 0 < t \le T,$$
$$w(1,t) = -\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad 0 < t \le T.$$

In this problem $f(x) = -1, x \in [0, 1], K_\alpha = 1$. The exact solution of the above problem is

$$w(x,t) = x(1-x) + (2x-3)\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)},$$

which may be verified by direct differentiation and substitution in the fractional differential equation, using the formula

$$D_t^{1-\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p+\alpha)} t^{p+\alpha-1}.$$

For $\alpha = 0.5$ and $T = 100$, by using the GPS method with $\lambda = 0.5$ and taking $M = 1$ and $N = 4$, we obtained the exact solution of this problem in $CPUtime = 0.0029$ s. We also presented in Table 1 the error $\|e^n\| = \left(h \sum_{i=1}^{M-1} (w(x_i, t_n) - w_{i,n})^2 \right)^{\frac{1}{2}}$ of the GL-BDIA and L1-CDIA methods (given in Chen et al[4]), and CPU time (s) of them by setting $h = k = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$.

TABLE 1 The error $\|e^n\|$ for the GL-BDIA and the L1-CDIA methods defined in Chen et al[4] for Example 5.1

k	h	$\|e^n\|$ for GL-BDIA	CPU Time, s	$\|e^n\|$ for L1-CDIA	CPU Time, s
$\frac{1}{5}$	$\frac{1}{5}$	1.68e-002	0.150	2.69e-006	0.101
$\frac{1}{10}$	$\frac{1}{10}$	8.77e-003	0.768	1.02e-006	0.475
$\frac{1}{20}$	$\frac{1}{20}$	4.49e-003	5.085	3.98e-007	2.629
$\frac{1}{40}$	$\frac{1}{40}$	2.27e-003	37.334	1.59e-007	19.167

TABLE 2 The maximum error and computational orders obtained by the Gegenbauer pseudospectral method for Example 5.2

M	N	$E_\infty(M,N)$	C-order	CPU Time, s
6	6	7.94e-004	…	0.205
6	8	9.33e-005	3.0892	0.512
8	8	8.82e-005	0.0811	1.191
8	10	5.86e-005	0.5898	2.530

TABLE 3 The maximum error of the predictor-corrector approach combined with the method of line[28] for Example 5.2

k	$\max \lvert W_{h,k}(x_i, 0.3) - w(x_i, 0.3)\rvert$
0.00012	2.04e-004
0.00006	1.46e-004
0.00003	1.18e-004
0.000015	1.040e-004

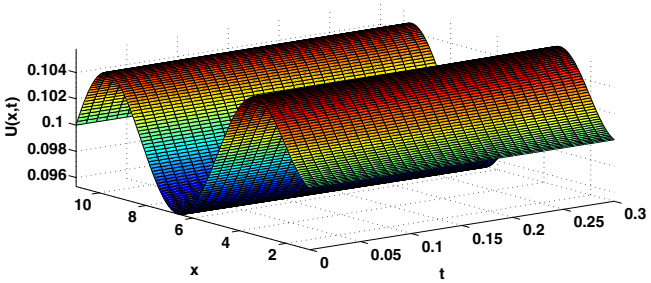

FIGURE 1 Plot of the approximate solution obtained by the Gegenbauer pseudospectral method with $N = 8, M = 6, \lambda = 0.2$, and $\alpha = 0.2$ for Example 5.2 [Colour figure can be viewed at wileyonlinelibrary.com]

Example 5.2. As the last example, we consider the following TFFPE[28]:

$$\frac{\partial w(x,t)}{\partial t} = D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\left(-\frac{\sin x + 6}{6}\right) + 2\frac{\partial^2}{\partial x^2}\right] w(x,t), \quad 1 \le x \le 11, \quad 0 < t \le T, \quad 0 < \alpha \le 1,$$

subject to the initial condition

$$w(x,0) = 0.10, \quad 1 \le x \le 11,$$

and the boundary conditions

$$w(0,t) = 0.10, \quad 0 < t \le T,$$

$$w(1,t) = 0.10, \quad 0 < t \le T,$$

In this problem, $f(x) = -\frac{\sin x+6}{6}, x \in [1, 11], K_\alpha = 2$. Here, we consider the numerical solution of L1-CDIA scheme, given in Chen et al[4] (with $k = 0.000015$ and $h = 0.03125$) as the exact solution of this problem. For comparison, we present the results obtained for $W(x_i, 0.3) = U(2x_i - 1, 1)$, in which $x_i = 1 + ih$, for $i = 1, 2, \dots, N$, where $N = \frac{b-a}{h}$. Table 2 represents the numerical results of the GPS method for $\alpha = 0.8$ and $T = 0.3$ with $\lambda = 0.8$ and different values of M and N. The results of the predictor-corrector approach combined with the method of line,[28] and the L1-FDIA method[4] for $\alpha = 0.8$, $h = 0.0625$, $T = 0.3$, and different time step sizes k, are presented in Tables 3 and 4, respectively. Moreover, the approximate solutions of the problem, obtained by the GPS method, with $\alpha = 0.2$ and $\lambda = 0.2$ for $N = 8, M = 6$ and $N = 12, M = 8$, are plotted in Figures 1 and 2, respectively. Tables 5 and 6 show that the numerical results have a stability behavior of GPS method. In Tables 5 and 6, we consider $\alpha = 0.8$ with $\lambda = 0.8$ and $\alpha = 0.2$ with $\lambda = 0.2$, respectively. Comparison of Legendre pseudospectral method ($\lambda = 0.5$) and GPS method with $\alpha = 0.2$ and $\lambda = 0.2$ are presented in Table 7. These results confirm the accuracy of the proposed spectral approach.

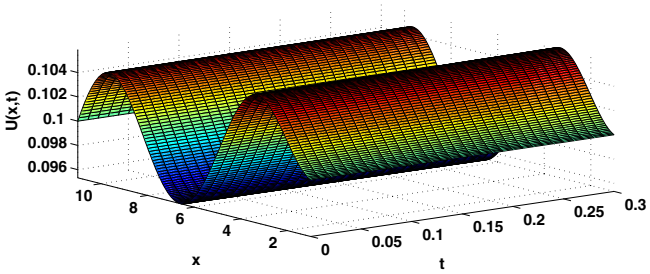

FIGURE 2 Plot of the approximate solution obtained by the Gegenbauer pseudospectral method with $N = 12, M = 8, \lambda = 0.2$, and $\alpha = 0.2$ for Example 5.2 [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 4 The maximum error of the L1-FDIA method for Example 5.2

k	$\max \lvert W_{h,k}(x_i, 0.3) - w(x_i, 0.3) \rvert$
0.00024	1.3936e-004
0.00012	1.3942e-004
0.00006	1.3949e-004
0.00003	1.3947e-004

TABLE 5 Numerical results for Example 5.2 with $\alpha = 0.8$ and $\lambda = 0.8$

x	GPS for $M = 8$ and $N = 8$	GPS for $M = 10$ and $N = 10$	Exact Results
1.5	0.100690	0.100676	0.100688
3.5	0.104495	0.104494	0.104499
5.5	0.097624	0.097593	0.097564
7.5	0.097653	0.097674	0.097702
9.5	0.104340	0.104338	0.104316

Abbreviation: GPS, Gegenbauer pseudospectral.

TABLE 6 Numerical results for Example 5.2 with $\alpha = 0.2$and $\lambda = 0.2$

x	GPS for $M = 8$ and $N = 8$	GPS for $M = 10$ and $N = 10$	Exact Results
1.5	0.101049	0.101045	0.101068
3.5	0.106167	0.106184	0.106230
5.5	0.098053	0.097976	0.097918
7.5	0.096852	0.096887	0.096909
9.5	0.104731	0.104748	0.104748

Abbreviation: GPS, Gegenbauer pseudospectral.

TABLE 7 Comparison of Legendre pseudospectral method($\lambda = 0.5$) and Gegenbauer pseudospectral method with $\alpha = 0.2$and $\lambda = 0.2$for Example 5.2

M	N	$E_\infty(M, N)$	C-order	CPU Time, s	Legendre Pseudospectral
6	6	1.1875e-003	…	0.201	1.1862e-003
6	8	1.5068e-004	2.9784	0.519	1.5107e-004
8	8	1.5024e-004	0.0042	1.238	1.5054e-004
8	10	6.1002e-005	1.3003	2.541	6.1415e-005

From Tables 3-5, we observed that the approximate solutions obtained by the GPS method are more accurate than those of the methods presented in.[4,28] So we can conclude that the GPS method is an effective method for solving the TFFPE.

6 | CONCLUSION

We have presented a GPS method for solving numerically a 1-dimensional TFFPE using Chebyshev spectral differentiation matrix in spatial direction. As we observed, the new method reduces the main problem to a generalized Sylvester matrix equation, which can be solved by the global GMRES(k) algorithm. Numerical illustrations show that the proposed method is effective and approximate solutions are satisfactory for small M and N. It is worth pointing out that the proposed GPS method can be extended for 2-dimensional fractional partial differential equations, which are our future works.

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