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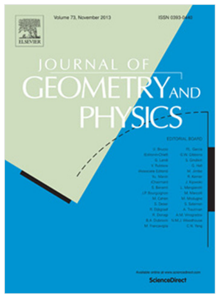

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2-CAPABILITY AND 2-NILPOTENT MULTIPLIER OF FINITE DIMENSIONAL NILPOTENT LIE ALGEBRAS

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Abstract. In the present context, we investigate to obtain some more results about 2-nilpotent multiplier $\mathcal{M}^{(2)}(L)$ of a finite dimensional nilpotent Lie algebra L. For instance, we characterize the structure of $\mathcal{M}^{(2)}(H)$ when H is a Heisenberg Lie algebra. Moreover, we give some inequalities on dim $\mathcal{M}^{(2)}(L)$ to reduce a well known upper bound on 2-nilpotent multiplier as much as possible. Finally, we show that $H(m)$ is 2-capable if and only if m=1.

1. Introduction

For a finite group G, let G be the quotient of a free group F by a normal subgroup R, then the c-nilpotent multiplier $\mathcal{M}^{(c)}(G)$ is defined as

$$R \cap \gamma_{c+1}(F)/\gamma_{c+1}[R, F],$$

in which $\gamma_{c+1}[R, F] = [\gamma_c[R, F], F]$ for $c \geq 1$. It is an especial case of the Baer invariant [3] with respect to the variety of nilpotent groups of class at most c. When $c = 1$, the abelian group $\mathcal{M}(G) = \mathcal{M}^{(1)}(G)$ is more known as the Schur multiplier of G and it is much more studied, for instance in [11, 14, 18].

Since determining the c-nilpotent multiplier of groups can be used for the classification of group into isoclinism classes(see [2]), there are multiple papers concerning this subject.

Recently, several authors investigated to develop some results on the group theory case to Lie algebra. In [22], analogues to the c-nilpotent multiplier of groups, for a given Lie algebra L, the c-nilpotent multiplier of L is defined as

$$\mathcal{M}^{(c)}(L) = R \cap F^{c+1}/[R, F]^{c+1},$$

in which L presented as the quotient of a free Lie algebra F by an ideal R, $F^{c+1} = \gamma_{c+1}(F)$ and $[R, F]^{c+1} = \gamma_{c+1}[R, F]$. Similarly, for the case $c = 1$, the abelian Lie algebra $\mathcal{M}(L) = \mathcal{M}^{(1)}(L)$ is more studied by the first author and the others (see for instance [4, 5, 6, 8, 9, 10, 15, 16, 17, 24]).

The c-nilpotent multiplier of a finite dimensional nilpotent Lie algebra L is a new field of interest in literature. The present context is involving the 2-nilpotent multiplier of a finite dimensional nilpotent Lie algebra L. The aim of the current paper is divided into several steps. In [22, Corollary 2.8], by a parallel result to the group theory result, showed for every finite nilpotent Lie algebra L, we have

$$\dim(\mathcal{M}^{(2)}(L)) + \dim(L^3) \leq \frac{1}{3}n(n-1)(n+1). \tag{1.1}$$

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Here we prove that abelian Lie algebras just attain the bound 1.1. It shows that always Ker $\theta = 0$ in [22, Corollary 2.8 (ii)a].

Since Heisenberg algebras $H(m)$ (a Lie algebra of dimension $2m+1$ with $L^2 = Z(L)$ and dim $(L^2) = 1$) have interest in several areas of Lie algebra, similar to the result of [5, Example 3] and [13, Theorem 24], by a quite different way, we give explicit structure of 2-nilpotent multiplier of these algebras. Among the other results since the Lie algebra which attained the upper bound 1.1 completely described in Lemma 2.6 (they are just abelian Lie algebras), by obtaining some new inequalities on dimension $\mathcal{M}^{(2)}(L)$, we reduce bound 1.1 for non abelian Lie algebras as much as possible.

Finally, among the class of Heisenberg algebras, we show that which of them is 2-capable. It means which of them is isomorphic to $H/Z_2(H)$ for a Lie algebra H. For more information about the capability of Lie algebras see [18, 21]. These generalized the recently results for the group theory case in [19].

2. Further investigation on 2-nilpotent multiplier of finite dimensional nilpotent Lie algebra

The present section illustrates to obtain further results on 2-nilpotent multiplier of finite dimensional nilpotent Lie algebra. At first we give basic definitions and known results for the seek of convenience the reader.

Let F be a free Lie algebra on an arbitrary totaly ordered set X. Recall from [25], the basic commutator on the set X, which is defined as follows, is a basis of F.

The elements of X are basic commutators of length one and ordered relative to the total order previously chosen. Suppose all the basic commutators a_i of length less than $k \geq 1$ have been defined and ordered. Then the basic commutators of length k to be defined as all commutators of the form $[a_i, a_j]$ such that the sum of lengths of a_i and a_j is k, $a_i > a_j$, and if $a_i = [a_s, a_t]$, then $a_j \geq a_t$. Also the number of basic commutators on X of length n, namely $l_d(n)$, is

$$\frac{1}{n}\sum_{m|n} \mu(m) d^{\frac{n}{m}},$$

where μ is the Möbius function.

From [8], let F be a fixed field, L, K be two Lie algebras and $[\ ,\ ]$ denote the Lie bracket. By an action of L on K we mean an F-bilinear map

$$(l, k) \in L \times K \mapsto {}^{l}k \in K \text{ satisfying}$$

${}^{[l,l']}k = {}^{l}({}^{l'}k) - {}^{l'}({}^{l}k)$ and ${}^{l}[k, k'] = [{}^{l}k, k'] + [k, {}^{l}k']$, for all $c \in F, l, l' \in L, k, k' \in K$.

When L is a subalgebra of a Lie algebra P and K is an ideal in P, then L acts on K by Lie multiplications ${}^{l}k = [l, k]$. A crossed module is a Lie homomorphism $\sigma : K \to L$ together with an action of L on K such that

$$\sigma({}^{l}k) = [l, \sigma(k)] \text{ and } {}^{\sigma(k)}k' = [k, k'] \text{ for all } k, k' \in K \text{ and } l \in L.$$

Let $\sigma : L \to M$ and $\eta : K \to M$ be two crossed modules, L and K act on each other and on themselves by Lie. Then these actions are called compatible provided that

$$^{{}^{k}l}k' = {}^{k'}({}^{l}k) \text{ and } ^{{}^{l}k}l' = {}^{l'}({}^{k}l).$$

The non-abelian tensor product $L \otimes K$ of L and K is the Lie algebra generated by the symbols $l \otimes k$ with defining relations

$$c(l \otimes k) = cl \otimes k = l \otimes ck, (l + l') \otimes k = l \otimes k + l' \otimes k,$$

$$l \otimes (k + k') = l \otimes k + l \otimes k',\ {}^{l}l' \otimes k = l \otimes {}^{l'}k - l' \otimes {}^{l}k,\ l \otimes {}^{k}k' = {}^{k'}l \otimes k - {}^{k}l \otimes k',$$

$$[l \otimes k, l' \otimes k'] = -{}^{k}l \otimes {}^{l'}k',\ \text{for all } c \in F, l, l' \in L, k, k' \in K.$$

The non-abelian tensor square of L is a special case of tensor product $L \otimes K$ when $K = L$. Note that we denote the usual abelian tensor product $L \otimes_{\mathbb{Z}} K$, when L and K are abelian and the actions are trivial.

Let $L \square K$ be the submodule of $L \otimes K$ generated by the elements $l \otimes k$ such that $\sigma(l) = \eta(k)$. The factor Lie algebra $L \wedge K \cong L \otimes K / L \square K$ is called the exterior product of L and K, and the image of $l \otimes k$ is denoted by $l \wedge k$ for all $l \in L, k \in K$. Throughout the paper Γ is denoted the universal quadratic functor (see [8]).

Recall from [18], the exterior centre of a Lie algebra $Z^{\wedge}(L) = \{l \in L \mid l \wedge l' = 1_{L \wedge L},\ \forall\ l' \in L\}$ of L. It is shown that in [18] the exterior centre L is a central ideal of L which allows us to decide when Lie algebra L is capable, that is, whether $L \cong H/Z(H)$ for a Lie algebra H.

The following Lemma is a consequence of [18, Lemma 3.1].

Lemma 2.1. *Let L be a finite dimensional Lie algebra, L is capable if and only if $Z^{\wedge}(L) = 0$.*

The next two lemmas are special cases of [22, Proposition 2.1 (i)] when $c = 2$ and that is useful for proving the next theorem.

Lemma 2.2. *Let I be an ideal in a Lie algebra L. Then the following sequences are exact.*

(i) $\mathrm{Ker}(\mu_I^2) \to \mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(L/I) \to \frac{I \cap L^3}{[[I,L],L]} \to 0.$

(ii) $(I \wedge L/L^3) \wedge L/L^3 \to \mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(L/I) \to I \cap L^3 \to 0,$ *when* $[[I, L], L] = 0.$

Lemma 2.3. *Let I be an ideal of L, and put $K = L/I$. Then*

(i) $\dim \mathcal{M}^{(2)}(K) \le \dim \mathcal{M}^{(2)}(L) + \dim \dfrac{I \cap L^3}{[[I, L], L]}.$

(ii) *Moreover, if I is a 2-central subalgebra. Then*

(a). $(I \wedge L) \wedge L \to \mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(K) \to \dim I \cap L^3 \to 0.$

(b). $\dim \mathcal{M}^{(2)}(L) + \dim I \cap L^3 \le \dim \mathcal{M}^{(2)}(K) + \dim (I \otimes L/L^3) \otimes L/L^3.$

Proof. (i). Using Lemma 2.2 (i).

$(ii)(a)$. Since $[I, L] \subseteq Z(L)$, Ker $\mu_I^2 = (I \wedge L) \wedge L$ and $[[I, L], L] = 0$ by Lemma 2.2. It follows the result.

$(ii)(b)$. Since there is a natural epimorphism $(I \otimes L/L^3) \otimes L/L^3 \to (I \wedge L/L^3) \wedge L/L^3$, the result deduces from Lemma 2.2 (ii). □

The following theorem gives the explicit structure of the Schur multiplier of all Heisenberg algebra.

Theorem 2.4. [5, Example 3] and [13, Theorem 24] *Let $H(m)$ be Heisenberg algebra of dimension $2m + 1$. Then*

(i) $\mathcal{M}(H(1)) \cong A(2).$

(ii) $\mathcal{M}(H(m)) = A(2m^2 - m - 1)$ *for all* $m \ge 2.$

The following result comes from [20, Theorem 2.8] and shows the behavior of 2-nilpotent multiplier respect to the direct sum of two Lie algebras.

Theorem 2.5. *Let A and B be finite dimensional Lie algebras. Then*

$$\begin{aligned}\mathcal{M}^{(2)}(A \oplus B)) &\cong \mathcal{M}^{(2)}(A) \oplus \mathcal{M}^{(2)}(B) \oplus \big((A/A^2 \otimes_{\mathbb{Z}} A/A^2) \otimes_{\mathbb{Z}} B/B^2\big) \\ &\oplus \big((B/B^2 \otimes_{\mathbb{Z}} B/B^2) \otimes_{\mathbb{Z}} A/A^2\big).\end{aligned}$$

The following theorem is proved in [22] and will be used in the next contribution. At this point, we may give a short proof with a quite different way of [22, Proposition 1.2] as follows.

Theorem 2.6. *Let $L = A(n)$ be an abelian Lie algebra of dimension L. Then $\mathcal{M}^{(2)}(L) \cong A(\frac{1}{3}n(n-1)(n+1))$.*

Proof. We perform induction on n. Assume $n = 2$. Then Theorem 2.5 allows us to conclude that

$$\begin{aligned}\mathcal{M}^{(2)}(L) &\cong \mathcal{M}^{(2)}(A(1)) \oplus \mathcal{M}^{(2)}(A(1)) \oplus \big(A(1) \otimes_{\mathbb{Z}} A(1) \otimes_{\mathbb{Z}} A(1)\big) \\ &\oplus \big(A(1) \otimes_{\mathbb{Z}} A(1)) \otimes_{\mathbb{Z}} A(1)\big) \cong A(1) \oplus A(1) \cong A(2).\end{aligned}$$

Now assume that $L \cong A(n) \cong A(n-1) \oplus A(1)$. By using induction hypothesis and Theorem 2.5, we have

$$\begin{aligned}\mathcal{M}^{(2)}(A(n-1) \oplus A(1)) &\cong \mathcal{M}^{(2)}(A(n-1)) \oplus \big(A(n-1) \otimes_{\mathbb{Z}} A(n-1) \otimes_{\mathbb{Z}} A(1)\big) \\ &\oplus \big(A(1) \otimes_{\mathbb{Z}} A(1) \otimes_{\mathbb{Z}} A(n-1)\big) \\ &\cong A(\tfrac{1}{3}n(n-1)(n-2)) \oplus A((n-1)^2) \oplus A(n-1) \\ &\cong A(\tfrac{1}{3}n(n-1)(n+1)).\end{aligned}$$

□

The main strategy, in the next contribution, is to give the similar argument of Theorem 2.4 for the 2-nilpotent multiplier. In the first theorem, we obtain the structure of $\mathcal{M}^{(2)}(L)$ when L is non-capable Heisenberg algebra.

Theorem 2.7. *Let $L = H(m)$ be a non-capable Heisenberg algebra. Then*

$$\mathcal{M}^{(2)}(H(m)) \cong A(\frac{8m^3 - 2m}{3}).$$

Proof. Since L is non-capable, Lemma 2.1 implies $Z^{\wedge}(L) = L^2 = Z(L)$. Invoking Lemma 2.3 by putting $I = Z^{\wedge}(L)$, we have $\mathcal{M}^{(2)}(H(m)) \cong \mathcal{M}^{(2)}(H(m)/H(m)^2)$. Now result follows from Theorem 2.6. □

The following theorem from [18, Theorem 3.4] shows in the class of all Heisenberg algebras which one is capable.

Theorem 2.8. *$H(m)$ is capable if and only if $m = 1$.*

Corollary 2.9. *$H(m)$ is not 2-capable for all $m \geq 2$.*

Proof. Since every 2-capable Lie algebra is capable, the result follows from Theorem 2.8. □

Since $H(m)$ for all $m \geq 2$ is not 2-capable, we just need to discus about the 2-capability of $H(1)$. Here, we obtain 2-nilpotent multiplier of $H(1)$ and in the next section we show $H(1)$ is 2-capable.

Theorem 2.10. *Let $L = H(1)$. Then*

$$\mathcal{M}^{(2)}(H(1)) \cong A(5).$$

Proof. We know that $H(1)$ is in fact the free nilpotent Lie algebra of rank 2 and class 2. That is $H(1) \cong F/F^3$ in which F is the free Lie algebra on 2 letters x, y. The second nilpotent multiplier of $H(1)$ is $F^4 \cap F^3/[F^3, F, F]$ which is isomorphic to F^3/F^5 ant the latter is the abelian Lie algebra on the set of all basic commutators of weights 3 and 4 which is the set $\{[y, x, x], [y, x, y], [y, x, x, x], [y, x, x, y], [y, x, y, y]\}$. So the result holds. □

We summarize our result as below

Theorem 2.11. *Let $H(m)$ be Heisenberg algebra of dimension $2m + 1$. Then*

(i) $\mathcal{M}^{(2)}(H(1)) \cong A(5)$.

(ii) $\mathcal{M}^{(2)}(H(m)) = A(\frac{8m^3-2m}{3})$ *for all* $m \geq 2$.

The following Lemma lets us to obtain the structure of the 2-nilpotent multiplier of all nilpotent Lie algebras with $\dim L^2 = 1$.

Lemma 2.12. [15, Lemma 3.3] *Let L be an n-dimensional Lie algebra and* $\dim L^2 = 1$. *Then*

$$L \cong H(m) \oplus A(n - 2m - 1).$$

Theorem 2.13. *Let L be an n-dimensional Lie algebra with* $\dim L^2 = 1$. *Then*

$$\mathcal{M}^{(2)}(L) \cong \begin{cases} A(\frac{1}{3}n(n-1)(n-2)) & if\ m > 1, \\ \\ A(\frac{1}{3}n(n-1)(n-2)+3) & if\ m = 1. \end{cases}$$

Proof. By using Lemma 2.12, we have $L \cong H(m) \oplus A(n - 2m - 1)$. Using the behavior of 2-nilpotent multiplier respect to direct sum

$$\begin{aligned} \mathcal{M}^{(2)}(L) \quad &\cong \quad \mathcal{M}^{(2)}(H(m)) \oplus \ \mathcal{M}^{(2)}(A(n-2m-1)) \\ &\oplus \quad \big((H(m)/H(m)^2 \otimes_{\mathbb{Z}} H(m)/H(m)^2) \otimes_{\mathbb{Z}} A(n-2m-1)\big) \\ &\oplus \quad \big((A(n-2m-1) \otimes_{\mathbb{Z}} A(n-2m-1)) \otimes_{\mathbb{Z}} H(m)/H(m)^2\big). \end{aligned}$$

First assume that $m = 1$, then by virtue of Theorems 2.6 and 2.11

$$\mathcal{M}^{(2)}(H(1)) \cong A(5) \text{ and } \mathcal{M}^{(2)}(A(n-3)) \cong A(\frac{1}{3}(n-2)(n-3)(n-4)).$$

Thus

$$\begin{aligned} \mathcal{M}^{(2)}(L) \quad &\cong \quad A(5) \oplus \ A(\tfrac{1}{3}(n-2)(n-3)(n-4)) \\ &\oplus \quad \big(A(2) \otimes_{\mathbb{Z}} A(2)) \otimes_{\mathbb{Z}} A(n-3)\big) \\ &\oplus \quad \big((A(n-3) \otimes_{\mathbb{Z}} A(n-3)) \otimes_{\mathbb{Z}} A(2)\big) \\ &\cong \quad A(\tfrac{1}{3}n(n-1)(n-2)+3). \end{aligned}$$

The case $m \geq 1$ is obtained by a similar fashion. □

Theorem 2.14. *Let L be a n-dimensional nilpotent Lie algebra such that* $\dim L^2 = m (m \geq 1)$. *Then*

$$\dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3}(n-m)\big((n+2m-2)(n-m-1)+3(m-1)\big)+3.$$

In particular, $\dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3}n(n-1)(n-2)+3$. *The equality holds in last inequality if and only if* $L \cong H(1) \oplus A(n-3)$.

Proof. We do induction on m. For $m=1$, the result follows from Theorem 2.13. Let $m \geq 2$, and taking I a 1-dimensional central ideal of L. Since I and L/L^3 act to each other trivially we have $(I \otimes L/L^3) \otimes L/L^3 \cong (I \otimes_{\mathbb{Z}} \frac{L/L^3}{(L/L^3)^2}) \otimes_{\mathbb{Z}} \frac{L/L^3}{(L/L^3)^2})$. Thus by Lemma 2.3 $(ii)(b)$

$$\dim \mathcal{M}^{(2)}(L) + \dim I \cap L^3 \leq \dim \mathcal{M}^{(2)}(L/I) + \dim\, (I \otimes_{\mathbb{Z}} \frac{L/L^3}{(L/L^3)^2}) \otimes_{\mathbb{Z}} \frac{L/L^3}{(L/L^3)^2}).$$

Since

$$\dim \mathcal{M}^{(2)}(L/I) \leq \frac{1}{3}(n-m)\big((n+2m-5)(n-m-1)+3(m-2)\big),$$

we have

$$\begin{aligned} \dim \mathcal{M}^{(2)}(L) &\leq \tfrac{1}{3}(n-m)\big((n+2m-5)(n-m-1)+3(m-2)\big)+3+(n-m)^2 \\ &= \tfrac{1}{3}(n-m)\big((n+2m-2)(n-m-1)+3(m-1)\big)+3, \end{aligned}$$

as required. □

The following corollary shows that the converse of [22, Proposition 1.2] for $c=2$ is also true. In fact it proves always $\operatorname{Ker} \theta = 0$ in [22, Corollary 2.8 (ii)a].

Corollary 2.15. *Let L be a n-dimensional nilpotent Lie algebra. If* $\dim \mathcal{M}^{(2)}(L) = \frac{1}{3}n(n-1)(n+1)$, *then* $L \cong A(n)$.

3. 2-capability of Lie algebras

Following the terminology of [7] for groups, a Lie algebra L is said to be 2-capable provided that $L \cong H/Z_2(H)$ for a Lie algebra H. The concept $Z_2^*(L)$ was defined in [23] and it was proved that if $\pi : F/[R,F,F] \to F/R$ be a natural Lie epimorphism then

$$Z_2^*(L) = \pi(Z_2(F/[[R,F],F])), \text{ for } c \geq 0$$

The following proposition gives the close relation between 2-capability and $Z_2^*(L)$.

Proposition 3.1. *A Lie algebra L is 2-capable if and only if* $Z_2^*(L) = 0$.

Proof. Let L has a free presentation F/R, and $Z_2^*(L) = 0$. Consider the natural epimorphism $\pi : F/[[R,F],F] \twoheadrightarrow F/R$. Obviously

$$\operatorname{Ker} \pi = R/[[R,F],F] = Z_2(F/[[R,F],F]),$$

and hence $L \cong F/[[R,F],F]/Z_2(F/[[R,F],F])$, which is a 2-capable.

Conversely, let L is 2-capable and so $H/Z_2(H) \cong L$ for a Lie algebra H. Put $F/R \cong H$ and $Z_2(H) \cong S/R$. There is natural epimorphism $\eta : F/[[S,F],F] \twoheadrightarrow F/S \cong L$. Since $Z_2(F/[[R,F],F]) \subseteq \operatorname{Ker} \eta$, $Z_2^*(L) = 0$, as required. □

The following Theorem gives an instrumental tools to present the main.

Theorem 3.2. *Let I be an ideal subalgebra of L such that $I \subseteq Z_2^*(L)$. Then the natural Lie homomorphism $\mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(L/I)$ is a monomorphism.*

Proof. Let S/R and F/R be two free presentations of L and I, respectively. Looking the natural homomorphism

$$\phi : \mathcal{M}^{(2)}(L) \cong R \cap F^2/[[R,F],F] \to \mathcal{M}^{(2)}(L/I) \cong R \cap S^2/[[S,F],F]$$

and the fact that $S/R \subseteq Z_2(F/R)$ show ϕ has trivial kernel. The result follows. □

Theorem 3.3. *A Heisenberg Lie algebra $H(m)$ is 2-capable if and only if $m = 1$.*

Proof. Let $m \geq 2$, by Corollary 2.9 $H(m)$ is not capable so it is not 2-capable as well. Hence we may assume that $L \cong H(1)$. Let I be an ideal of L od dimension 1. Then L/I is abelian of dimension 2, and hence dim $\mathcal{M}^{(2)}(L) = 2$. On the other hands, Theorem 2.11 implies dim $\mathcal{M}^{(2)}(L) = 5$, and Theorem 3.2 deduces $\mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(L/I)$ can not be a monomorphism, as required. □

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