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The non-abelian tensor square of p-groups of order p^4

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In this paper, in the class of p-groups of order p^4, we obtain the non-abelian exterior square, the exterior center, the non-abelian tensor square, the tensor center and the third homotopy group of suspension of an Eilenberg–MacLane space $k(G, 1)$ of such groups.

Keywords: Non-Abelian tensor square; p-group.

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1. Introduction and Motivation

The concept of non-abelian tensor square which was introduced by Brown and Loday [5], is an applied topic in the K-theory and the homotopy theory. Since then several papers published in this subject and the reader can find more information in [3, 5, 6, 10]. Following the terminology in [5], let G and H be two groups act on each other compatibly and on themselves by conjugation, then the non-abelian tensor product of G and H, is the group generated by the symbols $g \otimes h$ with defining relations

$$g_1 g \otimes h = ({}^{g_1}g \otimes {}^{g_1}h)(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h)({}^{h}g \otimes {}^{h}h_1)$$

for all $g, g_1, h, h_1 \in G$. The tensor square of G is the special case of the non-abelian tensor product of two groups G and H when $G = H$.

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The exterior square $G \wedge G$ is obtained by imposing the additional relation $g \otimes g = 1_{\otimes}$ on $G \otimes G$. The image of $g \otimes h$ in $G \wedge G$ is denoted by $g \wedge h$ for all $g, h \in G$. From the defining relations of $G \otimes G$, there exists the commutator map $\kappa : G \otimes G \to [G, G]$ given by $\kappa(g \otimes h) = [g, h]$ which is a homomorphism. The kernel of κ is denoted by $J_2(G)$. Brown and Loday in [6] describe the role of $J_2(G)$ in algebraic topology, they showed the third homotopy group of suspension of an Eilenberg–MacLane space $k(G, 1)$ satisfied the condition $\Pi_3(SK(G, 1)) \cong J_2(G)$. Clearly, κ has all elements $g \otimes g$ $(g \in G)$ in its kernel, hence it induces a homomorphism $\kappa' : G \wedge G \to G'$. The kernel of κ' which is isomorphic to $\mathcal{M}(G)$, is the Schur multiplier of G (see [4, 6] for more details).

Given an abelian group A, from [13], $\Gamma(A)$ is used to denote the abelian group with generators $\gamma(a)$, for $a \in A$, by defining relations

(i) $\gamma(a^{-1}) = \gamma(a)$.
(ii) $\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ca)$.

for all $a, b, c \in A$. Γ is called the Whitehead's universal quadratic functor. From [6], we have

Theorem 1.1. *Let G and H be abelian groups. Then*

(i) $\Gamma(G \times H) = \Gamma(G) \times \Gamma(H) \times (G \otimes H)$,
(ii) $\Gamma(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n & n \text{ is odd} \\ \mathbb{Z}_{2n} & n \text{ is even} \end{cases}$

The following diagram shows relation between the non-abelian tensor square, the non-abelian exterior square, the third integral homology group of G and the Whitehead functor Γ, [13].

It has exact rows and central extensions as columns (see [6] for details)

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & 1 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H_3(G) & \longrightarrow & \Gamma(G/G') & \longrightarrow & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 1 \\
 & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_3(G) & \longrightarrow & \Gamma(G/G') & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & & & \kappa \downarrow & & \kappa' \downarrow & & \\
 & & & & & & G' & = & G' & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 1 & & 1. & &
\end{array}$$

Recall from [7], the concept of tensor and exterior center, respectively,

$$Z^{\otimes}(G) = \{g \in G | g \otimes g_1 = 1_{G \otimes G}, \text{ for all } g, g_1 \in G\},$$

$$Z^{\wedge}(G) = \{g \in G | g \wedge g_1 = 1_{G \wedge G}, \text{ for all } g, g_1 \in G\}.$$

A group G is called capable if there exists a group H such that $G \cong H/Z(H)$. The epicenter of G which is denoted by $Z^*(G)$ is defined as follows:

Definition 1.2. Let $\psi : E \to G$ be an arbitrary surjective homomorphism with ker $\psi \subseteq Z(G)$. Then the intersection of all subgroups of the form $\psi(Z(G))$ is denoted by $Z^*(G)$.

$Z^*(G)$ has the property that G is capable if and only if $Z^*(G) = 1$. Beyl and Tappe proved $Z^*(G)$ is isomorphic to $Z^\wedge(G)$, though $Z^*(G)$ is defined in a different fashion. The next lemmas, show the relation between $Z^*(G)$, $Z^\otimes(G)$ and $Z^\wedge(G)$, which play important role in this paper

Lemma 1.3 ([2, p. 208]). *Let G be any group. Then $Z^*(G) \cong Z^\wedge(G)$.*

Lemma 1.4 ([7, Proposition 16]). *Let G be any group. Then $Z^\otimes(G) \leq Z^\wedge(G)$.*

We intend to obtain the structure of $G \wedge G, Z^\wedge(G), G \otimes G, Z^\otimes(G)$ and $\Pi_3(K(G,1))$, by using the presentation of G, here, we give the presentation of all p-groups of order p^4 as follows.

Theorem 1.5 ([9, Theorem 1.11]). *Let G be a group of order p^4, where p is a prime. Then G is isomorphic to exactly one of the following groups*:

$$G_1 \cong \mathbb{Z}_{p^4}, \quad G_2 \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p, \quad G_3 \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}, \quad G_4 \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p,$$

$$G_5 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, \quad G_6 \cong \mathbb{Z}_p \times E_1, \quad G_7 \cong \mathbb{Z}_p \times E_2,$$

$$G_8 \cong \langle x, y, z | x^p = y^p = z^{p^2} = 1, [x,z] = [y,z] = 1, [x,y] = z^p \rangle,$$

$$G_9 \cong \langle x, y | x^{p^3} = y^p = 1, x^y \cong x^{1+p^2} \rangle,$$

$$G_{10} \cong \langle x, y | x^{p^2} = y^p = 1, [x,y,x] = [x,y,y] = 1 \rangle,$$

$$G_{11} \cong \langle x, y | x^{p^2} = y^{p^2} = 1, [x,y,x] = [x,y,y] = 1, [x,y] = x^p \rangle,$$

$$G_{12} \cong \langle x, y | x^4 = y^4 = 1, [x,y,x] = [x,y,y] = 1, [x,y] = x^2y^2 \rangle,$$

$$G_{13} \cong \langle x, y | x^2 = y^8 = (xy)^2 = 1 \rangle, \quad G_{14} \cong \langle x, y | x^4 = y^2 = (xy)^2 \rangle,$$

$$G_{15} \cong \langle x, y | x^2 = 1, xyx = y^3 \rangle,$$

$$G_{16} \cong \langle x, y | x^{p^2} = y^p = 1, [x,y,x] = 1, [x,y,y] = x^p, [x,y,y,y] = 1 \rangle,$$

$$G_{17} \cong \langle x, y | x^{p^2} = y^p = 1, [x,y,x] = 1, [x,y,y] = x^{np}, [x,y,y,y] = 1 \rangle,$$

$$G_{18} \cong \langle x, y | x^9 = 1, x^3 = y^3, [x,y,x] = 1, [x,y,y] = x^6, [x,y,y,y] = 1 \rangle,$$

$$G_{19} \cong \langle x, y | x^p = 1, y^p = 1, [x,y,x] = [x,y,y,x] = [x,y,y,y] = 1 \rangle,$$

$$G_{20} \cong \langle x, y | x^p = 1, y^p = [x,y,y], [x,y,x] = [x,y,y,x] = [x,y,y,y] = 1 \rangle.$$

The next theorem gives some information on the nilpotency class, the center and the derived subgroup of groups of order p^4. Let $\mathrm{cl}(G)$ denote the nilpotency class of a group G, then

Proposition 1.6 ([12, Proposition 1.1]). *Let G be a non-abelian group of order p^4. Then*

(1) *G contains an abelian maximal subgroup.*
(2) *If $\mathrm{cl}(G) = 2$, then $|Z(G)| = p^2, |G'| = p$ and G contains exactly $p+1$ abelian maximal subgroups, which intersect at $Z(G)$.*
(3) *If $\mathrm{cl}(G) = 3$, then $Z(G) = \gamma_3(G)$ has order p, $Z_2(G) = G'$ has order p^2, and G contains a unique abelian maximal subgroup.*

2. Non-abelian Tensor Square and Nanabelian Exterior Square of groups of Order p^4

Zainal *et al.* [14] obtained the structure of non-abelian tensor square of groups of order p^4 when the class of nilpotency is two. Since most of p-groups of order p^4 have class three, in this section, we obtain the non-abelian tensor square of p-groups of order p^4 of nilpotency class three. Also, we give the structure of exterior square of p-groups of order p^4. The following theorem gives us the structure of non-abelian tensor square of groups of order p^4 of nilpotency class two, where p is an odd prime.

Theorem 2.1 ([14, Theorem 9]). *Let G be a group of order p^4, where p is an odd prime. Then*

$$G \otimes G = \begin{cases} \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_9, \\ \mathbb{Z}_p^{(9)} & \text{if } G \cong G_7 \text{ or } G_8, \\ \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong G_{11}, \\ \mathbb{Z}_p^{(11)} & \text{if } G \cong G_6, \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)} & \text{if } G \cong G_{10}. \end{cases}$$

The next theorem states the structure of tensor square of groups with respect to the direct product of two groups.

Theorem 2.2 ([6, Proposition 11]). *Let G and H be groups. Then $(G \times H) \otimes (G \times H) = (G \otimes G) \times (G^{ab} \otimes H^{ab}) \times (H^{ab} \otimes G^{ab}) \times (H \otimes H)$.*

Blyth *et al.* [3] give us the structure of non-abelian tensor square of a group G when G^{ab} is finitely generated as follows.

Theorem 2.3 ([3, Corollary 1.4]). *Let G be a group such that G^{ab} is finitely generated. If G^{ab} has no element of order two or if G' has no complement in G, then $G \otimes G = \Gamma(G^{ab}) \times G \wedge G$.*

The following theorem is a key tool to obtain the structure of $G \wedge G$.

Theorem 2.4 ([7, Proposition 16(iv)]). *Let G be a group and $N \trianglelefteq G$, then $G/N \wedge G/N \cong G \wedge G$ if and only if $N \leq Z^\wedge(G)$.*

For the non-abelian groups of order 16, we have

Theorem 2.5. *Let G be a group of order 16. Then*

$$G \wedge G \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_4 & \text{if } G \cong G_{12}, \\ \mathbb{Z}_8 & \text{if } G \cong G_{13}, \\ \mathbb{Z}_4 & \text{if } G \cong G_{14} \text{ or } G_{15}. \end{cases}$$

Proof. We have the result by the HAP package of GAP [8]. □

For abelian groups of order p^4, we have

Theorem 2.6. *Let G be an abelian groups of order p^4. Then*

$$G \wedge G \cong \begin{cases} 1 & \text{if } G \cong G_1, \\ \mathbb{Z}_p & \text{if } G \cong G_2, \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_3, \\ \mathbb{Z}_p^{(3)} & \text{if } G \cong G_4, \\ \mathbb{Z}_p^{(6)} & \text{if } G \cong G_5. \end{cases}$$

Proof. Since the derived subgroup of an abelian group is trivial, so $G \wedge G \cong \mathcal{M}(G)$, now we have the result by [9, Table 1]. □

For odd p, we have

Theorem 2.7. *Let G be a group of order p^4, where p is an odd prime. Then*

$$G \wedge G \cong \begin{cases} \mathbb{Z}_p & \text{if } G \cong G_9, \\ \mathbb{Z}_p^{(3)} & \text{if } G \cong G_7, G_8, G_{10}, G_{16}, G_{17}, G_{20}, \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_{11}, \\ \mathbb{Z}_p^{(5)} & \text{if } G \cong G_6, \\ \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{19}, \\ \mathbb{Z}_3^{(3)} & \text{if } G \cong G_{18}. \end{cases}$$

Proof. Let $G \cong G_6$, by Theorem 2.1 $G \otimes G \cong \mathbb{Z}_p^{(11)}$. On the other hand, $G^{ab} \cong \mathbb{Z}_p^{(3)}$, and $\Gamma(G^{ab}) \cong \mathbb{Z}_p^{(6)}$. So by Theorem 2.3, we have $G \wedge G \cong \mathbb{Z}_p^{(5)}$. The proof for the other cases except G_{16}, G_{17}, G_{19} and G_{20} is completely similar. When $G \cong G_{16}, G_{17}$ or G_{20}, then $Z^{\wedge}(G) = Z(G) \cong \mathbb{Z}_p$, so by Theorem 2.4, $G \wedge G \cong G/Z(G) \wedge G/Z(G) \cong E_1 \wedge E_1$, using [10, Lemma 2.1, Corollary 2.3], $G \wedge G \cong \mathbb{Z}_p^{(3)}$.

Let $G \cong G_{19}$. We know $|G \wedge G| = |\mathcal{M}(G)||G'|$. By [9, Table 3], $|\mathcal{M}(G)| = p^2$ and $|G'| = p^2$, $|G \wedge G| = p^4$. Using [11, Theorem 2], the exponent of $G \otimes G$ is p, since the exponent of G is p. Therefore, the exponent of $G \wedge G$ is p and hence $G \wedge G$ is an elementary abelian group so $G \wedge G \cong \mathbb{Z}_p^{(4)}$. □

For non-abelian groups of order 16, we have

Lemma 2.8. *Let G be a non-abelian group of order 16. Then $G \otimes G \cong \mathbb{Z}_2^{(3)} \oplus \mathbb{Z}_4^{(2)}$ if $G \cong G_{12}$ and $G \otimes G \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2^{(3)}$ if $G \cong G_{13}, G_{14}$ or G_{15}.*

Proof. The result is obtained directly by HAP package programming from GAP [8]. □

In the next theorem, we obtain the non-abelian tensor square of p-groups of order p^4 of nilpotency class three.

Theorem 2.9. *Let G be a group of order p^4, where p is an odd prime. Then*

$$G \otimes G \cong \begin{cases} \mathbb{Z}_p^{(6)} & \text{if } G \cong G_{16}, G_{17} \text{ or } G_{20}, \\ \mathbb{Z}_3^{(6)} & \text{if } G \cong G_{18}, \\ \mathbb{Z}_p^{(7)} & \text{if } G \cong G_{19}. \end{cases}$$

Proof. (i) Let G be isomorphic to one of groups G_{16}, G_{17} or G_{20}, we have $G' \cong \mathbb{Z}_p^{(2)}$ and $G^{ab} \cong \mathbb{Z}_p^{(2)}$. By Theorem 2.4, $G \wedge G \cong \mathbb{Z}_p^{(3)}$. On the other hand, by Theorem 1.1, we have $\Gamma(G^{ab}) \cong \Gamma(\mathbb{Z}_p^{(2)}) \cong \mathbb{Z}_p^{(3)}$. G^{ab} is a finitely generated abelian group with no element of order 2. Then by Theorem 2.3, we have $G \otimes G \cong \Gamma(G^{ab}) \times G \wedge G \cong \mathbb{Z}_p^{(6)}$.

(ii) Let $G \cong G_{18}$, which is defined for $p = 3$, by using HAP package programming from GAP [8] we have $G \otimes G \cong \mathbb{Z}_3^{(6)}$.

(iii) The proof for the case $G \cong G_{19}$ is completely similar to (i), except $\mathcal{M}(G) \cong \mathbb{Z}_p^{(2)}$, we have $G \otimes G \cong \mathbb{Z}_p^{(7)}$. □

3. Tensor Center and Exterior Center of Groups of Order p^4

This section is devoted to obtaining the structure of $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$ when G is a groups of order p^4. In the next theorem, the tensor center of an arbitrary finite abelian group is determined.

Lemma 3.1 ([7, Proposition 1.8]). *Let G be a finite abelian p-group of order p^4. Then $Z^{\otimes}(G) = 1$.*

Following theorem is an important tool for the next investigations.

Theorem 3.2 ([7, Proposition 16(v)]). *Let G be a group and $N \trianglelefteq G$, then $G/N \otimes G/N \cong G \otimes G$ if and only if $N \leq Z^{\otimes}(G)$*

The authors in [9, Theorem 4.20] calculated the epicenter of all groups of order p^4. Now by Lemma 1.3, $Z^*(G) \cong Z^{\wedge}(G)$ so the following corollary is trivial.

Corollary 3.3. *Let G be a group of order p^4, where p is prime. Then*

$$Z^{\wedge}(G) \cong \begin{cases} 1 & \text{if } G \cong G_3, G_5, G_6, G_{11}, G_{12}, G_{13} \text{ or } G_{19}, \\ \mathbb{Z}_p & \text{if } G \cong G_4, G_7, G_8, G_{10}, G_{16}, G_{17} \text{ or } G_{20}, \\ \mathbb{Z}_{p^4} & \text{if } G \cong G_1, \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_2 \text{ or } G_9, \\ \mathbb{Z}_2 & \text{if } G \cong G_{14} \text{ or } G_{15}, \\ \mathbb{Z}_3 & \text{if } G \cong G_{18}. \end{cases}$$

In the next theorem, we obtain tensor center of all groups of order p^4.

Theorem 3.4. *Let G be a group of order p^4, where p is prime. Then*

$$Z^{\otimes}(G) \cong \begin{cases} 1 & \text{if } G \cong G_1, G_2, G_3, G_4, G_5, G_6, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15} \text{ or } G_{19}, \\ \mathbb{Z}_p & \text{if } G \cong G_7, G_8, G_9, G_{16}, G_{17} \text{ or } G_{20}, \\ \mathbb{Z}_3 & \text{if } G \cong G_{18}. \end{cases}$$

Proof. (i) The groups G_1, G_2, G_3, G_4 and G_5 are all abelian so the result follows by Lemma 3.1. Let G be isomorphic to one of groups $G_6, G_{11}, G_{12}, G_{13}$ or G_{19}. Using Lemmas 1.4 and 3.3, G is capable and $Z^{\otimes}(G) \leq Z^{\wedge}(G) = Z^*(G) \cong 1$, so $Z^{\otimes}(G) \cong 1$.

(ii) Let $G \cong G_{14}$ or G_{15}, by looking at the table [7], we get the result.

(iii) Let $G \cong G_7$. Using Theorem 3.2 and putting $N = G' \cong \mathbb{Z}_p$, we have $G/N \otimes G/N \cong \mathbb{Z}_p^{(3)} \otimes \mathbb{Z}_p^{(3)} \cong \mathbb{Z}_p^{(9)}$. On the other hand, $G' \leq Z^{\otimes}(G)$ by Theorem 2.1. Again, using Lemmas 1.4 and 3.3, we have $Z^{\otimes}(G) \cong \mathbb{Z}_p$.

(iv) Let G be isomorphic to one of groups $G_8, G_9, G_{10}, G_{16}, G_{17}, G_{18}$, or G_{20}, the proof is completely similar to (iii). □

4. Third Homotopy Group of Groups of Order p^4

This section tries to devise the structure of the third homotopy group of all groups of order p^4. First of all, we present the structure of $\Pi_3(K(G, 1))$ when G is an abelian group of order p^4. It is useful to mention that $\Pi_3(SK(G, 1)) \cong J_2(G)$.

Lemma 4.1. *Let G be an abelian group of order p^4, where p is an odd prime. Then*

$$J_2(G) \cong \begin{cases} \mathbb{Z}_p^{(4)} & \text{if } G \cong G_1, \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_2, \\ \mathbb{Z}_{p^2}^{(4)} & \text{if } G \cong G_3, \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)} & \text{if } G \cong G_4, \\ \mathbb{Z}_p^{(16)} & \text{if } G \cong G_5. \end{cases}$$

Proof. It is trivial. □

For non-abelian groups of order 16, we have

Lemma 4.2. *Let G be a non-abelian group of order* 16. *Then*

$$J_2(G) \cong \begin{cases} \mathbb{Z}_2^{(4)} \oplus \mathbb{Z}_4 & \text{if } G \cong G_{12}, \\ \mathbb{Z}_2^{(4)} & \text{if } G \cong G_{13}, \\ \mathbb{Z}_2^{(2)} \oplus \mathbb{Z}_4 & \text{if } G \cong G_{14} \text{ or } G_{15}. \end{cases}$$

Proof. Thanks to GAP group system [8], we have the result. □

Lemma 4.3. *Let $G \cong G_{18}$. Then we have $J_2(G) \cong \mathbb{Z}_3^{(4)}$.*

Proof. The result is obtained by GAP [8]. □

Blyth *et al.* [3] proved the following theorem that helps us to obtain $J_2(G)$ for p-groups of order p^4.

Theorem 4.4 ([3, Corollary 1.4]). *Let G be a group such that G^{ab} is a finitely generated abelian group with no elements of order* 2. *Then $J_2(G) \cong \Gamma(G^{ab}) \times \mathcal{M}(G)$.*

Finally, we obtain the structure of $J_2(G)$, of non-abelian groups of order p^4 where p is an odd prime.

Lemma 4.5. *Let G be a non-abelian group of order p^4, where p is an odd prime. Then*

$$J_2(G) \cong \begin{cases} \mathbb{Z}_p^{(10)} & \text{if } G \cong G_6, \\ \mathbb{Z}_p^{(8)} & \text{if } G \cong G_7 \text{ or } G_8, \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong G_9, \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{10}, \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_{11}, \\ \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{16}, G_{17} \text{ or } G_{20}, \\ \mathbb{Z}_p^{(5)} & \text{if } G \cong G_{19}. \end{cases}$$

Proof. Using Theorem 4.4, we have the result. □

The authors in [9] gave some tables containing the structure of $\mathcal{M}(G), \mathcal{M}^{(2)}(G), Z^*(G)$ and $Z_2^*(G)$ for all groups of order p^4. Here, we complete these tables by adding the structure of $G \wedge G, Z^\wedge(G), G \otimes G, Z^\otimes(G)$ and $J_2(G)$. Table 1 contains results for abelian groups.

Table 2 contains results for p-groups when $p = 2, 3$. It is important to note that the group of type (18) is just defined for $p = 3$ and the group of type (20) for $p = 3$

Table 1.

	Type of G	$G \wedge G$	$G \otimes G$	$Z^{\wedge}(G)$	$Z^{\otimes}(G)$	$J_2(G)$
$p \geq 2$	G_1	1	$\mathbb{Z}_{p^4}$	$\mathbb{Z}_{p^4}$	1	$\mathbb{Z}_{p^4}$
$p \geq 2$	G_2	$\mathbb{Z}_p$	$\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}$	1	$\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p^{(3)}$
$p \geq 2$	G_3	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(4)}$	1	1	$\mathbb{Z}_{p^2}^{(4)}$
$p \geq 2$	G_4	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p$	1	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(8)}$
$p \geq 2$	G_5	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$	1	1	$\mathbb{Z}_p^{(16)}$

Table 2.

	Type of G	$G \wedge G$	$G \otimes G$	$Z^{\wedge}(G)$	$Z^{\otimes}(G)$	$J_2(G)$
$p = 2$	G_6	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(8)}$	1	1	$\mathbb{Z}_2^{(9)}$
$p = 2$	G_7	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4^{(2)} \oplus \mathbb{Z}_2^{(7)}$	$\mathbb{Z}_2$	1	$\mathbb{Z}_4^{(2)} \oplus \mathbb{Z}_2^{(6)}$
$p = 2$	G_8	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_2^{(9)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{(8)}$
$p = 2$	G_9	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4$	1	$\mathbb{Z}_8 \oplus \mathbb{Z}_2^{(2)}$
$p = 2$	G_{11}	$\mathbb{Z}_4$	$\mathbb{Z}_4^{(3)} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	1	$\mathbb{Z}_2^{(2)} \oplus \mathbb{Z}_4^{(2)}$
$p = 2$	G_{12}	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^{(3)} \oplus \mathbb{Z}_4^{(2)}$	1	1	$\mathbb{Z}_2^{(4)} \oplus \mathbb{Z}_4$
$p = 2$	G_{13}	$\mathbb{Z}_8$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2^{(3)}$	1	1	$\mathbb{Z}_2^{(4)}$
$p = 2$	G_{14}	$\mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2^{(3)}$	$\mathbb{Z}_2$	1	$\mathbb{Z}_2^{(2)} \oplus \mathbb{Z}_4$
$p = 2$	G_{15}	$\mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2^{(3)}$	$\mathbb{Z}_2$	1	$\mathbb{Z}_2^{(2)} \oplus \mathbb{Z}_4$
$p = 3$	G_{18}	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(4)}$
$p = 3$	G_{19}	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(4)}$
$p = 3$	G_{20}	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(4)}$

Table 3.

	Type of G	$G \wedge G$	$G \otimes G$	$Z^{\wedge}(G)$	$Z^{\otimes}(G)$	$J_2(G)$
$p > 2$	G_6	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	1	1	$\mathbb{Z}_p^{(10)}$
$p > 2$	G_7	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(8)}$
$p > 2$	G_8	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(8)}$
$p > 2$	G_9	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$
$p > 2$	G_{10}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p$	1	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(4)}$
$p > 2$	G_{11}	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(2)}$	1	1	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)}$
$p > 2$	G_{16}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$
$p > 2$	G_{17}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$
$p > 3$	G_{19}	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	1	1	$\mathbb{Z}_p^{(5)}$
$p > 3$	G_{20}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$

is isomorphic to the group of type (19). We obtain the results of this table by GAP [8].

Finally, Table 3 contains results of the remained p-groups of order p^4.

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