# A HYBRID METHOD BASED ON SPECTRAL METHOD AND FINITE ELEMENT METHOD FOR SECOND ORDER PDES

[1]M. KAFAEI RAZAVI, [2]A. KERAYECHIAN, [3]M. GACHPAZAN

1, 2,3Department of Applied Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail: [1]kafaei@staff.um.ac.ir, [2]krachian@um.ac.ir, [3]gachpazan@um.ac.ir

**Abstract-** Solving PDEs in particular elliptic equations on closed and bounded regions is of great importance. In this article the spectral elements method for second order partial differential equations with dirichlet boundary conditions on regular region is considered. Although the main goal is to solve elliptic problems with this kind of boundary conditions, however, we present the method on general closed and bounded regions. At the end we compare the results of proposed method with the finite element methods. The numerical results shows the efficiency of our method.

**Index terms-** Spectral Element Method, elliptic PDEs, Dirichlet boundary conditions, regular domains.

## I. INTRODUCTION

The spectral element method is actually an extension of spectral method inspired by the concepts used in finite element method. In this way, by using discrete primary region of study to a set of simpler sub-region, we try to make spectral method applicable for large regions as well as more complex, in terms of geometry, and convert coefficients matrix to a sparse matrix by using primer discrete. Spectral methods have been applied since 80s of last century in solving the ordinary and partial differential equations and nowadays, these methods are completely well-known. However, Spectral element method has been taken into consideration in this decade.
In this article, we present spectral element method for solving second order partial differential equations including two linear independent variable on square-rectangular regions on the base of works such as Taoli and Chuanju, 2005, Fischer, 1997, Fischer, 1996, Samson et all, 2012, Hussain, 2011.

## II. THE SPECTRAL ELEMENT METHOD FOR NUMERICAL SOLUTION OF ELLIPTIC SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

We consider linear second order partial differential equations with Dirichlet boundary conditions problem with the form of below, in which A, B, C, D, E, F, G and φ are definite and continues functions on Ω, and also assume that $u \in C2(\Omega)$.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0\ , \quad (x,y) \in \Omega,$$

$$u(x,y) = \phi(x,y), \qquad (x,y) \in \partial\Omega\ ,$$

We consider area Ω a square-rectangular area as follows:

$$\Omega = [a,b] \times [c,d].$$

Then, we divide each of the intervals [a, b] and [c, d] as follows:

$$a = \hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_{n+1} = b,$$

$$c = \hat{y}_1 < \hat{y}_2 < \cdots < \hat{y}_{m+1} = d,$$

Now, we consider the sets below:

$$\Omega_{i,j} = [\hat{x}_i, \hat{x}_{i+1}] \times [\hat{y}_j, \hat{y}_{j+1}], \qquad i = 1,2,\dots,n, \quad j = 1,2,\dots,m,$$

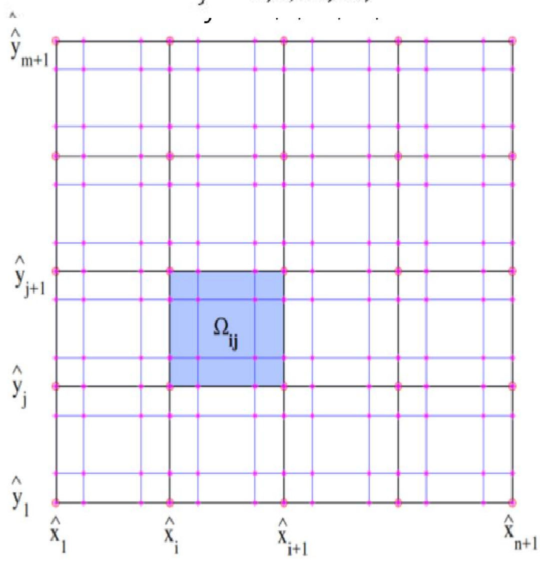


**Figure1: Regular square-rectangular region and the sub-regions $\Omega ij$**

It is obvious that

$$\Omega = \bigcup_{i=1}^{n} \left( \bigcup_{j=1}^{m} \Omega_{i,j} \right).$$

Now, on each of the intervals $[\hat{x}_i, \hat{x}_{i+1}]$ and $[\hat{y}_j, \hat{y}_{j+1}]$, we consider the following collocation network:

$$\hat{x}_i = x_{i,1} < x_{i,2} < \cdots < x_{i,N} = \hat{x}_{i+1}\ , \quad i = 1,2,\dots,n,$$

$$\hat{y}_j = y_{j,1} < y_{j,2} < \cdots < y_{j,M} = \hat{y}_{j+1}\ , \quad j = 1,2,\dots,m,$$

So that

$$x_{i,s} = \frac{\hat{x}_{i+1} - \hat{x}_i}{2}\xi_s + \frac{\hat{x}_i + \hat{x}_{i+1}}{2}, \qquad i = 1,2,\dots,n, \quad s = 1,2,\dots,N,$$

$$y_{j,t} = \frac{\hat{y}_{j+1} - \hat{y}_i}{2}\eta_t + \frac{\hat{y}_j + \hat{y}_{j+1}}{2}, \qquad j = 1,2,\dots,m, \quad t = 1,2,\dots,N,$$

Where $N > 1$ and

$$\xi_s = \cos\frac{(s-1)\pi}{N-1}, \qquad s = 1,2,\dots,N,$$

$$\eta_t = \cos\frac{(t-1)\pi}{N-1}, \qquad t = 1,2,\dots,N.$$

Now, a network from points is obtained, which generates the discrete set bellow:

$$n^* = n(N-1)+1, \qquad m^* = m(N-1)+1, \qquad N^* = n^* \times m^*.$$

$$X_k = x_{i,s}, \qquad Y_l = y_{j,t}, \qquad u_{k,l} = u(x_{i,s}, y_{j,t}),$$

$$k = (i-1)(N-1)+s, \qquad l = (j-1)(N-1)+t,$$

$$i = 1,2,\dots,n, \qquad j = 1,2,\dots,m, \qquad s,t = 1,2,\dots,N,$$

So $\Omega^*$ can also be expressed as follows, which are arranged on the base of the principle of counting grid points.

$$\Omega^* = \{(X_k, Y_l) \mid k = 1,2,\dots,n^*, \quad l = 1,2,\dots,m^*\}.$$

In fact, to be more precise, we count grid points $\Omega^*$ with the following rule:

$$P_1 = (X_1, Y_1) = (x_{1,1}, y_{1,1}) = (\hat{x}_1, \hat{y}_1),$$

$$P_2 = (X_2, Y_1) = (x_{1,2}, y_{1,1}),$$

$$\vdots$$

$$P_N = (X_N, Y_1) = (x_{1,N}, y_{1,1}) = (x_{2,1}, y_{1,1}) = (\hat{x}_2, \hat{y}_1),$$

$$\vdots$$

$$P_{n^*} = (X_{n^*}, Y_1) = (x_{n,N}, y_{1,1}) = (\hat{x}_{n+1}, \hat{y}_1),$$

$$P_{n^*+1} = (X_{n^*+1}, Y_2) = (x_{1,1}, y_{2,1}) = (\hat{x}_1, \hat{y}_2),$$

$$\vdots$$

$$P_{N^*} = (X_{n^*}, Y_{m^*}) = (x_{n,N}, y_{m,N}) = (\hat{x}_{n+1}, \hat{y}_{m+1}),$$

Now, we count all elements of $\Omega^*$, the $\Omega_{k,i}$, like following figure on the base of the principle of counting.

$$\Omega_1^* = \Omega_{1,1}, \qquad \Omega_2^* = \Omega_{2,1}, \dots, \Omega_n^* = \Omega_{n,1},$$

$$\Omega_{n+1}^* = \Omega_{1,2}, \dots, \Omega_{n\times m}^* = \Omega_{n,m}.$$

Now, we write the partial differential equation stated in each points of (X,Y) for k=1,2,…,n* and l=1,2,…,m*.

$$A(X_k,Y_l)u_{xx}(X_k,Y_l) + B(X_k,Y_l)u_{xy}(X_k,Y_l) + C(X_k,Y_l)u_{yy}(X_k,Y_l) + D(X_k,Y_l)u_x(X_k,Y_l) + E(X_k,Y_l)u_y(X_k,Y_l) + F(X_k,Y_l)u(X_k,Y_l) + G(X_k,Y_l) = 0. \qquad (*)$$

In each of the partial regions $\Omega_{i,j}$ as a forming elements of $\Omega^*$, we apply Chebyshev collocation method. For this purpose, we consider chebyshev derivative matrix D as follows:

$$D = (d_{i,j})_{N\times N}.$$

For this specified element, we determine the first and second order derivative matrices with respect to x and y as follows:

$$D^{(1,1)} = \frac{2}{\hat{x}_{i+1} - \hat{x}_i}D, \qquad D^{(1,2)} = \frac{2}{\hat{y}_{j+1} - \hat{y}_j}D,$$

$$D^{(2,1)} = \frac{4}{(\hat{x}_{i+1} - \hat{x}_i)^2}D^{(2)},$$

$$D^{(2,2)} = \frac{4}{(\hat{y}_{j+1} - \hat{y}_j)^2}D^{(2)},$$

In which, upper right hand side index with the value of 1 shows derivative with respect to x and with the value of 2 shows derivative with respect to y. Upper left hand side index shows order of derivative and

$$D^{(2)} = D^2 = D.D = (d_{i,j}^2)_{N\times N}.$$

For $(X_k, Y_l) \subset \Omega_j$, it is necessary to calculate partial derivative of $u_x(X_k, Y_l)$, $u_y(X_k, Y_l)$, $u_{xy}(X_k, Y_l)$, $u_{xx}(X_k, Y_l)$ and $u_{yy}(X_k, Y_l)$. Therefore, we need to determine that $(X_k, Y_l)$ is dependent to which of each elements of $\Omega_{i,j}$. In $(X_k, Y_l)$, k and l indexes are assumed as follows:

$$k = (i-1)(N-1)+s, \qquad i = 1,2,\dots,n, \quad s = 1,2,\dots,N,$$

$$l = (j-1)(N-1)+t, \qquad j = 1,2,\dots,m, \quad t = 1,2,\dots,N.$$

We use lagrangian two-variables interpolation for two-variables function of f. So we have:

$$f(x,y) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} l_h(x)l_g(y)f(x_h, y_g),$$

Where

$$l_h(x) = \prod_{\substack{j=1\\ j\neq h}}^{N}\frac{x - x_j}{x_h - x_j}, \qquad h = 1,2,\dots,N,$$

$$l_g(y) = \prod_{\substack{k=1\\ k\neq g}}^{N}\frac{y - y_k}{y_g - y_k}, \qquad g = 1,2,\dots,N,$$

Obtain the following approximate relations:

$$f_x(x,y) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} l'_h(x)l_g(y)f(x_h, y_g),$$

$$f_y(x,y) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} l_h(x)l'_g(y)f(x_h, y_g),$$

$$f_{xx}(x,y) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} l''_h(x)l_g(y)f(x_h, y_g),$$

$$f_{xy}(x,y) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} l'_h(x)l'_g(y)f(x_h, y_g),$$

$$f_{yy}(x,y) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} l_h(x)l''_g(y)f(x_h, y_g),$$

Where

$$l'_h(x) = \frac{d}{dx}l_h(x) = \frac{d}{d\xi}l_h(\xi)\frac{d\xi}{dx},$$

$$l'_g(y) = \frac{d}{dy}l_g(y) = \frac{d}{d\eta}l_g(\eta)\frac{d\eta}{dy},$$

And

$$l'_h(x_{i,s}) = d_{s,h}^{(1,1)}, \quad l'_g(y_{j,t}) = d_{t,g}^{(1,2)},$$
$$l''_h(x_{i,s}) = d_{s,h}^{(2,1)}, \quad l''_g(y_{j,t}) = d_{t,g}^{(2,2)}.$$

By applying chebyshev derivative matrices in the region of $\Omega_{i,j}$ as well as above relations, it can be written:

$$u_x(X_k, Y_l) = u_x(x_{i,s}, y_{j,t}) \simeq \sum_{h=1}^{N} d_{s,h}^{(1,1)} v_{h,t},$$
$$u_y(X_k, Y_l) = u_y(x_{i,s}, y_{j,t}) \simeq \sum_{g=1}^{N} d_{t,g}^{(1,2)} v_{s,g},$$
$$u_{xx}(X_k, Y_l) = u_{xx}(x_{i,s}, y_{j,t}) \simeq \sum_{h=1}^{N} d_{s,h}^{(2,1)} v_{h,t},$$
$$u_{xy}(X_k, Y_l) = u_{xy}(x_{i,s}, y_{j,t}) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} d_{s,h}^{(1,1)} d_{t,g}^{(1,2)} v_{h,g},$$
$$u_{yy}(X_k, Y_l) = u_{yy}(x_{i,s}, y_{j,t}) \simeq \sum_{g=1}^{N} d_{t,g}^{(2,2)} v_{s,g},$$

In that, $v_{s,t}$ is local values of u in $\Omega_{i,j}$ element, which specify as follows

$$v_{s,t} = u(X_k, Y_l) = u(x_{i,s}, y_{j,t}) = U_\kappa$$

$\kappa$ is the number of point in the principle of counting grid points:

$$\kappa = p(s,t) = \big((j-1)(N-1)+t-1\big)(n(N-1)+1) + (i-1)(N-1)+s = (l-1)(n(N-1)+1) + k, \quad (**)$$
$$i = 1,2,\dots,n, \quad j = 1,2,\dots,m, \quad s,t = 1,2,\dots,N.$$

We can rewrite partial derivative according to $U_{p(s,t)}$:

$$u_x(X_k, Y_l) \simeq \sum_{h=1}^{N} d_{s,h}^{(1,1)} U_{p(h,t)},$$
$$u_y(X_k, Y_l) \simeq \sum_{g=1}^{N} d_{t,g}^{(1,2)} U_{p(s,g)},$$
$$u_{xx}(X_k, Y_l) \simeq \sum_{h=1}^{N} d_{s,h}^{(2,1)} U_{p(h,t)},$$
$$u_{yy}(X_k, Y_l) \simeq \sum_{g=1}^{N} d_{t,g}^{(2,2)} U_{p(s,g)}$$
$$u_{xy}(X_k, Y_l) \simeq \sum_{h=1}^{N}\sum_{g=1}^{N} d_{s,h}^{(1,1)} d_{t,g}^{(1,2)} U_{p(h,g)}.$$

We contract:

$$A_{k,l} = A(X_k, Y_l), \quad B_{k,l} = B(X_k, Y_l), \quad C_{k,l} = C(X_k, Y_l),$$
$$D_{k,l} = D(X_k, Y_l), \quad E_{k,l} = E(X_k, Y_l), \quad F_{k,l} = F(X_k, Y_l),$$
$$G_{k,l} = G(X_k, Y_l), \quad \phi_{k,l} = \phi(X_k, Y_l),$$
$$u_{k,l} = u(X_k, Y_l) = U_\kappa = U_{p(s,t)}.$$

By inserting these values in equation (*) for interior points of $\Omega_{i,j}$, linear equations system will be obtained according to $U_\kappa$.

$$A_{k,l}\sum_{h=1}^{N} d_{s,h}^{(2,1)} U_{p(h,t)} + B_{k,l}\sum_{h=1}^{N}\sum_{g=1}^{N} d_{s,h}^{(1,1)} d_{t,g}^{(1,2)} U_{p(h,g)} + C_{k,l}\sum_{g=1}^{N} d_{t,g}^{(2,2)} U_{p(s,g)} + D_{k,l}\sum_{h=1}^{N} d_{s,h}^{(1,1)} U_{p(h,t)} + E_{k,l}\sum_{g=1}^{N} d_{t,g}^{(1,2)} U_{p(s,g)} + F_{k,l} U_{p(h,t)} + G_{k,l} = 0,$$
$$(X_k, Y_l) \in \Omega_{i,j}, \quad (X_k, Y_l) \notin \partial\Omega_{i,j},$$
$$k = (i-1)(N-1)+s, \quad i = 1,2,\dots,n, \quad s = 2,3,\dots,N-1,$$
$$l = (j-1)(N-1)+t, \quad j = 1,2,\dots,m, \quad t = 2,3,\dots,N-1,$$

For boundary conditions we have the following relations:

$$U_\kappa - \phi_{k,l} = 0, \quad (X_k, Y_l) \in \partial\Omega.$$

Now by forming equations, we use spectral element order and in each element, on the base of the principle of counting grid points, we perform counting. For passives, however, we use the same total order grid points.

If the point $(X_k, Y_l)$ for

$$k = (i-1)N+s, \quad i = 1,2,\dots,n, \quad s = 1,2,\dots,N,$$
$$l = (j-1)N+t, \quad j = 1,2,\dots,m, \quad t = 1,2,\dots,N,$$

Is a boundary point of $\Omega$, $k = 1$ or $k = n^*$ or $l = 1$ or $l = m^*$, then we define the following $H_\kappa$ equations in which $\kappa$ is achieved from relation (**):

$$H_\kappa(U_1, U_2, \dots, U_{N^*}) = U_\kappa - \phi_{k,l} = 0,$$

In non-boundary points of element $\Omega_{i,j}$ and boundary points of region $\Omega$, we insert:

$$H_\kappa(U_1, U_2, \dots, U_{N^*}) = A_{k,l}\sum_{h=1}^{N} d_{s,h}^{(2,1)} U_{p(h,t)} + B_{k,l}\sum_{h=1}^{N}\sum_{g=1}^{N} d_{s,h}^{(1,1)} d_{t,g}^{(1,2)} U_{p(h,g)} + C_{k,l}\sum_{g=1}^{N} d_{t,g}^{(2,2)} U_{p(s,g)} + D_{k,l}\sum_{h=1}^{N} d_{s,h}^{(1,1)} U_{p(h,t)} + E_{k,l}\sum_{g=1}^{N} d_{t,g}^{(1,2)} U_{p(s,g)} + F_{k,l} U_{p(h,t)} + G_{k,l} = 0.$$

In spectral element method offered, spectral method independently used on each element of $\Omega_{i,j}$. However, for the boundary points of $\Omega_{i,j}$, It should be noted that there is not the local boundary conditions which specifies the relation between this element with its adjacent elements. To establish a connection between the spectral points of adjacent elements to each other, theoretically application of the equation on the boundary points of sub-regions of Ω, means the sightly element can be sufficient. However, regarding that in these points, only directional and one-sided derivatives, in one direction vertically or horizontally, are used. Therefore, because of the lack of use of the adjacent element, coefficient matrix system of linear equations obtained is usually ill-conditioned (condition number of coefficient matrix is too large). So, to avoid this mentioned cases on the boundary points according to smoothing answer of the condition of the equality of right and left derivative for the derivative with respect to x as well as derivatives with respect to y by using the information related to the adjacent elements. Unless these points are its main boundary points of ∂Ω, for which the boundary conditions are available and applicable. These relations are stated as follows:

$$H_\kappa(U_1, U_2, \dots, U_{N^*}) - u_x(x_{i+1,1}, y_{j,t}) - u_x(x_{i,N}, y_{j,t}) = \sum_{h=1}^{N} d_{s,h}^{(1,1)} U_{p(h,t)}|_{\Omega_{i,j}} - \sum_{h=1}^{N} d_{s,h}^{(1,1)} U_{p(h,t)}|_{\Omega_{i,j}} = 0,$$

$$i = 1,2,\dots,n \quad 1, \qquad j = 1,2,\dots,m, \qquad t = 1,2,\dots,N,$$

$$H_\kappa(U_1, U_2, \dots, U_{N^*}) = u_y(x_{i,s}, y_{j+1,1}) - u_y(x_{i,s}, y_{j,N}) - \sum_{g=1}^{N} d_{1,g}^{(1,12)} U_{p(s,g)}|_{\Omega_{i,j+1}} - \sum_{g=1}^{N} d_{1,g}^{(1,12)} U_{p(s,g)}|_{\Omega_{i,j+1}} = 0,$$

$$i = 1,2,\dots,n, \qquad j = 1,2,\dots,m-1, \qquad s = 1,2,\dots,N.$$

Now, we solve the obtained linear equations systems from relations mentioned as well as boundary conditions by a suitable numerical method [9].

## III. THE NUMERICAL RESULTS

In this section, the results obtained from the proposed method is compared with the results of finite element method.

The differential equation has been considered under Dirichlet boundary conditions:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \qquad (x,y) \in \Omega,$$

$$u(x,y) = \phi(x,y), \qquad (x,y) \in \partial\Omega,$$

For the first problem, coefficient functions have been considered constant as follows:

$$A = C = 1, \qquad B = D = E = F = 0$$

For the second problem, coefficient functions have been considered non-constant as follows:

$$A = x^2, \qquad B = 0, \qquad C = y^2, \qquad D = x, \qquad E = y, \qquad F = 1$$

G and ϕ functions are given separatly in each issue. In the first two problem, we have:

$$u_{xx} + u_{yy} = \nabla^2 u = \Delta u = -G, \qquad (x,y) \in \Omega \subset \mathbb{R}^2,$$

$$u(x,y) = \phi(x,y), \qquad (x,y) \in \partial\Omega$$

Problem 1.

$$\Delta u = 4, \qquad (x,y) \in \Omega \subset \mathbb{R}^2,$$

$$u(x,y) = x^2 + y^2, \qquad (x,y) \in \partial\Omega$$

Problem 2.

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y + u = -5x^2 - 5y^2, \qquad (x,y) \in \Omega \subset \mathbb{R}^2,$$

$$u(x,y) = -x^2 - y^2, \qquad (x,y) \in \partial\Omega$$

That u(x.y) are exact solutions of the problem. The numerical results are presented in the following tables.

**Table1. The results of spectral element method on $\Omega = [-2,2] \times [-2,2]$ to solve the problem 1, the number of grid points: 169 and the maximum error: $1.8652\times10^{-14}$**

| $x$ | $y$ | $u$ | $u^*$ | $e$ |
|---|---|---|---|---|
| −1.75 | -2 | 7.063 | 7.063 | 0 |
| -1 | -2 | 5 | 5 | 0 |
| 0 | -1 | 1 | 1 | $-6.439\times10^{-15}$ |
| 0.75 | -1 | 1.563 | 1.563 | $-6.439\times10^{-15}$ |
| 1.25 | -1 | 2.563 | 2.563 | $-4.441\times10^{-15}$ |
| -2 | 0 | 4 | 4 | 0 |
| 0 | −0.25 | 0.0625 | 0.0625 | $-7.716\times10^{-15}$ |
| 0.75 | −0.25 | 0.625 | 0.625 | $-5.551\times10^{-15}$ |
| 1.25 | −0.25 | 1.625 | 1.625 | $-3.997\times10^{-15}$ |
| -2 | 0.75 | 4.563 | 4.563 | 0 |
| 0 | 0.25 | 0.0625 | 0.0625 | $-5.579\times10^{-15}$ |
| 0.75 | 0.25 | 0.625 | 0.625 | $-4.108\times10^{-15}$ |
| 1.25 | 0.25 | 1.625 | 1.625 | $-3.997\times10^{-15}$ |
| -2 | 1.25 | 5.563 | 5.563 | 0 |
| 0 | 1 | 1 | 1 | $-4.885\times10^{-15}$ |
| 0.75 | 1 | 1.563 | 1.563 | $-9.97\times10^{-15}$ |
| 1.25 | 1 | 2.563 | 2.563 | $-1.066\times10^{-14}$ |

**Table 2. The results of finite element method on $\Omega = [-2,2] \times [-2,2]$ to solve the problem 1, the number of grid points: 169 and the maximum error: 0.027912**

| $x$ | $y$ | $u$ | $u^*$ | $e$ |
|---|---|---|---|---|
| 2 | -2 | 8 | 8 | 0 |
| 2 | 0 | 4 | 4 | 0 |
| −2 | −102 | 5.44 | 5.44 | 0 |
| 0.5239 | −0.9761 | 1.236 | 1.227 | −0.008317 |
| 1.001 | 1.67 | 3.792 | 3.79 | −0.002393 |
| −1.678 | 1.434 | 4.874 | 4.872 | −0.001496 |
| −0.5903 | −1.678 | 3.168 | 3.166 | −0.002575 |
| 0.6295 | 1.683 | 3.234 | 3.23 | −0.004503 |
| −0.8729 | 1.15 | 2.089 | 2.083 | −0.005562 |
| −1.717 | 0.6726 | 3.417 | 3.401 | −0.01553 |
| 1.433 | −1.827 | 5.416 | 5.39 | −0.02643 |
| 0.4794 | −0.1496 | 0.2645 | 0.2522 | −0.01235 |

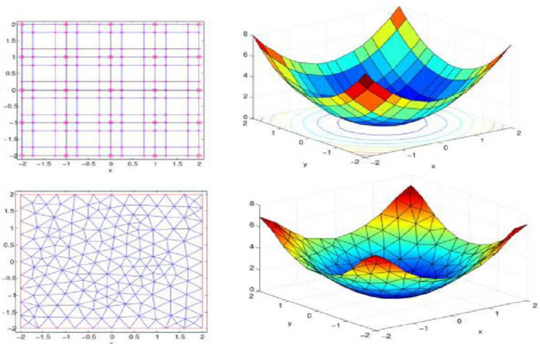

**Figure 2: The results of spectral element method (upper) and finite element method (lower) to solve the problem 1**

**Table 3. The results of spectral element method on $\Omega = [-2,2] \times [-2,2]$ to solve the problem 2, the number of grid points: 169 and the maximum error: 1/0172×10$^{-13}$**

| $x$ | $y$ | $u$ | $u^*$ | $e$ |
|---|---|---|---|---|
| −1.75 | -2 | 7.٠63 | −7.٠63 | −5.329 × 1٠$^{-15}$ |
| -1 | -2 | -5 | -5 | ٠ |
| ٠ | -1 | -1 | -1 | −4.441 × 1٠$^{-16}$ |
| ٠.75 | -1 | −1.563 | −1.563 | 1.332 × 1٠$^{-15}$ |
| 1.25 | -1 | −2.563 | −2.563 | 2.665 × 1٠$^{-15}$ |
| -2 | ٠ | -4 | -4 | ٠ |
| ٠ | − ٠.25 | − ٠.٠625 | − ٠.٠625 | −7.827 × 1٠$^{-15}$ |
| ٠.75 | − ٠.25 | − ٠.625 | − ٠.625 | −5.44 × 1٠$^{-14}$ |
| 1.25 | − ٠.25 | −1.625 | −1.625 | −1.776 × 1٠$^{-15}$ |
| -2 | ٠.75 | −4.563 | −4.563 | 2.487 × 1٠$^{-14}$ |
| ٠ | ٠.25 | − ٠.٠625 | − ٠.٠625 | 9.159 × 1٠$^{-16}$ |
| ٠.75 | ٠.25 | − ٠.625 | − ٠.625 | −2.554 × 1٠$^{-15}$ |
| 1.25 | ٠.25 | −1.625 | −1.625 | 7.1٠5 × 1٠$^{-15}$ |
| -2 | 1.25 | −5.563 | −5.563 | ٠ |
| ٠ | 1 | -1 | -1 | 8.438 × 1٠$^{-15}$ |
| ٠.75 | 1 | −1.563 | −1.563 | −1.599 × 1٠$^{-14}$ |
| 1.25 | 1 | −2.563 | −2.563 | −2.442 × 1٠$^{-14}$ |

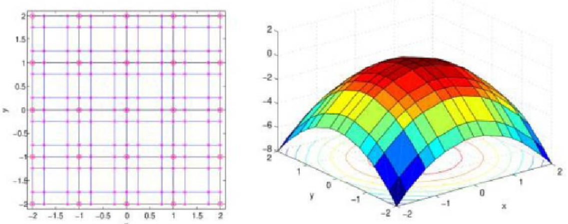

**Figure 3: The results of spectral element method to solve the problem**

## CONCLUSION

In this paper the spectral elements method for solving second order PDEs with two independent variables was used. The spectral method on a discretized region in the form of sub-regions consist of element and spectral points was used. Forming equations on sub-regions was led to a system of algebraic equations where by solving a system by a suitable numerical method, the solution was obtained. To investigate the efficiency of the method, various test problems were considered and the results were compared with the exact solu.

## REFERENCES


[1] Atkinson. K, D. Chien, O. Hansen, 2010, A spectral method for elliptic equations: the Dirichlet problem, Advances in Computational Mathematics, Volume 33, Number 2 169-189.

[2] Atkinson. K, O. Hansen, D. Chien, 2011, A spectral method for elliptic equations: the Neumann problem, Math, 34:295-317.

[3] Boyd. J.P, 2001, Chebyshev and Fourier spectral methods, 2nd edition, Dover, Mineola.

[4] Boyd. J.P, F. Yu, 2011, Comparing seven spectral methods for interpolation and for solving the Poisson equation in a disk: Zernike polynomials, Logan-Shepp ridge polynomials, Chebyshev-Fourier Series, cylindrical Robert functions, Bessel-Fourier expansions, squareto-disk conformal mapping and radial basis functions, Journal of Computational Physics 230, 1408-1438.

[5] Fischer. P.F, 1997, An overlapping Schwarz method for spectral element solution of the incompressible Navier-Stokes equations, J. Comput. Phys. 133, 84-101.

[6] Fischer. P.F, 1996, Parallel multi-level solvers for spectral element methods, in Proc. Intl. Conf. on Spectral and High-Order Methods, 95, Houston, TX, A. V. Ilin and L.R. Scott, eds., Houston J. Math., pp. 595-604.

[7] Gottlieb. D, S.A. Orszag, 1986, Numerical analysis of spectral methods, Theory and applications, SIAM.

[8] Husain. A, 2011, h − p spectral element methods for three dimensional elliptic problems on non-smooth domains using parallel computers, arXiv e-print:1110.2316 math.NA, IIT Kanpur, India.

[9] Kafaei Razavi, M., Kerayechian, A.,Gachpazan, M., and Shateyi, S., 2014, “A New Iterative Method for Finding Approximate Inverses of Complex Matrices,” Abstract and Applied Analysis, Article ID 563787, 7 pages, 2014.

[10] Lions. J.L, G. Stampacchia, 1967, Variational inequlities, Comm. Pure Appl. Math. 20, 493 - 519.

[11] Liu, Y., Teng, K., Xu, T., Badal, K., 2017, Higher-order triangular spectral element method with optimized cubature points for seismic wavefield modeling, Journal of Computational Physics, Vol. 336, 458-480.

[12] Logan. B.F, L. Shepp, 1975, Optimal reconstruction of a function from its projections, Duke Mathematics Journal, 42, 645 - 659.

[13] Ragozin. D, 1971, Constructive polynomial approximation on spheres and projective spaces, Trans. Amer. Math. Soc. 162, 157-170.

[14] Samson. M.D, H. Li, Li-Lian Wang, Submitted on 21 Apr 2012, A New Triangular Spectral Element Method I: Implementation and Analysis on a Triangle.

[15] Silberman. I, 1954, Planetary waves in the atmosphere, J. Meteorol, 11, pp 27-34.

[16] Stoer. J, R. Bulirsch, 1993, Introduction to Numerical Analysis, Springer-Verlag New York Inc.

[17] Taoli. H, X. Chuanju, 2005, A MPI Parallel Preconditioned Spectral Element Method for the Helmholtz Equation, Numerical Mathematics -English Series-, Vol. 14(1), 56-66.

[18] Trefethen. L.N, 2000, Spectral Methods in MATLAB, SIAM, Philadelphia, PA.

[19] Xu. Y, 2006, Analysis on the unit ball and on the simplex, Electronic Trans. on Numer. Anal. 25, 284-301.

[20] Zienkiewicz. O.C, Y.K. Cheung, 1967, The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill, London.


★★★