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# EXISTENCE, UNIQUENESS AND STABILITY OF FUZZY DELAY DIFFERENTIAL EQUATIONS WITH LOCAL LIPSCHITZ AND LINEAR GROWTH CONDITIONS

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Abstract. Fuzzy delay differential equation (FDDE) is a type of functional differential equations which are drive by Liu's process. In this paper, we are going to provide and prove a novel existence and uniqueness theorem for the solutions of FDDEs under Local Lipschitz and Linear growth conditions. We will investigate the stability of solutions to FDDEs by theorem as well. Finally, to illustrate the main results we give some examples.

## 1. Introduction

Liu introduced credibility theory and presented for the first time the concept of credibility measure to facilitate measuring of fuzzy events [1]. It is worthy to note that this measure is a powerful tool for dealing with fuzzy phenomena and is based on normality, monotonicity, self-duality, and maximality axioms. The main goal of this paper is to provide some weaker conditions to study the existence and uniqueness of solution to the fuzzy delay differential equations. To this end, we prove a new existence and uniqueness theorem under the Local Lipschitz and Linear growth conditions. we will study the stability.

**Definition 1.1.** ([1]) The set function **Cr** is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

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**Definition 1.2.** ([1]) Let $\Theta$ be a nonempty set, $\mathcal{P}$ the power set of $\Theta$, and $\mathbf{Cr}$ a credibility measure. The triple $(\Theta, \mathcal{P}, \mathbf{Cr})$ is called a credibility space.

**Definition 1.3.** ([1]) A fuzzy process $\mathbf{C}_t$ is said to be a Liu's process if

(1) $\mathbf{C}_0 = 0$,
(2) $\mathbf{C}_t$ has stationary and independent increments,
(3) every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed fuzzy variable with expected value $\mathbf{et}$ and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2(1 + \exp(\tfrac{\pi|x-et|}{\sqrt{6}\sigma t}))^{-1}, \ -\infty < x < +\infty.$$

## 2. Main results

We introduce as a plus term $\tau$ and consider the following FDDEs:

$$\begin{cases} dx(t) = f(x(t-\tau),t)dt + g(x(t-\tau),t)d\mathbf{C}_t, & t \geq \tau, \\ x(t) = \phi(t), & t < \tau. \end{cases} \tag{2.1}$$

where $\mathbf{C}_t$ is a standard Liu's process, $\tau \geq 0$ and denote by $\mathbf{C}([-\tau, \mathbf{R}], \mathbf{R})$ the family of continuous function $\phi$. Then is $\mathbf{C}([-\tau, \mathbf{R}], \mathbf{R})$ Banach space with norm $\|\phi\| = \sup|\phi(t)|$. Function $f, g : \mathbf{C}([-\tau, \mathbf{R}], \mathbf{R}) \times \mathbf{R} \to \mathbf{R}$. In this section, a suitable condition propose an uniqueness solution FDDEs.

Suppose that there exists some positive constant $\mathbf{L}$ such that
(**I**) Lipschitz condition there exists a positive number $\mathbf{L}$ such that:

$$|f(x(t),t) - f(y(t),t)| + |g(x(t),t) - g(y(t),t)| \leq \mathbf{L}|x(t) - y(t)|$$

and
(**II**) Linear growth condition there exists a positive number $\mathbf{L}$ such that:

$$|f(x(t),t)| + |g(x(t),t)| \leq \mathbf{L}|1 + x(t)|.$$

**Lemma 2.1.** *There exists $c > 0$ such that for any $t \in [0, T)$, the FDDE (2.1) has a unique solution on the interval $[t, t+c]$ (setting $t + c = \mathbf{T}$ if $t + c > \mathbf{T}$) if the coefficients $f(x(t),t)$ and $g(x(t),t)$ satisfy* **I** *and* **II**.

**Proof**: We have prove in the main paper.

**Theorem 2.2.** *The FDDE (2.1) has a unique solution on the interval $[0, T]$ if the coefficients $f(x(t-\tau),t)$ and $g(x(t-\tau),t)$ satisfy* **I** *and* **II**.

*Proof.* Suppose that $[0, c], [c, 2c], \ldots [kc, T]$ are the subsets of $[0, T]$ with $kc < T \leq (k+1)c$. For any, it follows from Lemma 2.1 that the equation (2.1) has a unique solution $x^i(t, \theta)$ on the interval $[(i-1)c, ic]$

for $i = 1, 2, \ldots k+1$ and setting $(k+1)c = T$. Therefore, the FDDE (2.1) has a unique solution $x(t)$ on the interval $[0, T]$ by setting

$$x(t,\theta) = \begin{cases} x^1(t,\theta), & t \in [0, c], \\ \vdots & \\ x^k(t,\theta), & t \in [(k-1)c, kc] \\ x^{k+1}(t,\theta), & t \in [kc, T]. \end{cases}$$

This result becomes evident in the case when both $f$ and $g$ are independent of the present state $x(t)$, namely for the equation

$$dx(t) = f(x(t-\tau), t)dt - g(x(t-\tau), t)d\mathbf{C}_t.$$

In this case we have explicitly that

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s-\tau), s)ds + \int_{t_0}^{t} g(x(s-\tau), s)d\mathbf{C}_s$$

$$\xi(t_0) + \int_{t_0}^{t} f(\xi(s-t_0-\tau), s)ds + \int_{t_0}^{t} g(\xi(s-t_0-\tau), s)d\mathbf{C}_s$$

for $t_0 \le t \le t_0 + \tau$. Then, for $t_0 + \tau < t < t_0 + 2\tau$,

$$x(t) = x(t_0+\tau) + \int_{t_0+\tau}^{t} f(x(s-\tau), s)ds + \int_{t_0+\tau}^{t} g(x(s-\tau), s)d\mathbf{C}_s.$$

Iterating this procedure over the intervals $[t_0 + 2\tau, t_0 + 3\tau]$ etc. $\square$

**Theorem 2.3.** *The FDDE (2.1) is stable if the coefficients $f(x,t)$ and $g(x,t)$ satisfy* **I** *and* **II**.

**Example 2.4.** Consider the following FDDE

$$\begin{cases} dx(t) = \mu(t-\tau)dt + \sigma d\mathbf{C}_t, \\ \\ x(t) = 1, \end{cases}$$

where $\mu$ and $\sigma$ are constants, and $\mathbf{C}_t$ is a standard Liu process.
If $0 \le t \le \tau$, then $-\tau \le t - \tau \le 0$, and so $x(t-\tau) = 1$.
Therefore, for $0 \le t \le \tau$, we have to solve the following fuzzy differential equation

$$\begin{cases} dx(t) = \mu dt + \sigma d\mathbf{C}_t, \\ \\ x(0) = 1, \end{cases} \tag{2.2}$$

we obtain

$$x(t) = x(0) + \int_0^t \mu dt + \sigma \mathbf{C}_t$$

and so $x(t) = x_0 + \mu t + \sigma \mathbf{C}_t$ since the two solution (2.2) are

$$x(t) = x_0 + \mu t + \sigma \mathbf{C}_t$$

$$y(t) = y_0 + \mu t + \sigma \mathbf{C}_t$$

we have $|x(t) - y(t)| = |x(0) - y(0)|$ then for any given $\varepsilon > 0$, taking $\varepsilon = \delta$, we have

$$\mathbf{Cr}\{|x(t) - y(t)| \geq \varepsilon\} = |x(0) - y(0)| = 0 < \varepsilon$$

for any $0 \leq t \leq \tau$. Hence the FDDE is stable.
Now, if $\tau \leq t \leq 2\tau$, then $0 \leq t - \tau \leq \tau$, and so $x(t-\tau) = 1 + \mu(t-\tau) + \sigma\mathbf{C}_{t-\tau}$.
Therefore, for $\tau \leq t \leq 2\tau$, we have to solve the following fuzzy differential equation

$$\begin{cases} dx(t) = \mu[1 + \mu(t-\tau) + \sigma\mathbf{C}_{t-\tau}] + \sigma d\mathbf{C}_t, \\ \\ x(\tau) = 1 + \mu(\tau) + \sigma\mathbf{C}_\tau, \end{cases} \tag{2.3}$$

we obtain

$$x(t) = x(\tau) + \int_\tau^t \mu[1 + \mu(s-\tau) + \sigma\mathbf{C}_{s-\tau}]ds + \sigma\mathbf{C}_t$$

and so $x(t) = x_0 + \mu^2\frac{(t-\tau)^2}{2} + \mu(t-\tau) + \mu\sigma\int_\tau^t \mathbf{C}_{s-t}ds$ since the two solution (2.3) are

$$x(t) = x_0 + \mu^2\frac{(t-\tau)^2}{2} + \mu(t-\tau) + \mu\sigma\int_\tau^t \mathbf{C}_{s-t}ds$$

$$y(t) = y_0 + \mu^2\frac{(t-\tau)^2}{2} + \mu(t-\tau) + \mu\sigma\int_\tau^t \mathbf{C}_{s-t}ds$$

we have $|x(t) - y(t)| = |x(0) - y(0)|$ then for any given $\varepsilon > 0$, taking $\varepsilon = \delta$, we have

$$\mathbf{Cr}\{|x(t) - y(t)| \geq \varepsilon\} = |x(0) - y(0)| = 0 < \varepsilon$$

for any $\tau \leq t \leq 2\tau$. Hence the FDDE (2.3) is stable.

## References

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