Bull. Malays. Math. Sci. Soc.
DOI 10.1007/s40840-017-0540-6

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Decomposition of the Nonabelian Tensor Product of Lie Algebras via the Diagonal Ideal

Peyman Niroomand[1] · **Farangis Johari**[2] · **Mohsen Parvizi**[2] · **Francesco G. Russo**[3]

Received: 1 May 2016 / Revised: 27 April 2017

Abstract We prove a theorem of splitting for the nonabelian tensor product $L \otimes N$ of a pair (L, N) of Lie algebras L and N in terms of its diagonal ideal $L \square N$ and of the nonabelian exterior product $L \wedge N$. A similar circumstance was described few years ago in the special case $N = L$. The interest is due to the fact that the size of $L \square N$ influences strongly the structure of $L \otimes N$.

Keywords Lie algebras · Schur multiplier · Homology · Nonabelian tensor product

Mathematics Subject Classification 17B30 · 17B60 · 17B99

Communicated by Miin Huey Ang.

✉ Peyman Niroomand
p_niroomand@yahoo.com

Farangis Johari
farangisjohary@yahoo.com

Mohsen Parvizi
parvizi@math.um.ac.ir

Francesco G. Russo
francescog.russo@yahoo.com

1 School of Mathematics and Computer Science, Damghan University, Damghan, Iran

2 Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran

3 Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X1, Rondebosch, Cape Town 7701, South Africa

Published online: 30 August 2017

1 Introduction

A large part of the homology theory of Lie algebras is based on the notion of abelian tensor product. The classical reference of Knapp [9] devotes several chapters to explain the deep relations between the theory of extensions of Lie algebras and the theory of Schur multipliers of Lie algebras. These relations are indeed investigated since a long time in algebra and topology (see [17]).

The notion of nonabelian tensor product is more recent and is originally due to Ellis [5] for Lie algebras. One of the main problems, concerning the study of nonabelian tensor products of Lie algebras, is related to the size of the so-called diagonal ideal (see the following section for details), because this gives a measure of how far is our Lie algebra from being abelian. Roughly speaking, given the nonabelian tensor product $L \otimes N = \langle l \otimes n \mid l \in L, n \in N \rangle$ of a Lie algebra L by an ideal N (acting in a certain way on L), we call "diagonal ideal" the set $L \square N = \langle l \otimes l \mid l \in N \rangle$. This is indeed an abelian ideal of $L \otimes N$, which measure in a certain sense "how much" $L \otimes N$ is abelian.

The main results of the present contribution deal with the size of $L \square N$ under suitable assumptions of finite dimension on N. We begin with a result of decomposition for $L \otimes N$ in Theorem 3.6, and a precise condition of splitting is successively shown in Theorem 3.12. Topological interpretations of Theorems 3.6 and 3.12 are given in Corollary 3.13, where one can find connections of homotopical nature with the suspension of an Eilenberg–MacLane space. In order to prove these main results, we use some homological methods and some computations of linear algebra in a series of preparatory lemmas. Section 2 introduces the formal definitions and the notions which will be used in Sect. 3, where the main results are placed.

2 Compatible Actions and Nonabelian Tensor Products

Let F be a fixed field, L, K be two Lie algebras, $c \in F, l, l' \in L, k, k' \in K$ and $[\ ,\ ]$ be the usual Lie bracket. We say that L acts on K if there is an F—bilinear map $(l, k) \in L \times K \mapsto {}^{l}k = [l, k] \in K$ satisfying ${}^{[l,l']}k = {}^{l}({}^{l'}k) - {}^{l'}({}^{l}k)$ and ${}^{l}[k, k'] = [{}^{l}k, k'] + [k, {}^{l}k']$. This notation is standard and follows [4–6]. Of course, if L is a subalgebra of some Lie algebra P and K is an ideal in P, then the Lie multiplication in P induces an action of L on K. In fact, l acts on k by ${}^{l}k = [l, k]$. Let L and K be Lie algebras acting on each other, and on themselves by Lie multiplication. Then these actions are said to be *compatible* if ${}^{{}^{k}l}k' = {}^{k'}({}^{l}k)$ and ${}^{{}^{l}k}l' = {}^{l'}({}^{k}l)$. Now if L and K are both ideals of some Lie algebra, then the Lie multiplication gives rise to compatible actions. The *nonabelian tensor product* $L \otimes K$ of L and K is the Lie algebra generated by the symbols $l \otimes k$ with defining relations

$$
\begin{gathered}
c(l \otimes k) = cl \otimes k = l \otimes ck, \quad (l + l') \otimes k = l \otimes k + l' \otimes k, \\
l \otimes (k + k') = l \otimes k + l \otimes k', \quad {}^{l}l' \otimes k = l \otimes {}^{l'}k - l' \otimes {}^{l}k, \\
l \otimes {}^{k}k' = {}^{k'}l \otimes k - {}^{k}l \otimes k', \quad [l \otimes k, l' \otimes k'] = -{}^{k}l \otimes {}^{l'}k'.
\end{gathered}
$$

A nonabelian tensor product $L \otimes L$ is called *nonabelian tensor square* of L. On the other hand, if L is abelian, the tensor product we defined above becomes the normal tensor product of vector spaces. We will concentrate on $L \otimes N$, in which $N = K$ is an ideal of L.

Following [5, Sect. 2],

$$L\square N = \langle l \otimes l \mid l \in N\rangle$$

is an ideal of $L\otimes N$ (called *diagonal ideal* of $L\otimes N$) and lies in the center $Z(L\otimes N)$ of $L \otimes N$. Denoting by $l \wedge n$ the coset $l \otimes n + (L\square N)$, we may consider the Lie algebra quotient $L \wedge N = L \otimes N/(L\square N) = \langle l \wedge n \mid l \in L, n \in N\rangle$, called *nonabelian exterior product* of L by N.

The *Schur multiplier* $M(L, N)$ *of the pair* (L, N) is defined to be the abelian Lie algebra, appearing in the following natural exact sequence of Mayer–Vietoris type

$$H_3(L) \longrightarrow H_3(L/N) \longrightarrow M(L, N) \longrightarrow M(L) \longrightarrow M(L/N) \longrightarrow$$
$$\longrightarrow \frac{L}{[L, N]} \longrightarrow \frac{L}{[L, L]} \longrightarrow \frac{L}{[L, L] + N} \longrightarrow 0,$$

where $M(-)$ and $H_3(-)$ are the Schur multiplier and the third homology of a Lie algebra, respectively (see [1–5,8,13,14]).

We recall from [4,5] that it is possible to get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\Gamma\left(\frac{N}{[N,L]}\right) & \longrightarrow & J_2(L, N) & \longrightarrow & M(L, N) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\Gamma\left(\frac{N}{[N,L]}\right) & \xrightarrow{\psi} & L \otimes N & \xrightarrow{\varepsilon_{L,N}} & L \wedge N & \longrightarrow & 0 \\
 & & \kappa_{L,N}\downarrow & & \kappa'_{L,N}\downarrow & & \\
 & & [L, N] & = & [L, N] & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{2.1}$$

where

$$\kappa_{L,N} : l \otimes n \in L \otimes N \longmapsto [l, n] \in [L, N]$$

is an epimorphism such that $J_2(L, N) = \ker \kappa_{L,N} \subseteq Z(L \otimes N)$ and both

$$\varepsilon_{L,N} : l \otimes n \in L \otimes N \mapsto (l \otimes n) + L\square N \in L \otimes N/L\square N$$

and

$$\kappa'_{L,N} : l \wedge n \in L \wedge N \longmapsto [l, n] \in [L, N]$$

are epimorphisms. Moreover, $M(L, N) \simeq \ker \kappa'_{L,N} \subseteq Z(L \wedge N)$. Note that $J_2(L, L) = J_2(L)$ was described in [5, pp. 109–110]. We also note that the columns of (2.1) are central extensions, while the rows of (2.1) form two long exact sequences. In fact, Γ denotes the *quadratic Whitehead functor* and ψ is properly defined in [5, Definition, p.107] (see also [18] for a categorical definition of Γ and ψ). The properties of (2.1) are discussed in various contributions and in different perspectives (see [4–7,10–16]).

3 Splitting of $\otimes$ via $\square$ and $\wedge$

Since the notion of *dimension* for a Lie algebra is in a certain sense "more geometric than algebraic," we cannot expect full analogies with respect to the results in [6,8,10,16], when we replace this notion with that of order of a group. In fact, we investigated the role of the homological invariants between Lie algebras and finite groups in [11–14]. We begin to look for information on the bases of those Lie algebras, which will be involved in the main results of the present paper. The abelian case plays a fundamental role and is discussed in the next result.

Proposition 3.1 *Let N be an ideal of dimension m with basis $\{x_1, x_2, \ldots, x_m\}$ of an abelian Lie algebra L of dimension n with basis $\{y_1, y_2, \ldots, y_m, y_{m+1}, y_{m+2}, \ldots, y_n\}$, where $y_i = x_i$ for $1 \le i \le m$. Then*

$$L \otimes N \cong (L \square N) \oplus \langle y_j \otimes x_t \mid 1 \le t \le m,\ 1 \le t < j \le n \rangle.$$

Proof The decomposition of $L \otimes N$ follows directly from the fact that in this case it is the normal tensor product of vector spaces. In fact, we have

$$L \cong \bigoplus_{i=m+1}^{n} \langle y_i \rangle \oplus \bigoplus_{j=1}^{m} \langle x_j \rangle, \text{ where } N \cong \bigoplus_{j=1}^{m} \langle x_j \rangle,$$

$$L \otimes N \cong \langle x_i \otimes x_t + x_t \otimes x_i, x_i \otimes x_i \mid 1 \le i < t \le m \rangle$$
$$\oplus \langle y_j \otimes x_t \mid 1 \le t \le m,\ 1 \le t < j \le n \rangle,$$

$$L \square N \cong \langle x_i \otimes x_t + x_t \otimes x_i, x_i \otimes x_i \mid 1 \le i < t \le m \rangle.$$

□

A first result of splitting may be formulated in the abelian case.

Corollary 3.2 *Let N be an ideal of a finite-dimensional abelian Lie algebra L. Then*

$$L \otimes N \cong (L \square N) \oplus (L \wedge N).$$

Proof This follows from Proposition 3.1. □

We are ready to prove a crucial result of the present section.

Proposition 3.3 *Let N be an ideal of a Lie algebra L such that $N/[N, L]$ is of dimension m with basis $\{y_1 + [N, L], y_2 + [N, L], \ldots, y_m + [N, L]\}$ and $L/[N, L]$ is of dimension n with basis $\{y_1 + [N, L], y_2 + [N, L], \ldots, y_n + [N, L]\}$. Then*

$$\frac{L}{[N,L]} \otimes \frac{N}{[N,L]} \cong \left(\frac{L}{[N,L]} \square \frac{N}{[N,L]}\right)$$
$$\oplus \langle (y_j + [N, L]) \otimes (y_i + [N, L]) \mid 1 \le i \le m, 1 \le i < j \le n\rangle.$$

Proof Essentially, we do two observations. The first is that $[L/[N, L], N/[N, L]] = 0$. The second is that the actions of $L/[N, L]$ and $N/[N, L]$ on each other are given by Lie bracket and so are trivial. Then we may conclude

$$\frac{L}{[N,L]} \otimes \frac{N}{[N,L]} \cong \left(\frac{L}{[N,L]}\right)^{ab} \otimes \frac{N}{[N,L]},$$

that is, the nonabelian tensor product $L/[N, L] \otimes N/[N, L]$ is isomorphic to the usual abelian tensor product $(L/[N, L])^{ab} \otimes N/[N, L]$. Therefore, we apply Corollary 3.2

$$\left(\frac{L}{[N,L]}\right)^{ab} \otimes \frac{N}{[N,L]} \cong \left(\left(\frac{L}{[N,L]}\right)^{ab} \square \frac{N}{[N,L]}\right) \oplus \left(\left(\frac{L}{[N,L]}\right)^{ab} \wedge \frac{N}{[N,L]}\right),$$

where the abelian Lie algebra factor $(L/[N, L])^{ab} \wedge N/[N, L]$ admits a basis exactly of the form $\{(y_j + [L, L]) \otimes (y_i + [N, L]) \mid 1 \le i \le m, 1 \le i < j \le n\}$ (note that $(L/[N, L])^{ab} = (L/[N, L])/([L, L]/[N, L]) \cong L/[L, L]$) and the other abelian Lie algebra factor is $(L/[N, L])^{ab} \square N/[N, L] \cong L/[N, L] \square N/[N, L]$. Then the result follows from Proposition 3.1. □

We may reformulate the proposition above in the following way.

Corollary 3.4 *Let N be an ideal of a Lie algebra L such that $L/[N, L]$ is finite-dimensional. Then*

$$L/[N, L] \otimes N/[N, L] \cong (L/[N, L] \square N/[N, L]) \oplus (L/[N, L] \wedge N/[N, L]).$$

A crucial step, which is fundamental for our aims, deals with the description of the natural epimorphism

$$\pi : l \otimes n \in L \otimes N \longmapsto (l + [N, L]) \otimes (n + [N, L]) \in \frac{L}{[N,L]} \otimes \frac{N}{[N,L]} \quad (3.1)$$

and its restriction

$$\pi_| : n \otimes n \in L \square N \longmapsto (n+[N, L]) \otimes (n+[N, L]) \in \frac{L}{[N,L]} \square \frac{N}{[N,L]}. \quad (3.2)$$

Assuming that N is an ideal in L, there are two homomorphisms

$$\tau_1 : l \otimes [n, l'] \in L \otimes [N, L] \longmapsto l \otimes [n, l'] \in L \otimes N, \tag{3.3}$$

$$\tau_2 : [n, l'] \otimes n' \in [N, L] \otimes N \longmapsto [n, l'] \otimes n' \in L \otimes N$$

which are very useful in the proof of the next lemma. The kernel of (3.1) is studied in the next result. We inform the reader that we will use the notion of *Lie pairing* in the next proof (see [5]).

Lemma 3.5 *Let N be an ideal of a Lie algebra L, π as in* (3.1)*, τ_1, τ_2 as in* (3.3) *and* $M = \mathrm{Im}\ \tau_1 + \mathrm{Im}\ \tau_2$. *Then* $\ker \pi = M$.

Proof Since M is an ideal of $L \otimes N$, we may consider the map

$$\bar{\pi} : (l \otimes n) + M \in \frac{L \otimes N}{M} \mapsto (l + [N, L]) \otimes (n + [N, L]) \in \frac{L}{[N, L]} \otimes \frac{N}{[N, L]}.$$

On the other hand, it is well defined the map

$$\alpha : (l + [N, L], n + [N, L]) \in \frac{L}{[N, L]} \times \frac{N}{[N, L]} \mapsto (l \otimes n) + M \in \frac{L \otimes N}{M}.$$

Now for all $l_1, l_2 \in L$ and $n_1, n_2 \in N$

$$\begin{aligned} &\alpha([l_1 + [N, L], l_2 + [N, L]]\ ,\ n_1 + [N, L]) \\ &\quad = ([l_1, l_2] \otimes n_1) + M = (l_1 \otimes [l_2, n_1] - l_2 \otimes [l_1, n_1]) + M \\ &\quad = \alpha(l_1 + [N, L], [l_2, n_1] + [N, L]) - \alpha(l_2 + [N, L], [l_1, n_1] + [N, L]). \end{aligned}$$

Similarly, it is easy to see that

$$\begin{aligned} &\alpha({}^{n_1+[N,L]}l_1 + [N, L]\ ,\ {}^{l_1+[N,L]}n_2 + [N, L]) \\ &\quad = -[\alpha(l_1 + [N, L], n_1 + [N, L])\ ,\ \alpha(l_2 + [N, L], n_2 + [N, L])] \end{aligned}$$

and

$$\begin{aligned} &\alpha(l_1 + [N, L]\ ,\ [n_1, n_2] + [N, L]) \\ &\quad = \alpha({}^{n_2+[N,L]}l_1 + [N, L]\ ,\ n_1 + [N, L]) - \alpha({}^{n_1+[N,L]}l_1 + [N, L]\ ,\ n_2 + [N, L]). \end{aligned}$$

Thus α is a Lie pairing and induces the homomorphism

$$\bar{\alpha} : \frac{L}{[N, L]} \otimes \frac{N}{[N, L]} \longrightarrow \frac{L \otimes N}{M}$$

such that $\bar{\pi} \circ \bar{\alpha} = \bar{\alpha} \circ \bar{\pi} = 1$. Therefore,

$$\frac{L}{[N,L]} \otimes \frac{N}{[N,L]} \simeq \frac{L \otimes N}{M}$$

and the result follows. □

The first main result of this paper is the following.

Theorem 3.6 *Let N be an ideal of a Lie algebra L such that $N/[N,L]$ is of dimension m with basis $\{y_1+[N,L], y_2+[N,L], \ldots, y_m+[N,L]\}$ and $L/[N,L]$ is of dimension n with basis $\{y_1+[N,L], y_2+[N,L], \ldots, y_n+[N,L]\}$. Then*

$$L \otimes N \simeq (L \,\square\, N) + \langle y_j \otimes y_i \mid 1 \leq i \leq m, 1 \leq i < j \leq n \rangle + M,$$

where $\{y_i \otimes y_t + y_t \otimes y_i, y_i \otimes y_i \mid 1 \leq i < t \leq m\}$ is a basis of $L \,\square\, N$, π as in (3.1) and $M = \ker \pi$. Moreover, $\dim(L \otimes N) \leq m(m+1)/2$.

Proof We consider π in (3.1) and apply Proposition 3.3, concluding that

$$\begin{aligned}\frac{L}{[N,L]} \otimes \frac{N}{[N,L]} &\cong \left(\frac{L}{[N,L]} \,\square\, \frac{N}{[N,L]}\right)\\ &\oplus \langle (y_j + [N,L]) \otimes (y_i + [N,L]) \mid 1 \leq j \leq m, 1 \leq j < i \leq n \rangle\\ &= \pi\left((L \square N) + \langle y_j \otimes y_i \mid 1 \leq j \leq m, 1 \leq j < i \leq n \rangle \right).\end{aligned}$$

Since (3.1) is surjective, we have just shown that

$$\pi(L \otimes N) \simeq \pi\left((L \square N) + \langle y_j \otimes y_i \mid 1 \leq j \leq m, 1 \leq j < i \leq n \rangle \right)$$

and knowing $M = \ker \pi$ by Lemma 3.5, this means

$$L \otimes N \simeq (L \square N) + (\langle y_j \otimes y_i \mid 1 \leq j \leq m, 1 \leq j < i \leq n \rangle + M).$$

The first part of the result follows. About the rest, assume that $X = \{x_\alpha \mid \alpha \in I\}$ is a basis of $[N,L]$ (eventually with infinite $|I|$) and $B = \{y_1, y_2, \ldots, y_m\}$ for some $y_1, y_2, \ldots, y_m \in L$. Then $D = X \cup B$ generates N and $L \,\square\, N$ is generated by $\{d_1 \otimes d_1, (d_1 \otimes d_2) + (d_2 \otimes d_1) \mid d_1, d_2 \in D\}$. On the other hand, we have $d \otimes d = 0$ if $d \in X$ and $(a \otimes b) + (b \otimes a) = 0$ if at least one among a and b lies in X. Therefore, $\{d_1 \otimes d_1, (d_1 \otimes d_2) + (d_2 \otimes d_1) \mid d_1, d_2 \in X\}$ is also a basis of $L \,\square\, N$. From this, we may conclude that $\{y_i \otimes y_t + y_t \otimes y_i, y_i \otimes y_i \mid 1 \leq i < t \leq m\}$ is a basis of $L \,\square\, N$ and that $\dim(L \otimes N) \leq m(m+1)/2$. □

We can be more precise with the description of the kernel of (3.1), when we consider the restriction (3.2). The following result is in this direction.

Lemma 3.7 *Let N be an ideal of a Lie algebra L such that $N/[N,L]$ is of dimension m with basis $\{y_1+[N,L], y_2+[N,L], \ldots, y_m+[N,L]\}$ and $L/[N,L]$ is of dimension n with basis $\{y_1+[N,L], y_2+[N,L], \ldots, y_n+[N,L]\}$. If $\pi_|$ is as in (3.2) and M as in Lemma 3.5, then* $\ker \pi_| = (L \square N) \cap (\langle y_j \otimes y_i \mid 1 \leq i \leq m, 1 \leq i < j \leq n \rangle + M)$.

Proof Since $\pi_|((L\square N)\cap(\langle y_j\otimes y_i \mid 1\leq i\leq m, 1\leq i<j\leq n\rangle+M))=0$, it is clear that $(L\square N)\cap(\langle y_j\otimes y_i \mid 1\leq i\leq m, 1\leq i<j\leq n\rangle+M)\subseteq\ker\pi_|$. Vice versa, we know from Lemma 3.5 that $\ker\pi = M = M\cap(L\square N)$; thus, $\ker\pi_|\subseteq(\langle y_j\otimes y_i \mid 1\leq i\leq m, 1\leq i<j\leq n\rangle+M)\cap(L\square N)$. The result follows. □

Then Theorem 3.6 may be reformulated in an alternative way.

Corollary 3.8 *In the same assumptions of Theorem 3.6 and with* $\pi_|$ *as in* (3.2),

$$(L\otimes N)/\ker\pi_|\simeq(L/[N,L]\square N/[N,L])\oplus(L\wedge N).$$

Proof First of all, Lemma 3.7 implies $\ker\pi_|\subseteq L\square N$ and we know that $L\square N\subseteq Z(L\otimes N)$, so $\ker\pi_|$ is normal in $L\otimes N$ and we may consider the quotient $L\otimes N/\ker\pi_|$. Now we apply Theorem 3.6 and get

$$\begin{aligned}\frac{L\otimes N}{\ker\pi_|}&\cong\frac{(L\square N)\ \oplus\ (\langle y_j\otimes y_i\mid 1\leq i\leq m, 1\leq i<j\leq n\rangle+M)}{\ker\pi_|}\\&\cong\frac{L\ \square\ N}{\ker\pi_|}\oplus\frac{\langle y_j\otimes y_i\mid 1\leq i\leq m, 1\leq i<j\leq n\rangle+M}{\ker\pi_|}.\end{aligned}$$

Note that $L\square N/\ker\pi_|\cong(L/[N,L]\square N/[N,L])$ and $L\wedge N\cong(\langle y_j\otimes y_i\mid 1\leq i\leq m, 1\leq i<j\leq n\rangle+M)/\ker\pi_|$. This completes the result. □

In order to proceed with the proof of our second main theorem, we recall some elementary facts, which follow from the exactness of (2.1). The following lemma is a direct consequence of [5, Propositions 16 and 17] when $N\neq L$.

Lemma 3.9 *[See [5], Propositions 16 and 17] Let* K *be an abelian Lie algebra of* $\dim K=m$. *Then*

$$\dim\Gamma(K)=\dim K\square K=m(m+1)/2.$$

An application of the above lemma can be found in the next result and it involves again the epimorphism $\pi_|$ in (3.2).

Lemma 3.10 *Let* N *be an ideal of a Lie algebra* L *such that* $\dim N/[N,L]=m$ *and* $\dim L/[N,L]$ *is finite. If* $\dim L/[N,L]\square N/[N,L]=m(m+1)/2$, *then* $\ker\pi_|=0$ *and*

$$L/[N,L]\square N/[N,L]\cong L\square N\cong\Gamma(N/[N,L]).$$

Proof We inform the reader that the notation for the elements $\gamma(n+[N,L])$ of $\Gamma\left(\frac{N}{[N,L]}\right)$ follows the original notation of Whitehead in [18] (see also [4,5]). By Lemma 3.9, $\dim\Gamma(N/[N,L])=m(m+1)/2$. Thus the composition

$$\pi_| \circ \psi : \gamma(n + [N, L]) \in \Gamma\left(\frac{N}{[N, L]}\right)$$
$$\longmapsto (n + [N, L])\Box(n + [N, L]) \in \frac{L}{[N, L]}\Box\frac{N}{[N, L]}$$

maps a basis of $\Gamma(N/[N, L])$ injectively into a part of a basis of $L/[L, N]\Box N/[L, N]$. Then $\pi_| \circ \psi$ is an isomorphism. Therefore, ψ is injective. We conclude $\dim L\Box N = \dim L/[L, N]\Box N/[L, N] = \dim \Gamma(N/[N, L]) = m(m+1)/2$. Hence $\ker \pi_| = 0$ and the result follows. □

In general, the columns of (2.1) are short exact sequences, but not the rows. However, even the rows of (2.1) become short exact sequences.

Corollary 3.11 *Let N be an ideal of a Lie algebra L such that* $\dim N/[N, L] = m$ *and* $\dim L/[N, L]$ *is finite. If* $\dim L/[N, L]\Box N/[N, L] = m(m+1)/2$, *then the following*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma\left(\frac{N}{[N,L]}\right) & \longrightarrow & J_2(L, N) & \longrightarrow & M(L, N) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma\left(\frac{N}{[N,L]}\right) & \longrightarrow & L \otimes N & \longrightarrow & L \wedge N & \longrightarrow & 0
\end{array}$$

is a commutative diagram with short exact sequences as rows.

Proof Application of Lemma 3.10 to (2.1). □

Of course, if L is a Lie algebra of finite dimension, then its factors $L/[N, L]$ and $N/[N, L]$ are of finite dimension (and consequently $L/[N, L]\Box N/[N, L]$). Then Corollary 3.11 is true, even if the whole L is of finite dimension. The second main result of this section may be formulated again with the weak restriction of finite dimension on the factor $L/[N, L]$ (and not on the whole L).

Theorem 3.12 *Let N be an ideal of a Lie algebra L such that* $\dim N/[N, L] = m$ *and* $\dim L/[N, L]$ *is finite. If* $\dim L/[N, L]\Box N/[N, L] = m(m+1)/2$, *then*

$$L \otimes N \cong (L \Box N) \oplus (L \wedge N).$$

Proof From Lemma 3.10, $\ker \pi_| = 0$ and so we may apply Corollary 3.8, getting $L \otimes N \cong (L/[N, L]\Box N/[N, L]) \oplus (L \wedge N)$. Now one has to note that $L\Box N \cong L/[N, L]\Box N/[N, L]$ again by Lemma 3.10 and the result follows. □

The role of the Lie algebra $J_2(L, N)$ has been investigated in [5] when $N = L$ and it is related to the homotopy theory in the sense of [5, Theorems 27, 28]. There is not a version of [5, Theorems 27, 28], when $N \neq L$, even if some ideas can be found in [7, Theorems 1, 2, 4, 5] for the case of groups. This emphasizes the following consequence of Theorem 3.12.

Corollary 3.13 *Let N be an ideal of a Lie algebra L such that* $\dim N/[N, L] = m$ *and* $\dim L/[N, L]$ *is finite. If* $\dim L/[N, L]\square N/[N, L] = m(m+1)/2$, *then*

$$J_2(L, N) \cong (L \square N) \oplus M(L, N).$$

Proof By Theorem 3.12,

$$\begin{aligned} J_2(L, N) &= \big((L \square N) \oplus (\langle y_j \otimes y_i \mid 1 \le i \le m, 1 \le i < j \le n\rangle + M)\big) \cap J_2(L, N) \\ &= (L \square N) \oplus \big((\langle y_j \otimes y_i \mid 1 \le i \le m, 1 \le i < j \le n\rangle + M\big) \cap J_2(L, N)\big). \end{aligned}$$

The rest follows by diagram (2.1), specifically by $J_2(L, N)/(L \square N) \cong M(L, N)$. □

Acknowledgements We thank the referees for their comments. The fourth author (F.G.R.) acknowledges NRF (South Africa) for the Grant Nos. CPRR 14071175245 and CSUR 93652. The third author was supported by a grant from Ferdowsi University of Madhhad-Graduate Studys (No. 31659).

References

1. Araskhan, M., Rismanchian, M.R.: Some properties on the Schur multiplier of a pair of Lie algebras. J. Algebra Appl. **11**, 1250011 (2012)
2. Araskhan, M., Rismanchian, M.R.: Some inequalities for the dimension of the Schur multiplier of a pair of (nilpotent) Lie algebras. J. Algebra **352**, 173–179 (2012)
3. Edalatzadeh, B., Saeedi, F., Salemkar, A.R.: The commutator subalgebra and Schur multiplier of a pair of nilpotent Lie algebras. J. Lie Theory **21**, 491–498 (2011)
4. Ellis, G.J.: Nonabelian exterior products of Lie algebras and an exact sequence in the homology of Lie algebras. J. Pure Appl. Algebra **46**, 111–115 (1987)
5. Ellis, G.J.: A nonabelian tensor product of Lie algebras. Glasg. Math. J. **33**, 101–120 (1991)
6. Ellis, G.J.: The Schur multiplier of a pair of groups. Appl. Categ. Struct. **6**, 355–371 (1998)
7. Ellis, G.J.: Relative derived functors and the homology of groups. Cah. Topol. Géom. Différ. Catég. **31**, 121–135 (1990)
8. Guin, D.: Cohomologie et homologie nonabéliennes des groupes. J. Pure Appl. Algebra **50**, 109–137 (1988)
9. Knapp, A.W.: Lie Groups, Lie Algebras and Cohomology. Princeton University Press, Princeton (1988)
10. Mashayekhy, B., Mirebrahimi, H., Vasagh, Z.: A topological approach to the nilpotent multipliers of a free product. Georgian Math. J. **21**, 89–96 (2014)
11. Niroomand, P., Russo, F.G.: A restriction on the Schur multiplier of a nilpotent Lie algebra. Electron. J. Linear Algebra **22**, 1–9 (2011)
12. Niroomand, P.: Some properties on the tensor square of Lie algebras. J. Algebra Appl. **11**, 1250085 (2012)
13. Niroomand, P., Parvizi, M., Russo, F.G.: Some criteria for detecting capable Lie algebras. J. Algebra **384**, 36–44 (2013)
14. Niroomand, P., Russo, F.G.: Some restrictions on the Betti numbers of a nilpotent Lie algebra. Bull. Belg. Math. Soc. Simon Stevin **21**, 403–413 (2014)
15. Niroomand, P., Russo, F.G.: On the size of the third homotopy group of the suspension of an Eilenberg–MacLane space. Turk. J. Math. **38**, 664–671 (2014)
16. Otera, D.E., Russo, F.G., Tanasi, C.: Some algebraic and topological properties of the nonabelian tensor product. Bull. Korean Math. Soc. **50**, 1069–1077 (2013)
17. Rotman, J.: An Introduction to Homological Algebra. Academic Press, San Diego (1979)
18. Whitehead, J.H.C.: A certain exact sequence. Ann. Math. **52**, 51–110 (1950)