Perfect Lattice Paths in the Plane

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Abstract

Consider an $m \times n$ table T and latices paths $\nu_1, \ldots, \nu_k$ in T such that each step $\nu_{i+1} - \nu_i = (1,1)$, $(1,0)$ or $(1,-1)$. The number of paths from the $(1,i)$-cell (resp. first column) to the (s,t)-cell is denoted by $\mathcal{D}^i(s,t)$ (resp. $\mathcal{D}(s,t)$). Also, the number of all paths form the first column to the las column is denoted by $\mathcal{I}_m(n)$. We give explicit formulas for the numbers $\mathcal{D}^1(s,t)$ and $\mathcal{D}(s,t)$.

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1 Introduction

A lattice path in $\mathbb{Z}^2$ is the drawing in $\mathbb{Z}^2$ of a sum of vectors from a fixed finite subset S of $\mathbb{Z}^2$, starting from a given point, say $(0,0)$ of $\mathbb{Z}^2$. A typical problem in lattice paths is the enumeration of all S-lattice paths (lattice paths with respect to the set S) with a given initial and terminal point satisfying possibly some further constraints. A nontrivial simple case is the problem of finding the number of lattice paths starting from the origin $(0,0)$ and ending at a point (m,n) using only right step $(1,0)$ and up step $(0,1)$ (i.e., $\mathbf{S} = \{(1,0),(0,1)\}$). The number of such paths are known to be the the binomial coefficient $\binom{m+n}{n}$. Yet another example, known as the ballot problem, is to find the number of lattice paths from $(1,0)$ to (m,n) with $m > n$, using the same steps as above, that never touch the line $y = x$. The number of such paths, known as ballot number, equals $\frac{m-n}{m+n}\binom{m+n}{n}$. In the special case where $m = n+1$, the ballot number is indeed the Catalan number C_n.

Let $T = T_{m,n}$ denote the $m \times n$ table in the plane and (x,y) be the cell in the columns x and row y (and refer to it as the (x,y)-cell). The set of lattice paths from the (i,j)-cell to the (s,t)-cell, with steps belonging to a finite set S, is denoted by $L(i,j;s,t:\mathbf{S})$, and the number of those paths is denoted by $l(i,j;s,t:\mathbf{S})$, where $1 \leqslant i,s \leqslant m$ and $1 \leqslant j,t \leqslant n$.

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Throughout this paper, we set $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$, and the corresponding lattice paths starting from the first column and ending at the last column are called *perfect lattice paths*. The number of all perfect lattice paths is denoted by $\mathcal{I}_m(n)$, that is,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^{m} l(1,i;n,j;\mathbf{S}).$$

Sometimes it is more convenient to name each step of lattice paths by a letter, and hence every lattice path will be encoded as a *lattice word*. We label the steps of the set $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$ by letters $\mathbf{u} = (1,1)$, $\mathbf{r} = (1,0)$, and $\mathbf{d} = (1,-1)$; also if $\mathbf{h}$ is a letter of the word $\mathcal{W}$, order or size of $\mathbf{h}$ in $\mathcal{W}$ is the number of times the letter $\mathbf{h}$ appears in the word $\mathcal{W}$ and it is denoted by $|\mathbf{h}| = |\mathbf{h}|_{\mathcal{W}}$.

2 Main Results

Clearly, $\mathcal{I}_m(n)$ is the number of words $a_1a_2\ldots a_{n-1}a_n$ ($a_i \in \{1,\ldots,m\}$) such that $|a_{i+1} - a_i| \leqslant 1$ for all $i = 1,\ldots,n-1$. In this section, we give formulas for the number $\mathcal{I}_m(n)$ in the cases where $n+1 \leqslant m \leqslant 2n$ and $2n \leqslant m$. To achieve this goal, we must recall some further notations from [1]. The number of lattice paths from the $(1,i)$-cell to the (s,t)-cell is denoted by $\mathcal{D}^i(s,t)$. Indeed, $\mathcal{D}^i(s,t) = l(1,i;s,t:\mathbf{S})$. Also, the number of lattice paths from the first column to the (s,t)-cell is denoted by $\mathcal{D}_{m,n}(s,t)$, or $\mathcal{D}(s,t)$ if there is no confusion, that is,

$$\mathcal{D}_{m,n}(s,t) = \sum_{i=1}^{m} \mathcal{D}^i(s,t).$$

In what follows, the number of lattice paths from $(1,1)$-cell to (s,t)-cell ($1 \leqslant s \leqslant n$ and $1 \leqslant t \leqslant m$), using just the two steps $(1,1)$ and $(1,-1)$, is denoted by $\mathcal{A}(s,t)$. In other words, $\mathcal{A}(s,t) = l(1,1;s,t:\mathbf{S}')$, where $\mathbf{S}' = \{(1,1),(1,-1)\}$. Clearly $\mathcal{A}(s,t) = 0$ for $s < t$, and that $\mathcal{A}(s,t)$ is the number lattice paths from the $(1,1)$-cell to (s,t)-cell not sliding above the line $y = x$. One observe that $\mathcal{A}(s,t) = 0$ if s,t have distinct parities as the paths counted by $\mathcal{A}(s,t)$ begins from $(1,1)$ and every step in $\mathbf{S}'$ keeps the parities of entries so that such paths never meet (s,t)-cells with (s,t) having distinct parities. Using the symbols $\mathbf{u}$ and $\mathbf{d}$, the number $\mathcal{A}(s,t)$ counts the words of length $s-1$ on $\{\mathbf{u},\mathbf{d}\}$ whose all initial subwords have more or equal $\mathbf{u}$ than $\mathbf{d}$. Analogous to $\mathcal{A}(s,t)$, the number $\mathcal{D}^1(s,t)$ counts the words $a_1a_2\ldots a_i$ with $1 \leqslant a_i \leqslant t$ such that $|a_{i+1} - a_i| \leqslant 1$ for all $1 \leqslant i \leqslant s-1$. In other words, $\mathcal{D}^1(s,t)$ counts the number of words of length $s-1$ on $\{\mathbf{u},\mathbf{r},\mathbf{d}\}$ whose all initial subwords have more or equal $\mathbf{u}$ than $\mathbf{d}$.

Theorem 2.1. *For all* $1 \leqslant s,t \leqslant m$, *we have*

$$\mathcal{D}^1(s,t) = \sum_{i=0}^{\lfloor \frac{s-t}{2} \rfloor} \binom{s-1}{s-t-2i} \mathcal{A}(t+2i,t).$$

Example 2.2. Using theorem 2.1, we can compute $\mathcal{D}^1(8,4)$ as

$$\mathcal{D}^1(8,4) = \sum_{i=0}^{\lfloor \frac{8-4}{2} \rfloor} \binom{8-1}{8-4-2i} \mathcal{A}(2i+4,4) = 133.$$

The numbers $\mathcal{A}(s,t)$ are indeed computed as in the ballot problem were the paths can touch the $y=x$ line but never go above it. The number of such ballot paths from $(1,0)$ to (m,n) is $\frac{m-n+1}{m+1}\binom{m+n}{m}$. Recall that $\mathcal{A}(s,t)$ is the number of words $\mathcal{W}$ of length $s-1$ on $\{\mathtt{u},\mathtt{d}\}$ with more or equal u than d in any initial subword, hence $\mathcal{A}(s,t)$ is equal to the above number with $m := |\mathtt{u}|_{\mathcal{W}}$ and $n := |\mathtt{d}|_{\mathcal{W}}$. Now since $|\mathtt{u}|_{\mathcal{W}} + |\mathtt{d}|_{\mathcal{W}} = s-1$ and $|\mathtt{u}|_{\mathcal{W}} - |\mathtt{d}|_{\mathcal{W}} = t-1$, it follows that $m=(s+t)/2-1$ and $n=(s-t)/2$. Hence we obtain the following

Lemma 2.3. *Inside the $n\times n$ table, we have*

$$\mathcal{A}(s,t)=\frac{2t}{s+t}\binom{s-1}{\frac{s-t}{2}}.$$

Hence

$$\mathcal{D}^1(s,t)=\sum_{i=0}^{\lfloor\frac{s-t}{2}\rfloor}\frac{t}{t+i}\binom{s-1}{s-t-2i}\binom{t+2i-1}{i}.$$

for all $1\leqslant s,t\leqslant n$.

In [1], we have computed the number $\mathcal{I}_n(n)$ for all $n\geqslant 1$. In what follows, we shall give formulas for $\mathcal{I}_m(n)$, where $n+1\leqslant m\leqslant 2n$. To achieve this goal, we use the numbers $\mathcal{H}(s,t)$ inside the $m\times n$ table defined as $\mathcal{H}(s,t)=\sum_{i=1}^{t}\mathcal{D}^1(s,i)$, where $1\leqslant s\leqslant n$ and $1\leqslant t\leqslant m$.

Lemma 2.4. *Inside the $m\times n$ table with $m\leqslant n\leqslant 2m$, we have*

$$\mathcal{H}(n,m)=\mathcal{D}(n,n)-\sum_{i=m}^{n-1}3^{n-i-1}\mathcal{D}^1(i,m).$$

Example 2.5. Using Lemma 2.4, we can calculate $\mathcal{H}(9,5)$ as

$$\mathcal{H}(9,5)=\mathcal{D}(9,9)-\sum_{i=5}^{8}3^{8-i}\mathcal{D}^1(i,5)=1931.$$

Lemma 2.6. *Inside the $m\times n$ table, we have*

$$\mathcal{D}^1(n,m)=\sum_{i=1}^{m}\mathcal{D}^1(s,i)\times\mathcal{D}^1(n-s+1,m-i+1).$$

for all $1\leqslant s\leqslant n$.

Example 2.7. Lemma 2.6 gives a way to compute $\mathcal{D}^1(9,5)$ in the following:

$$\mathcal{D}^1(9,5)=\sum_{i=1}^{5}\mathcal{D}^1(5,i)\mathcal{D}^1(9-5+1,5-i+1)=195.$$

Lemma 2.8. *Inside the $m\times n$ table, we have*

$$\mathcal{D}(s,t)=3^{s-1}-\sum_{i=t+1}^{s-1}3^{s-i-1}\mathcal{D}^1(i,t)-\sum_{i=m+2-t}^{s-1}3^{s-i-1}\mathcal{D}^1(i,m+1-t)$$

for all $s\leqslant n+2$.

Theorem 2.9. *Inside the $m \times n$ table, we have*

$$\mathcal{I}_m(n) = \sum_{i=1}^{m} \mathcal{D}(a,i)\mathcal{D}(b,i)$$

for all $a, b \geqslant 1$ such that $a + b = n + 1$. In other words, the inner product of columns a and b equals $\mathcal{I}_m(n)$. In particular, if $n = 2k - 1$ is odd, then

$$\mathcal{I}_m(n) = \sum_{i=1}^{m} \mathcal{D}_{k,i}^2.$$

Now, we calculate $S(x, y)$, the number of all perfect lattice paths from $(1,1)$-cell to (x, y)-cell in the whole space (not restricted to a table):

Theorem 2.10. *The number $S(x, y)$ is given by*

$$S(x,y) = \sum_{r=0}^{x-1} \binom{x-1}{r} \binom{x-r-1}{\frac{x-y-r}{2}} = \sum_{d=0}^{\lfloor \frac{x-y}{2} \rfloor} \binom{x-1}{d} \binom{x-d-1}{x-y-2d}.$$

Let $T = T_{m,n}$ and $S_{(a,b)}(x, y)$ denote the number of all perfect lattice path from (a, b)-cell to (x, y)-cell without leaving the table T. As before one can compute $S_{(a,b)}(x, y)$ by subtracting the number of all those paths starting from (a, b)-cell and ending at (x, y)-cell and leave the table from the total number of such paths. We have

Theorem 2.11. *Inside the $m \times n$ table, for all $a \leqslant x$ and $b \leqslant y$, we have*

$$\begin{aligned} S_{a,b}(x,y) =& S(x-a+1, y-b+1) - \sum_{x'=a+b}^{x-y} \mathcal{D}^1(x'-a, b) S(x-x'+1, y+1) \\ & - \sum_{x'=m+a-b+1}^{x+y-m-1} \mathcal{D}^1(x'-a, m-b+1) S(x-x'+1, m-y). \end{aligned}$$

Example 2.12. Utilizing Theorems 2.10 and 2.11, we see that, inside the 8×8 table,

$$S_{(2,1)}(7,3) = S(6,3) = 25.$$

References

[1] D. Yaqubi, M. Farrokhi D. G., and H. Ghasemian Zoeram, Lattice paths inside a table I, *Submitted*.

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Certificate

To whom it may concern:

This letter certifies that **A. Jafarzadeh** attended "10th Graph Theory and Algebraic Combinatorics Conference (GTACC10)" that was held on 17 and 18 January 2018 at Yazd University, Yazd, Iran; and gave a presentation **(joint with D. Yaqubi)** entitle **"Perfect Lattice Paths in the Plane"** at the conference.

Best regards,
Saeid Alikhani
Chair of Conference

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