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Isomorphic g-noncommuting graphs of finite groups

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Abstract. Let G be a finite non-abelian group and g be a fixed element of G. In 2014, Tolue *et al.* introduced the g-noncommuting graph of G (denoted by Γ_G^g) with vertex set G and two distinct vertices x and y join by an edge if $[x, y] \neq g$ and g^{-1}. In this paper, we consider an induced subgraph of Γ_G^g with vertex set $G \setminus Z(G)$ which is denoted by Δ_G^g. We state some properties of Δ_G^g and prove that two groups with isomorphic g-noncommuting graphs have the same order.

1. Introduction

Recently, joining graph theory and group theory together form a topic which is one of the most interest to some authors. There are many graphs associated to groups, rings or some algebraic structures. We may refer to works on non-commuting graphs [2], relative non-commuting graphs [16], Engel graphs [1] and non-cyclic graphs [3]. One of the important graphs associated to a group is the non-commuting graph. This graph, first introduced by Paul Erdős [12], was denoted by Γ_G and is a graph with $G \setminus Z(G)$ as the vertex set and two distinct vertices x and y join, whenever $xy \neq yx$. The concept of non-commuting graphs has been generalized in some different ways. One of them is the generalized non-commuting graph related to a subgroup H of G (see [16]) or even related to two subgroups H and K (see [8]). Moreover, there is another generalization of non-commuting graphs via an automorphism (see [5]). Now, we are going to consider the new generalization of non-commuting graphs called g-noncommuting graphs,

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which is associated to a fixed element g of group G, given by Tolue *et al.* in [15] as the following.

Definition 1.1. For any non-abelian group G and fixed element g in G, the g-noncommuting graph of G is the graph with vertex set G and two distinct vertices x and y join by an edge if $[x, y] \neq g$ and g^{-1}.

There are some results on g-noncommuting graphs. For instance, some graph theoretical invariants, planarity and regularity are stated in [15]. In this paper, we would like to consider the induced subgraph of g-noncommuting graphs on $G \setminus Z(G)$ which is denoted by Δ_G^g. It is obvious that if g is an identity element, then Δ_G^g coincides with the known non-commuting graph of G. Recall that $K(G) = \{[x, y] : x, y \in G\}$ is the set of commutators of G and $G' = \langle K(G)\rangle$. Kappe *et al.* [10] concluded with a status report on what is now called the Ore Conjecture, stating that every element in a finite non-abelian simple group is a commutator, and so $G' = K(G)$ in this case. It is clear that Δ_G^g is a complete graph whenever $g \notin K(G)$, and so everything is known. Thus, we always assume that $e \neq g \in K(G)$.

In Sections 2 and 3, we investigate some graph theoretical properties of Δ_G^g like clique number, regularity, planarity and connectivity.

In Section 4, we prove that for any two non-abelian finite groups G and H such that $\Delta_G^g \cong \Delta_H^h$, it holds that $|G| = |H|$ where $g \in G$ and $h \in H$. Moreover, we state a conjecture about the above graph isomorphism, and some of our attempts are also given at the end. Most of our notations and terminologies are standard and can be found in [6].

2. Some properties of g-noncommuting graphs

In this section, we may investigate some graph theoretical properties of Δ_G^g. Let us start with mentioning some relations between the new graph Δ_G^g and a commuting graph.

Lemma 2.1. *The commuting graph of group G is a spanning subgraph of Δ_G^g.*

Proof. It is straightforward. $\square$

Lemma 2.2. *If $K(G) = \{e, g\}$ or $\{e, g, g^{-1}\}$, then Δ_G^g is equal to a commuting graph.*

PROOF. In the first case, if x and y are adjacent in Δ_G^g, then $[x,y] \neq g$. Since $[x,y] \in K(G)$, $[x,y] = e$ and x and y are adjacent in the commuting graph. Now, suppose that $K(G) = \{e, g, g^{-1}\}$, then, if $[x,y] \neq g$ and g^{-1}, we should have $[x,y] = e$. Hence, the proof is completed. □

We know that the clique number of commuting graphs is equal to $|A| - |A \cap Z(G)|$, where A is an abelian subgroup of maximal order of G. So, the clique number of commuting graphs is a lower bound for the clique number of g-noncommuting graphs, and we have the following result:

Theorem 2.3. *Let G be a non-abelian group, and A be an abelian subgroup of maximal order of G. Then $\omega(\Delta_G^g) \geq |A| - |A \cap Z(G)|$.*

In [15], the authors gave a formula for the degree of vertices in Γ_G^g. Now, we can state it for Δ_G^g as follows. The proof is very similar to Lemma 2.2 in [15] and we omit here.

Lemma 2.4. *Let $x \in G \setminus Z(G)$.*

(i) *If $g^2 \neq e$, then $\deg(x) = |G| - |Z(G)| - \epsilon |C_G(x)| - 1$, where $\epsilon = 1$ if x is conjugate to xg or xg^{-1}, but not to both, and $\epsilon = 2$ if x is conjugate to xg and xg^{-1}.*

(ii) *If $g^2 = e$ and $g \neq e$, then $\deg(x) = |G| - |Z(G)| - |C_G(x)| - 1$, whenever xg is conjugate to x.*

(iii) *If xg and xg^{-1} are not conjugate to x, then $\deg(x) = |G| - |Z(G)| - 1$.*

Lemma 2.5. *If G is a group of odd order and Δ_G^g is a regular graph, then G is nilpotent.*

PROOF. Since $g \in K(G)$, the graph in not complete, so for every $x, y \in G \setminus Z(G)$ we have $|C_G(x)| = |C_G(y)|$. Therefore, the conjugacy classes of G have only two sizes, and by [9, Theorem 1] G is nilpotent. □

The planarity of Γ_G^g has been investigated in [15]. Here we deal with the planarity of Δ_G^g, indeed, we classify all groups of which the g-noncommuting graph is planar.

Theorem 2.6. *Let G be a finite non-abelian group. Then Δ_G^g is planar if and only if G is isomorphic to one of the following groups:*

(1) *S_3, D_8, Q_8, D_{10}, D_{12}, $D_8 \times \mathbb{Z}_2$, $Q_8 \times \mathbb{Z}_2$;*

(2) *$< a, b : a^3 = b^4 = e, a^b = a^{-1} > \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$;*

(3) *$< a, b : a^4 = b^4 = e, a^b = a^{-1} > \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$;*

(4) $< a, b : a^8 = b^2 = e, a^b = a^{-3} >\cong \mathbb{Z}_8 \rtimes \mathbb{Z}_2$;

(5) $< a, b : a^4 = b^2 = (ab)^4 = [a^2, b] = e >\cong (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$;

(6) $< a, b, c : a^2 = b^2 = c^4 = [a, c] = [b, c] = e, [a, b] = c^2 >\cong (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

Proof. If $|Z(G)| \geq 5$, then we have a clique of size 5. Thus, the planarity of Δ_G^g implies that $|Z(G)| \leq 4$. Also, if there exists an element $x \in G \setminus Z(G)$ such that $x^2 \notin Z(G)$ and $z_1, z_2 \in Z(G)$, then there is a clique with vertices $\{x, x^{-1}, xz_1, xz_2, x^{-1}z_1\}$. So $|Z(G)| \leq 2$ in this case. It is clear that if the degree of all vertices of Δ_G^g is greater than 5, then Δ_G^g will not be planar. Thus, there exists an element $x \in G \setminus Z(G)$ such that $\deg(x) \leq 5$. By Lemma 2.4, $|G| - \epsilon|C_G(x)| \leq 6 + |Z(G)|$, where $\epsilon = 1$ or 2. We know that $\deg(x) \geq 0$, therefore,

$$(\epsilon + 1)|C_G(x)| \leq |G| \implies |G| - \frac{\epsilon}{\epsilon + 1}|G| \leq |G| - \epsilon|C_G(x)| \leq 10.$$

Thus $|G| \leq 10(\epsilon + 1)$, where $\epsilon = 1$ or 2. So $|G| \leq 30$. Also, the commuting graph is a spanning subgraph of Δ_G^g, so it is enough to investigate groups of order less than 30 in [4, Theorem 2.2]. By using the group theory package GAP, the degrees of vertices of the graph associated to the above groups are computed and the proof is completed. □

3. Connectivity of g-noncommuting graphs

In this section, we focus on the connectivity of g-noncommuting graphs. Let us start with the following lemma:

Lemma 3.1. *Let g be a non-central element of G.*

(i) *If $g^2 = e$, then* $\operatorname{diam}(\Delta_G^g) = 2$.

(ii) *If $g^2 \neq e$ and $g^3 \neq e$, then* $\operatorname{diam}(\Delta_G^g) \leq 3$.

Proof. (i) Suppose that $x \neq g$ is a vertex of Δ_G^g. It is clear that $[x, g] \neq g$. If $g^2 = e$, then $[x, g] \neq g^{-1}$. Consequently, x is adjacent to g, or shortly, $x \sim g$, and so $\operatorname{diam}(\Delta_G^g) \leq 2$. Since $g = [x_1, x_2]$ for some $x_1, x_2 \in G \setminus Z(G)$, we have $d(x_1, x_2) \geq 2$. Therefore, $\operatorname{diam}(\Delta_G^g) = 2$.

(ii) Assume that $g^2 \neq e$ and $g^3 \neq e$. If $[x, g] \neq g^{-1}$, then $x \sim g$, and if $[x, g] = g^{-1}$, then we have

$$[x, g^2] = [x, g][x, g]^g = g^{-1}(g^{-1})^g = g^{-2}.$$

Since $[x, g^2] \neq g, g^{-1}$, it follows that $x \sim g^2$. Thus, every vertex x must join g or g^2. Now, for any two arbitrary vertices x and y, we can easily see that $d(x, y) \leq 2$ when x and y join g or g^2. If $x \sim g$ and $y \sim g^2$ or $y \sim g$ and $x \sim g^2$, then $d(x, y) \leq 3$. Hence $\operatorname{diam}(\Delta_G^g) \leq 3$. □

Theorem 3.2. *Let $|Z(G)| = 1$, $e \neq g \in G$ and $|C_G(g)| \neq 3$. Then Δ_G^g is connected.*

Proof. If $g^3 \neq e$, then the result holds by Lemma 3.1. If $g^3 = e$, then we can consider $g^2 \neq e$. Since $|C_G(g)| \neq 3$, it follows that $|C_G(g)| > 3$. Thus, there is an element $a \in C_G(g)$ such that $a \neq e, g$ and g^{-1}. Now, we can assume that $x_0 \in G \setminus Z(G)$ such that $[x_0, g^2] = g$ and $[x_0, g] = g^{-1}$. It is easy to see that $[x_0, ga] = [x_0, a](g^{-1})^a$ and $[x_0, g^2 a] = [x_0, a]g^{-1}[x_0, g]^a = [x_0, a]g^{-2}$. If $[x_0, a] \neq g, g^{-1}$, then a is adjacent to x_0 and g, and the graph is connected. In the case that $[x_0, a] = g$, it holds $[x_0, ga] = e$. Now, if $ga \in Z(G)$, then $a = g^{-1}$, a contradiction. Thus $ga \in G \setminus Z(G)$, and so ga is adjacent to x_0 and g. Hence the graph is connected. If $[x_0, a] = g^{-1}$, then $[x_0, g^2 a] = e$. In the case that $g^2 a \in Z(G)$, it holds $a = g^{-2} = g$, a contradiction. Thus $g^2 a$ is adjacent to x_0 and g. So the graph is connected and the proof is completed. □

As a consequence of the above corollary, we can state that if $|C_G(g)| \neq 3$, then Δ_G^g has no isolated vertex. First, we recall the following theorem from [13], which will be used in Proposition 3.4. We omit the proof.

Theorem 3.3. *Let G be a finite simple group, and $x \in G$ be an involution. Then $C_G(x) \neq G$, and if $|C_G(x)| = m$, then $|G| \leq (m(m+1)/2)!$.*

Proposition 3.4. *Let G be a non-abelian simple group. Then Δ_G^g has no isolated vertices.*

Proof. Let $x \in G \setminus Z(G)$. If $x^2 \neq e$, then x and x^{-1} are adjacent. If x is an involution and $|C_G(x)| = m$, then by Theorem 3.3, we must have $m \geq 3$. Thus, there is an element $t \in C_G(x)$ such that $t \neq e, x$, and so t is adjacent to x. Hence, the proof is completed. □

4. Isomorphism between g-noncommuting graphs

It is clear that if two groups G and H are isomorphic, then, obviously, $\Delta_G^g \cong \Delta_H^h$, but the converse is not true and it is interesting to find some conditions for the groups G and H to have $G \cong H$ or even $|G| = |H|$. This section involves the above isomorphism between g-noncommuting graphs. First, let us state the following important lemma which plays an important role in the proof of Theorem 4.2.

Lemma 4.1. *Let x be a non-isolated vertex in Δ_G^g such that $\deg(x) \neq |G|-|Z(G)|-1$, where g is an arbitrary fixed element in $K(G)$. If H is a group such that $\Delta_G^g \cong \Delta_H^h$ for some $h \in K(H)$, then $|Z(H)|$ divides $(|G|-|Z(G)|, |C_G(x)|)$ or $(|G|-|Z(G)|, 2|C_G(x)|)$.*

Proof. Assume that ϕ is an isomorphism between graphs Δ_G^g and Δ_H^h, and $\phi(x) = y$. Then by Lemma 2.4, $\deg(x) = |G| - |Z(G)| - \epsilon|C_G(x)| - 1$, where $\epsilon = 1$ or 2. Also, we have

$$|G| - |Z(G)| = |H| - |Z(H)| = |Z(H)| \left(\frac{|H|}{|Z(H)|} - 1 \right).$$

Since $\deg(x) = \deg(y)$, if $\deg(x) = |G| - |Z(G)| - |C_G(x)| - 1$, we have

$$|G| - |Z(G)| - |C_G(x)| = \begin{cases} |Z(H)| \left(\frac{|H|}{|Z(H)|} - \frac{|C_H(y)|}{|Z(H)|} - 1 \right) \text{ or} \\ |Z(H)| \left(\frac{|H|}{|Z(H)|} - 2\frac{|C_H(y)|}{|Z(H)|} - 1 \right). \end{cases}$$

Thus, $|Z(H)|$ divides $(|G|-|Z(G)|, |C_G(x)|)$. Similarly, if $\deg(x) = |G|-|Z(G)|-2|C_G(x)|-1$, then $|Z(H)|$ will divide $(|G|-|Z(G)|, 2|C_G(x)|)$, and the proof is completed. □

Now, we are in a position to prove the main theorem.

Theorem 4.2. *Let G and H be two non-abelian finite groups such that $\Delta_G^g \cong \Delta_H^h$, for some non-identity element $h \in H$. Then $|G| = |H|$.*

Proof. Assume that ϕ is an isomorphism between graphs Δ_G^g and Δ_H^h. Since $\Delta_G^g \cong \Delta_H^h$, we have $|G| - |Z(G)| = |H| - |Z(H)|$, and it is enough to prove $|Z(G)| = |Z(H)|$. Since $e \neq g \in K(G)$, there are vertices $x, y \in G \setminus Z(G)$ such that $[x, y] = g$. So x cannot be adjacent to y, and $\deg(x) \neq |G| - |Z(G)| - 1$. First, suppose that $|Z(G)| \neq 1$, then Δ_G^g has no isolated vertex because every non-central element of G, like t, is adjacent to tz for some $z \in Z(G)$. Thus $|Z(H)|$ divides $|Z(G)|(((|G|/|Z(G)|) - 1, |C_G(x)|/|Z(G)|)$ or $|Z(G)|(((|G|/|Z(G)|) - 1, 2|C_G(x)|/|Z(G)|)$, by Lemma 4.1. In the first case, we may put $d = (|G|/|Z(G)|-1, |C_G(x)|/|Z(G)|)$, and so d divides $|C_G(x)|/|Z(G)|$ and $|G|/|Z(G)| - 1$. Hence, $d \mid (|G|/|Z(G)| - 1, |G|/|Z(G)|) = 1$ and we should have $d = 1$. Therefore, $|Z(H)| \mid |Z(G)|$. In the second case, we may consider

$d = (|G|/|Z(G)| - 1, 2|C_G(x)|/|Z(G)|)$, and by a similar argument, $d \mid 2$, which implies that $|Z(H)| \mid 2|Z(G)|$.

If $\phi(x) = x'$, then $\deg(x) = \deg(x')$, and we have

$$|G| - |Z(G)| - \epsilon|C_G(x)| - 1 = |H| - |Z(H)| - \epsilon^{'}|C_H(x')| - 1, \quad \epsilon, \epsilon^{'} = 1 \text{ or } 2$$

Thus, if $\epsilon = 1$, then $|C_G(x)| = |C_H(x')|$ or $2|C_H(x')|$, and if $\epsilon = 2$, then $|C_G(x)| = |C_H(x')|$ or $\frac{1}{2}\ |C_H(x')|$. Now, we consider the following cases:

Case 1. $\epsilon = 1$.

If $|C_G(x)| = |C_H(x')|$ and $|Z(G)| \neq |Z(H)|$, then $|Z(H)| \leq \frac{1}{2}\ |Z(G)|$. Hence

$$|C_H(x')| = |C_G(x)| \text{ divides } |H| = |G| - |Z(G)| + |Z(H)|,$$

and $|Z(G)| < |C_G(x)|$, so $|C_G(x)| \mid |Z(G)| - |Z(H)|$. Thus $0 < |Z(G)| - |Z(H)| < |Z(G)|$, which is a contradiction. Hence, $|Z(G)| = |Z(H)|$ in this case. If $|C_G(x)| = 2|C_H(x')|$, then

$$\frac{1}{2}|C_G(x)| = |C_H(x')| \text{ divides } |H| = |G| - |Z(G)| + |Z(H)|.$$

Since $|Z(G)| \mid |C_G(x)|$ and $Z(G) \lneqq C_G(x)$, it follows that $|Z(G)| \leq \frac{1}{2}\ |C_G(x)|$. Consequently, $\frac{1}{2}\ |C_G(x)| \mid |G|$ implies that $\frac{1}{2}\ |C_G(x)| \mid |Z(G)| - |Z(H)|$, which is impossible. Hence $|Z(G)| = |Z(H)|$.

Case 2. $\epsilon = 2$.

We have $|Z(H)| \mid 2|Z(G)|$. If $|C_G(x)| = |C_H(x')|$ or $2|C_G(x)| = |C_H(x')|$, then

$$|C_G(x)| \mid |G| - |Z(G)| + |Z(H)|.$$

Thus $|C_G(x)| \mid |Z(G)| - |Z(H)|$. If $|Z(H)| = 2|Z(G)|$, then $|C_G(x)|$ divides $|Z(G)|$, a contradiction. Therefore, $|Z(H)| \leq |Z(G)|$, and so $|C_G(x)| \mid |Z(G)| - |Z(H)|$. Thus again we should have $|Z(G)| = |Z(H)|$ in this case.

Now, assume that $|Z(G)| = 1$, then there exists a non-central element t in G such that $t^2 \neq 1$. Thus t and t^{-1} are adjacent. If x or y are not isolated vertices, then, by a similar proof as above, we again have $|Z(G)| = |Z(H)|$. If x and y are isolated, then t and x are not adjacent. Therefore, $\deg(t) = |G| - |Z(G)| - |C_G(t)| - 1$ or $|G| - |Z(G)| - 2|C_G(t)| - 1$, and we can replace the vertex x by t. Thus the proof is completed. $\square$

Corollary 4.3. *Let $\Delta_G^g \cong \Delta_H^h$ with the same condition as in Theorem 4.2. If $|G|$ is odd and x is a vertex in Δ_G^g with $\deg(x) \neq |G| - |Z(G)| - 1$, then $|C_G(x)| = |C_H(\phi(x))|$, where ϕ is an isomorphism between the above two graphs.*

Proof. Since $|G|$ is odd, it holds $\deg(x) \neq 0$. We have $\deg(x) = |G| - |Z(G)| - \epsilon|C_G(x)| - 1$ and $\deg(y) = |H| - |Z(H)| - \epsilon'|C_H(y)| - 1$, where ϵ and $\epsilon' = 1$ or 2, and $y = \phi(x)$. If $\epsilon = \epsilon'$, then we have nothing to prove. Otherwise, $|G| = |H|$ is an even number, which is a contradiction. $\square$

In the next theorem, we will state some conditions under which the isomorphism between two graphs Δ_G^g and Δ_H^h deduces that if G is nilpotent, then H is nilpotent as well. We remind that $N(G)$ stands for the set $\{n \in \mathbb{N} |$ G has a conjugacy class of size n$\}$, and a group G is called an extra-special p-group if G is a p-group and $|G'| = |Z(G)| = p$.

Theorem 4.4. *Let G be a finite non-abelian group of odd order, and assume that Δ_G^g has no vertex adjacent to all other vertices. If $\Delta_G^g \cong \Delta_H^h$, then $N(G) = N(H)$, and if G is nilpotent, then H is nilpotent.*

Proof. Clearly, $N(G) = N(H)$, by Corollary 4.3. By the main result of [7], we know that if the number of conjugacy classes of size i for the nilpotent group G is equal to the number of conjugacy classes of size i of H for each i, then H is nilpotent. Theorem 4.2 implies that $|Z(G)| = |Z(H)|$. Now, if $x \in G \setminus Z(G)$, then by Corollary 4.3, we have $|C_G(x)| = |C_H(\phi(x))|$, where ϕ is the isomorphism between two graphs. Hence the proof is completed. $\square$

Lemma 4.5. *Let G be an extra-special p-group and $\Delta_G^g \cong \Delta_H^h$. If H is a nilpotent group of class 2, then H is also an extra-special p-group and $N(G) = N(H)$.*

Proof. By Theorem 4.2, $|Z(G)| = |Z(H)| = p$. Since the nilpotency class of H is 2, it follows that $H/Z(H)$ is an abelian group, and therefore, $H' \leq Z(H)$. So $|H'| = |Z(H)| = p$. Hence H is an extra-special p-group. Now, by [11, Theorem 3], the conjugacy classes of G and H have orders 1 or p. Thus the proof is completed. $\square$

Finally, it can be easily seen that if G is a p-group of order p^n with $|Z(G)| = p^{n-2}$ and $\Delta_G^g \cong \Delta_H^h$, then $N(G) = N(H)$. Furthermore, if G is a non-abelian simple group satisfying the Thompson's conjecture, $\Delta_G^g \cong \Delta_H^h$ and $N(G) = N(H)$, then $G \cong H$.

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