The 7^{th} Algebraic Combinatorics Conference of Iran
October 16-17, 2014, Ferdowsi University of Mashhad, Iran

ON THE RELATIVE NON-COMMUTING GRAPH

MEHDI HASSANKHANI[1*], ABBAS JAFARZADEH[1]
AND ABBAS MOHAMMADIAN[1]

[1] *Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.*

ABSTRACT. Let G be a non-abelian group and H a subgroup of G. We associate a graph $\Gamma_{(H,G)}$ to H and G as follows: Take $G \setminus C_G(H)$ as the vertices of $\Gamma_{(H,G)}$ and join two distinct vertices x and y, whenever x or y in H and $[x,y] \neq 1$. In this talk we prove that $\Gamma_{(H,G)}$ neither planar nor regular. Moreover, we show that if $\Gamma_{(H_1,G_1)} \cong \Gamma_{(H_2,G_2)}$, then $\Gamma_{H_1} \cong \Gamma_{H_2}$.

1. INTRODUCTION

In [3], the authors introduce a simple graph $\Gamma_{(H,G)}$ to H and G whose vertex set is $G \setminus C_G(H)$ and two distinct vertices x and y, whenever x or y in H are adjacent if and only if $[x,y] \neq 1$.

In this article, we intend to improve the results obtained in paper [3]. Moreover, we show that if $\Gamma_{(H_1,G_1)} \cong \Gamma_{(H_2,G_2)}$, then $\Gamma_{H_1} \cong \Gamma_{H_2}$. Also, we prove that there is no finite group K such that $\Gamma_{(H,G)} \cong \Gamma_K$. Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [2]. For any graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $deg(v)$ of a vertex v in Γ is the number of edges incident to v. Γ is regular if the degrees of all vertices of Γ are the same. A subset X of the vertices of Γ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is

---

2010 *Mathematics Subject Classification.* Primary 05C25; Secondary 20P05.
Key words and phrases. relative non-commuting graph, planar, regular.
* Speaker.

called the clique number of Γ and denoted by $\omega(\Gamma)$. A subset X of the vertices of Γ is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(G)$. The length of the shortest cycle in a graph Γ is called girth of Γ and denoted by $gr(\Gamma)$. A Hamilton cycle of Γ is a cycle that contains every vertex of Γ. A planar graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident.

In this article, G is a finite non-abelian group and H is a non-abelian subgroup of G. we denote the symmetric group and the alternating group on n letters by S_n and A_n, respectively. Also Q_8 and D_{2n} are used for the quaternion group with 8 elements and the dihedral group of order $2n$ ($n > 2$), respectively.

In Section 2, we study some graph properties of the relative non-commuting graph $\Gamma_{(H,G)}$ of a non-abelian subgroup H of G. We prove that $\Gamma_{(H,G)}$ is a connected graph and moreover, $diam(\Gamma_{(H,G)}) = 2$ and $gr(\Gamma_{(H,G)}) = 3$. Also, for any group G, $\Gamma_{H,G}$ neither planar nor regular.

In section 3, we give a positive answer to the following question:

Question 1.1. Let G_1 and G_2 be two finite non-abelian groups and H_1 and H_2 are non-abelian subgroup of G_1 and G_2, respectively. Is $\Gamma_{(H_1,G_1)} \cong \Gamma_{(H_2,G_2)}$ implies that $\Gamma_{H_1} \cong \Gamma_{H_2}$?

2. Some properties of relative non-commuting graphs

Proposition 2.1. *$diam(\Gamma_{(H,G)}) = 2$ and $gr(\Gamma_{(H,G)}) = 3$.*

Proposition 2.2. *The graph $\Gamma_{(H,G)}$ is not regular.*

Proposition 2.3. *The graph $\Gamma_{(H,G)}$ is not planar.*

Proposition 2.4. *In the graph $\Gamma_{(H,G)}$,*

(i) *$V_{(G\setminus H)}$ is only maximal independent set. In particular, $\alpha(\Gamma_{(H,G)}) = V_{(G\setminus H)} = G \setminus (H \cup C_G(H))$.*
(ii) *$\omega(\Gamma_{(H,G)}) = \omega(\Gamma_H) + 1$.*

Theorem 2.5. *Let $\Gamma_{(H,G)}$ be a Hamiltonian graph. Then the following hold:*

(i) *$[G : H] = 2$ and $Z(H) \leq C_G(H) = Z(G)$.*
(ii) *$G = HZ(G)$ and $[Z(G) : Z(H)] = 2$.*
(iii) *If $V_H = \{h_1, h_2, \ldots, h_n\}$, then $V_{G\setminus H} = \{h_1z, h_2z, \ldots, h_nz\}$ for some $z \in Z(G) \setminus H$. Moreover, $deg(h_i) = 2deg(h_iz)$, for every $1 \leq i \leq n$.*
(iv) *$|V_{G\setminus H}| = |V_H| = \frac{1}{2}(|G| - |Z(G)|)$.*

Remark 2.6. Note that the converse of Theorem 2.5 is true only if $|V_H|$ is odd. Indeed, let $|V_H| = 2k+1$ and $\{h_1, h_2, \dots, h_{2k+1}\}$ is a Hamiltonian cycle in the graph Γ_H. Then by Theorem 2.5, $V_{G\setminus H} = \{h_1 z, h_2 z, \dots, h_{2k+1} z\}$, for some $z \in Z(G)$. Therefore

$$\{h_1, h_2 z, h_3 \dots, h_{2k} z, h_{2k+1}, h_1 z, h_2, h_3 z \dots, h_{2k}, h_{2k+1} z\}$$

is a Hamiltonian cycle in the graph $\Gamma_{(H,G)}$.

3. Groups with the same relative non-commuting graphs

In this section we consider the non-abelian groups with isomorphic relative non-commuting graphs.

Theorem 3.1. *If $\Gamma_{(H,G)} \cong \Gamma_{(H',G')}$, then $\Gamma_{H_1} \cong \Gamma_{H_2}$.*

In the following theorem we show that if H is a subgroup of finite group G, then there is no finite group K such that $\Gamma_{(H,G)} \cong \Gamma_K$.

Theorem 3.2. *There is no finite group K such that $\Gamma_{(H,G)} \cong \Gamma_K$.*

In Theorem 3.1, if we put $G' = S_n$ and $H' = A_n$, then we can prove that the following theorem:

Theorem 3.3. *Let $\Gamma_{(A_n,S_n)} \cong \Gamma_{(H,G)}$, where G is a non-abelian group and H is a proper non-abelian subgroup of G. Then $|G| = |S_n|$ and $H \cong A_n$.*

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, Non-commuting graph of a group, *J. Algebra 298* (2006), 468–492.

[2] J.A. Bondy, U.S.R. Murty, *Graph Theory with applications*, American Elsevier, New York, 1976.

[3] B. Toloue, A. Erfanian, Relative non-commuting graphs, *J. Algebra Appl. 2* (2013),1–11