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On θ-commutators and the corresponding non-commuting graphs

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Abstract: The θ-commutators of elements of a group with respect to an automorphism are introduced and their properties are investigated. Also, corresponding to θ-commutators, we define the θ-non-commuting graphs of groups and study their correlations with other notions. Furthermore, we study independent sets in θ-non-commuting graphs, which enable us to evaluate the chromatic number of such graphs.

Keywords: Commutator, Automorphism, Independence number, Chromatic number, Multipartite graph

MSC: 05C25, 05C15, 05C69, 20D45, 20F12

1 Introduction

The *commutator* of two elements x and y of a group G is defined usually as $[x, y] := x^{-1}y^{-1}xy$. The influence of commutators in the theory of groups is inevitable and the analogy of computations encouraged some authors to define and study modifications of the ordinary commutators to include automorphisms or more generally endormorphisms of the underlying groups. The first of those is due to Ree [1] who generalizes the conjugation of x by y with respect to an endomorphism θ of G as $y^{-1}x\theta(y)$ and uses it to make relationships between the corresponding conjugacy classes with special ordinary conjugacy classes and irreducible characters of the group. Later, Acher [2] invokes a very similar generalization of conjugation as to that of Ree and studies the corresponding generalized conjugacy classes, centralizers and the center of groups in a more abstract way. Writing the commutators as $[x, y] = x^{-1}I_y(x)$, I_y being the inner automorphism associate to y, one may generalize them in a natural way to $[x, \theta] = x^{-1}\theta(x)$, in which θ is an endomorphism of the underlying group. The element $[x, \theta]$, called the *autocommutator* of the element x and automorphism θ when θ is an automorphism, seems to appear first in Gorenstein's book [3, p. 33] while it first appears in practice in the pioneering papers [4, 5] of Hegarty.

According to Ree's definition of conjugation, the commutator of two elements x and y of a group G with respect to an endomorphism θ will be $[x, y]_\theta := x^{-1}y^{-1}x\theta(y)$. One observes that $[x, y]_\theta = 1$ if and only if $\theta(y) = y^x$. Hence $[x, y]_\theta = 1$ does not guarantee in general that $[y, x]_\theta = 1$. The aim of this paper is to introduce a new generalization of commutators, as a minor modification to that of Ree, in order to obtain a new commutator behaving more like the ordinary commutators. Indeed, we define the conjugation of x by y via θ as $\theta(y)^{-1}\theta(x)y$, which is simply the image of $y^{-1}x\theta^{-1}(y)$, the conjugate of x by y via θ^{-1} in the sense of Ree's, under θ. Hence the corresponding commutators, we call them the *θ-commutators*, will be

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$[x,_\theta y] := x^{-1}\theta(y)^{-1}\theta(x)y$ and we observe that $[x,_\theta y] = 1$ if and only if $[y,_\theta x] = 1$. This property of θ-commutators, as we will see later, remains unchanged modulo a shift of elements by left multiplication corresponding to automorphisms which are congruent modulo the group of inner automorphisms. Therefore, all inner automorphisms give rise to same θ-commutators modulo a shift of elements by left multiplication.

The paper is organized as follows: Section 2 initiates the study of θ-commutators by generalizing the ordinary commutator identities as well as centralizers and the center of a group, and determines the structure of θ-centralizers and θ-center of the groups under investigation. In section 3, we shall define the θ-non-commuting graph associated to θ-commutators of a group and describe some of its basic properties and its correlations with other notions, namely fixed-point-free and class preserving automorphisms. Sections 4 and 5 are devoted to the study of independent subsets of θ-non-commuting graphs where we give an explicit structural theorem for them and apply them to see under which conditions the θ-non-commuting graphs are union of particular independent sets.

Throughout this paper, we use the following notations: given a graph Γ, the set of its vertices and edges are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. For every vertex $v \in V(\Gamma)$, the neighbor of v in Γ is denoted by $N_\Gamma(v)$ and the degree of v is given by $\deg_\Gamma(v)$. For convenience, we usually drop the index Γ and write $N(v)$ and $\deg(v)$ for the neighbor and degree of the vertex v, respectively. A subset of $V(\Gamma)$ with no edges among its vertices is an *independent set*. The maximum size of an independent set in Γ is denoted by $\alpha(\Gamma)$ and called the *independence number* of Γ. Also, the minimum number of independent sets required to cover all vertices of Γ is the *chromatic number* of Γ and it is denoted by $\chi(\Gamma)$. All other notations regarding groups, their subgraphs and automorphisms are standard and follow that of [6].

2 Some basic results

Recall that θ-commutator of two elements x and y of a group G with respect to an automorphism θ of G is defined as $[x,_\theta y] := x^{-1}\theta(y)^{-1}\theta(x)y$. Also, the *autocommutator* of x and θ is known to be $[\theta, x]^{-1} = [x, \theta] := x^{-1}\theta(x)$. We begin with the following lemma, which gives a θ-commutator analogue of some well-known commutator identities.

Lemma 2.1. *Let G be a group, x, y, z be elements of G and θ be an automorphism of G .Then*
(1) $[x,_\theta y]^{-1} = [y,_\theta x]$;
(2) $\theta(x)^{\theta(y)} = x[x,_\theta y][y, \theta]$;
(3) $[x,_\theta yz] = [x,_\theta z][\theta, x]^z[x,_\theta y]^z$;
(4) $[xy,_\theta z] = [x,_\theta z]^y[z, \theta]^y[y,_\theta z]$; *and*
(5) $[x,_\theta y^{-1}] = [x, \theta][yx,_\theta x]^{(yx)^{-1}}$.

The θ-centralizer of elements as well as the θ-center of a group can be defined analogously as follows:

Definition 2.2. *Let G be a group and θ be an automorphism of G. The θ-centralizer of an element $x \in G$, denoted by $C_G^\theta(x)$, is defined as*

$$C_G^\theta(x) = \{y \in G \mid [x,_\theta y] = 1\}.$$

Utilizing θ-centralizers, the θ-center of G is defined simply as

$$Z_\theta(G) = \bigcap_{x\in G} C_G^\theta(x) = \{y \in G \mid [x,_\theta y] = 1, x \in G\}.$$

In contrast to natural centralizers and the center of a group, θ-centralizers and the θ-center of a group G need not be subgroups of G. For example, if $G = \langle x\rangle \cong C_3$ and θ is the nontrivial automorphism of G, then $Z_\theta(G) = \emptyset$ and $C_G^\theta(x) = \{x\}$. In what follows, we discuss the situations that θ-centralizers and the θ-center of a group turn into subgroups.

Theorem 2.3. *Let G be a group and θ be an automorphism of G. Then*
(1) $C_G^\theta(1) = \mathrm{Fix}(\theta)$*;*
(2) $x^{-1}C_G^\theta(x)$ *is a subgroup of G for all $x \in G$;*
(3) $C_G^\theta(x)$ *is a subgroup of G if and only if $x^2 \in \mathrm{Fix}(\theta)$; and*
(4) $|C_G^\theta(x)|$ *divides* $|G|$.

Proof. (1) It is obvious.

(2) Let $y, z \in C_G^\theta(x)$. Then $\theta(x^{-1}y) = (x^{-1}y)^{x^{-1}}$ and $\theta(x^{-1}z) = (x^{-1}z)^{x^{-1}}$ so that $\theta(x^{-1}yx^{-1}z) = (x^{-1}yx^{-1}z)^{x^{-1}}$. Hence $yx^{-1}z \in C_G^\theta(x)$, that is, $(x^{-1}y)(x^{-1}z) \in x^{-1}C_G^\theta(x)$. On the other hand, $xy^{-1}x \in C_G^\theta(x)$, from which it follows that $(x^{-1}y)^{-1} = x^{-1}xy^{-1}x \in x^{-1}C_G^\theta(x)$. Therefore, $x^{-1}C_G^\theta(x)$ is a subgroup of G.

(3) From (2) it follows that $C_G^\theta(x)$ is a subgroup of G if and only if $x^{-1} \in C_G^\theta(x)$ and this holds if and only if $x^2 \in \mathrm{Fix}(\theta)$.

(4) It follows from (2). □

Lemma 2.4. *Let G be a group and θ be an automorphism of G .Then*
(1) $Z_\theta(G) \neq \emptyset$ *if and only if $\theta \in \mathrm{Inn}(G)$; and*
(2) $Z_\theta(G) = Z(G)g^{-1}$ *whenever $\theta = I_g \in \mathrm{Inn}(G)$.*
As a result, $Z_\theta(G)$ is a subgroup of G if and only if θ is the identity automorphism.

Proof. (1) If $x \in Z_\theta(G)$, then $[x,_\theta x^{-1}y] = 1$ for all $y \in G$, from which it follows that $\theta(y) = xyx^{-1}$ for all $y \in G$. Hence $\theta = I_{x^{-1}} \in \mathrm{Inn}(G)$. Conversely, if $\theta = I_{x^{-1}}$ for some $x \in G$, then $\theta(y) = xyx^{-1}$ so that $[x,_\theta y] = 1$ for all $y \in G$. Thus $x \in Z_\theta(G)$.

(2) Assume $x \in Z_\theta(G)$. We are going to show that $x \in Z(G)g^{-1}$ or equivalently $gx \in Z(G)$. We first observe that $\theta(x) = x$ and hence $xg = gx$. Now, for $y \in G$ we have

$$[x,_\theta y] = 1 \Leftrightarrow x^{-1}\theta(y)^{-1}\theta(x)y = 1 \Leftrightarrow x^{-1}(g^{-1}yg)^{-1}xy = 1 \Leftrightarrow gxy = ygx.$$

Hence $gx \in Z(G)$ and consequently $Z_\theta(G) \subseteq Z(G)g^{-1}$. Conversely, if $x \in Z(G)g^{-1}$, then $gx \in Z(G)$ and the above argument shows that $[x,_\theta y] = 1$ for all $y \in G$. Therefore, $Z(G)g^{-1} \subseteq Z_\theta(G)$ and the result follows. □

The above lemma states that $Z_\theta(G) = \emptyset$ if and only if θ is a non-inner automorphism of G. This fact will be used frequently in the sequel.

3 The θ-non-commuting graphs

Having defined the θ-commutators, we can now define and study the θ-non-commuting graph analog of the non-commuting graphs. In this section, some primary properties if such graphs and their relationship to other notions will be established.

Definition 3.1. *Let G be a group and θ be an automorphism of G. The θ-non-commuting graph of G, denoted by Γ_G^θ, is a simple undirected graph whose vertices are elements of $G \setminus Z_\theta(G)$ and two distinct vertices x and y are adjacent if $[x,_\theta y] \neq 1$.*

Clearly, the θ-non-commuting graph of a group coincides with the ordinary non-commuting graph whenever θ is the identity automorphism. Indeed, the map $\Theta : V(\Gamma_G^\theta) \dashrightarrow V(\Gamma_G^{I_g\theta})$ defined by $\Theta(x) = g^{-1}x$, for all $x \in V(\Gamma_G^\theta)$, presents an isomorphism between Γ_G^θ and $\Gamma_G^{I_g\theta}$. Hence, every two automorphisms in the same cosets of $\mathrm{Inn}(G)$ in $\mathrm{Aut}(G)$ give rise to the same graphs.

The following two results will be used in order to prove Theorem 3.4.

Lemma 3.2. *Let X be a subset of G with $|X| \leq |G|/2$. If there exists a vertex x in Γ_G^θ such that $[x,_\theta y] = 1$ for all $y \in G \setminus X$, then $|X| = |G|/2$.*

Proof. Assume that $[x,_\theta y] = 1$ for all $y \in G \setminus X$. We claim that $\langle x^{-1}(G \setminus X)\rangle$ is a proper subgroup of G. Suppose on the contrary that $\langle x^{-1}(G \setminus X)\rangle = G$. One can easily see that $\theta(x^{-1}y) = (x^{-1}y)^{x^{-1}}$ for all $y \in G \setminus X$. Hence $\theta = I_{x^{-1}}$, which implies that $Z_\theta(G) = Z(G)x$ by Lemma 2.4. But then $x \in Z_\theta(G)$, which is a contradiction. Thus, $|G \setminus X| \leq |\langle x^{-1}(G \setminus X)\rangle| \leq |G|/2$ and consequently $|X| = |G|/2$, as required. □

Corollary 3.3. *For every $x \in G$ we have* $\deg(x) \geq |G|/2$.

Theorem 3.4. *We have* $\operatorname{diam}(\Gamma_G^\theta) \leq 2$.

Proof. If $\operatorname{diam}(\Gamma_G^\theta) > 2$, then there exist two vertices x and y such that $d(x, y) > 2$. Thus $N(x) \cap N(y) = \emptyset$ so that $|N(x)| = |N(y)| = |G|/2$ by Corollary 3.3. Consequently, $G = N(x) \dot\cup N(y)$, which implies that $y \in N(x)$, that is, x and y are adjacent, a contradiction. □

Theorem 3.5. *We have* $\operatorname{girth}(\Gamma_G^\theta) \leq 4$ *and equality holds if and only if G is an abelian group,* $[G, \operatorname{Fix}(\theta)] = 2$ *and* $[G, \theta]^2 = 1$.

Proof. Suppose $\operatorname{girth}(\Gamma_G^\theta) > 3$. We show that $\operatorname{girth}(\Gamma_G^\theta) = 4$. Since $\operatorname{girth}(\Gamma_G^\theta) > 3$ $N(x_1) \cap N(x_2) = \emptyset$ for every edge $\{x_1, x_2\} \in E(\Gamma_G^\theta)$. Moreover, $|N(x_1)|, |N(x_2)| \geq |G|/2$ by Corollary 3.3, from which it follows that $|N(x_1)| = |N(x_2)| = |G|/2$, hence $G = N(x_1) \dot\cup N(x_2)$. Since for $y \in N(x_i)$ $(i = 1, 2)$ we have $G = N(x_i) \dot\cup N(y)$ as well, it follows that $N(y) = N(x_{3-i})$. Therefore, Γ_G^θ is a complete bipartite graph with the bipartition $(N(x_1), N(x_2))$, which yields $\operatorname{girth}(\Gamma_G^\theta) = 4$. To prove the second part, assume $\operatorname{girth}(\Gamma_G^\theta) = 4$. Then $G = N(x) \dot\cup N(y)$ is an equally partition for each $\{x, y\} \in E(\Gamma_G^\theta)$. Suppose $1 \in N(x)$. Then

$$g \in N(x) \Leftrightarrow [g,_\theta 1] = 1 \Leftrightarrow \theta(g) = g \Leftrightarrow g \in \operatorname{Fix}(\theta),$$

that is, $N(x) = \operatorname{Fix}(\theta)$ is a subgroup of G. Furthermore, $N(x)$ is abelian as $[a,_\theta b] = 1$ or equivalently $ab = ba$ for all distinct elements $a, b \in N(x)$. Now let $g \in G \setminus N(x)$. Clearly, $N(y) = N(x)g$. Since $g, ag \in N(y)$ for all $a \in N(x)$, it follows that $[g,_\theta ag] = 1$ or equivalently $a\theta(g) = \theta(g)a$. Therefore, $G = \langle N(x), \theta(g)\rangle$ is abelian. As $g^2 \in N(x)$ we have $\theta(g^2) = g^2$ so that $[g, \theta]^2 = 1$. Hence $[G, \theta]^2 = 1$, as required. The converse is straightforward. □

In what follows, we obtain some criterion for an automorphism to be fixed-point-free (or regular) or class-preserving. Remind that an automorphism θ of G is *fixed-point-free* if the only fixed point of θ is the trivial element, that is, $\operatorname{Fix}(\theta) = \langle 1\rangle$ is the trivial subgroup of G.

Theorem 3.6. *The graph Γ_G^θ is complete if and only if θ is a fixed-point-free automorphism of G.*

Proof. Assume Γ_G^θ is a complete graph. Then θ is non-inner and $[x,_\theta y] \neq 1$ for all vertices x and y in Γ_G^θ. If θ is not fixed-point-free, then there exists an element $x \in G$ such that $\theta(x) = x$. But then $[x,_\theta 1] = 1$, which is impossible. Thus θ is fixed-point-free. Conversely, suppose that θ is a fixed-point-free automorphism. By [6, 10.5.1(iii)], $\theta(g) \notin g^G$ for all $g \in G \setminus \{1\}$. Hence $\theta(x^{-1}y) \neq (x^{-1}y)^{x^{-1}}$ for all distinct vertices x and y, which implies that $[x,_\theta y] \neq 1$, that is, x and y are adjacent. The proof is complete. □

An automorphism θ of G is called *class preserving* if $\theta(g^G) = g^G$ for every conjugacy class g^G of G.

Theorem 3.7. *Let $k(G)$ denote the number of conjugacy classes of G. Then*

$$|E(\Gamma_G^\theta)| \geq \frac{1}{2}|G|(|G| - k(G))$$

and the equality holds if and only if θ is a class preserving automorphism of G.

Proof. If θ is an inner automorphism, then $|E(\Gamma_G^\theta)| = \frac{1}{2}|G|(|G| - k(G))$ and we are done. Hence, assume that θ is a non-inner automorphism. By Lemma 2.4, we observe that $V(\Gamma_G^\theta) = G$. Then

$$\left|E\left((\Gamma_G^\theta)^c\right)\right| = \frac{1}{2}\sum_{x\in G}|C_G^\theta(x)| - \frac{1}{2}|G|$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{x\in G} |\{y \in G \mid \theta(x^{-1}y) = (x^{-1}y)^{x^{-1}}\}| - \frac{1}{2}|G| \\
&= \frac{1}{2}\sum_{x\in G} |\{y \in G \mid \theta(y) = y^{x^{-1}}\}| - \frac{1}{2}|G| \\
&= \frac{1}{2}\sum_{y\in G} |\{x \in G \mid \theta(y) = y^{x^{-1}}\}| - \frac{1}{2}|G| \\
&\leq \frac{1}{2}\sum_{y\in G} |C_G(y)| - \frac{1}{2}|G| \\
&= \frac{1}{2}|G|k(G) - \frac{1}{2}|G|,
\end{aligned}
$$

from which, in conjunction with the fact that $\left|E\left((\Gamma_G^\theta)^c\right)\right| = \frac{1}{2}|G|(|G|-1) - |E(\Gamma_G^\theta)|$, the result follows. □

4 Independent sets

In the section, we give a description of independent subsets of the graph Γ_G^θ, which enables us to compute the independence number of Γ_G^θ. We begin with the easier case of abelian groups. Indeed, utilizing the following lemma, we can determine the structure of Γ_G^θ precisely when G is an abelian group.

Lemma 4.1. *Let G be a group and θ be an automorphism of G. Then*
(1) If x, y are in the same coset of $\mathrm{Fix}(\theta)$*, then $x \sim y$ if and only if $xy \neq yx$.*
(2) If x, y are in different cosets of $\mathrm{Fix}(\theta)$*, then $x \sim y$ if $xy = yx$.*

Proof. (1) By assumption $y^{-1}x \in \mathrm{Fix}(\theta)$. Thus

$$xy \neq yx \Leftrightarrow y^{-1}x \neq xy^{-1} \Leftrightarrow \theta(y^{-1}x) \neq xy^{-1} \Leftrightarrow [x,_\theta y] \neq 1 \Leftrightarrow x \sim y.$$

(2) We have $y^{-1}x \notin \mathrm{Fix}(\theta)$ and consequently

$$xy = yx \Rightarrow y^{-1}x = xy^{-1} \Rightarrow \theta(y^{-1}x) \neq xy^{-1} \Rightarrow [x,_\theta y] \neq 1 \Rightarrow x \sim y,$$

as required. □

Corollary 4.2. *A coset of* $\mathrm{Fix}(\theta)$ *is an independent set if and only if it is an abelian set.*

Corollary 4.3. *For every abelian group G, we have $\Gamma_G^\theta \cong K_{|F|,\ldots,|F|}$ is a complete m-partite graph in which $F = \mathrm{Fix}(\theta)$ and $m = [G : \mathrm{Fix}(\theta)]$.*

As we have seen in Lemma 4.1, there is a close relationship between independence and commutativity of vertices in the graph Γ_G^θ. The following key lemma illustrates this relationship in a much suitable form.

Lemma 4.4. *Let I be an independent subset of Γ_G^θ. Then*
(1) $I^{-1}I$ is an abelian set; and
(2) if I is non-abelian, then I is a product-free set.

Proof. (1) Let $x, y, z, w \in I$. Then

$$\theta(z^{-1}w) = \theta(x^{-1}z)^{-1}\theta(x^{-1}w) = (x^{-1}z)^{-x^{-1}}(x^{-1}w)^{x^{-1}} = (z^{-1}w)^{x^{-1}}.$$

Similarly, we have $\theta(z^{-1}w) = (z^{-1}w)^{y^{-1}}$, from which the result follows.

(2) Suppose on the contrary that I is not product-free so that $ab \in I$ for some $a, b \in I$. For $x \in I$ we have

$$(x^{-1}ab)^{x^{-1}} = \theta(x^{-1}ab) = \theta(x^{-1}a)\theta(x)\theta(x^{-1}b) = (x^{-1}a)^{x^{-1}}\theta(x)(x^{-1}b)^{x^{-1}},$$

from which we get $\theta(x) = x$. Hence $[x, y] = [x, _\theta y] = 1$ for all $x, y \in I$. Therefore I is abelian, which is a contradiction. □

Now we can state our structural description of arbitrary independent sets in the graph Γ_G^θ.

Theorem 4.5. *Let G be a group and I be a subset of G. Then I is an independent (resp. a maximal independent) subset of Γ_G^θ if and only if $I \subseteq gA$ (resp. $I = gA$) for every $g \in I$, in which A is an abelian (resp. a maximal abelain) subgroup of* $\mathrm{Fix}(I_g\theta)$.

Proof. First observe that if I is an independent subset of Γ_G^θ, then $A = \langle I^{-1}I\rangle$ is an abelian subgroup of $\mathrm{Fix}(I_g\theta)$ and $I \subseteq gA$ for every $g \in I$ by Lemma 4.4(1). Also, $I = gA$ and A is a maximal abelian subgroup of $\mathrm{Fix}(I_g\theta)$ whenever I is a maximal independent subset of Γ_G^θ. Clearly, every subset of gA is an independent set in Γ_G^θ. To complete the proof, we must show that any two independent sets $xA \subseteq yB$ in which A and B are maximal abelian subgroups of $\mathrm{Fix}(I_x\theta)$ and $\mathrm{Fix}(I_y\theta)$, respectively, coincide. First observe that $A \subseteq x^{-1}yB$ so that $x^{-1}y = b_0 \in B$. Hence $A \subseteq B$. Now, for every $b \in B$, we have $\theta(b) = b^{y^{-1}} = b^{b_0^{-1}x^{-1}} = b^{x^{-1}}$, which implies that $B \subseteq \mathrm{Fix}(I_x\theta)$. The maximality of A yields $A = B$ and consequently $xA = yB$, as required. □

Corollary 4.6. *We have*

$$\alpha(\Gamma_G^\theta) = \max\{|A| \mid A \leq \mathrm{Fix}(I_g\theta) \text{ is abelian}, \ g \in G\}.$$

Corollary 4.7. *The graph Γ_G^θ is empty if and only if G is abelian and θ is the identity automorphism, in which case Γ_G^θ is the null graph.*

Corollary 4.8. *Let G be a finite group and θ be an automorphism of G. If either G is non-abelian or θ is non-identity, then $\alpha(\Gamma_G^\theta) \leq |G|/2$ and the equality holds if and only if* $\mathrm{Fix}(I_g\theta)$ *is an abelian subgroup of G of index 2 for some element $g \in G$.*

Proof. If $\alpha(\Gamma_G^\theta) > |G|/2$, then Corollary 4.6 gives an element $g \in G$ such that $G = \mathrm{Fix}(I_g\theta)$ is abelian. But then $\theta = I_{g^{-1}} = I$, which is a contradiction. Hence $\alpha(\Gamma_G^\theta) \leq |G|/2$. Now if the equality holds, by using Corollary 4.6 once more, we observe that $\mathrm{Fix}(I_g\theta)$ is an abelian subgroup of G of index 2 for some $g \in G$. The converse is straightforward. □

5 Chromatic number

The results of section 4 on the independence number can be applied to study the chromatic number of θ-non-commuting graphs. Since every maximal independent set in Γ_G^θ is a left coset to an abelian group, the evaluation of the chromatic number of Γ_G^θ relies on the theory of covering groups by left cosets of their proper subgroups. In this regard, the following result of Tomkinson plays an important role.

Theorem 5.1 (Tomkinson [7])**.** *Let G be covered by some cosets g_iH_i for $i = 1, \ldots, n$. If the cover is irredundant, then $[G : \bigcap_{i=1}^n H_i] \leq n!$.*

Tomkinson's theorem has the following immediate result connecting the chromatic number of Γ_G^θ to the number of fixed points of θ.

Corollary 5.2. *For any group G, we have*

$$[G : \mathrm{Fix}(\theta)] \leq \chi(\Gamma_G^\theta)!$$

and the equality holds only if $\mathrm{Fix}(\theta) \subseteq Z(G)$.

Proof. Let $G = I_1 \cup \cdots \cup I_\chi$ be the union of independent sets $I_1, \ldots, I_\chi$ in which $I_i \subseteq g_i A_i$ and A_i is an abelian subgroup of $\mathrm{Fix}(I_{g_i}\theta)$, for $i = 1, \ldots, \chi = \chi(\Gamma_G^\theta)$. If $1 \in I_k$, then $g_k \in A_k$, which implies that $A_k \subseteq \mathrm{Fix}(\theta)$. Thus

$$[G : \mathrm{Fix}(\theta)] \leq [G : A_k] \leq [G : \bigcap_{i=1}^{n} A_i] \leq \chi!.$$

Now assume the equality holds. Then $\mathrm{Fix}(\theta) = A_k \subseteq A_i$, for all $i = 1, \ldots, \chi$. Since A_i are abelian, $\mathrm{Fix}(\theta)$ commutes with all elements of $A_1, \ldots, A_\chi$. On the other hand, as $A_k \subseteq A_i$, we have $a = \theta(a) = a^{g_i^{-1}}$, for all $a \in A_k$ and $i = 1, \ldots, \chi$, which implies that $\mathrm{Fix}(\theta)$ commutes with $g_1, \ldots, g_\chi$ as well. Therefore $\mathrm{Fix}(\theta) \subseteq Z(G)$, as required. □

In the sequel, we shall characterize those graphs having small chromatic numbers.

Theorem 5.3. *Let G be a finite group. Then $\chi(\Gamma_G^\theta) = 2$ if and only if G is abelian and $[G : \mathrm{Fix}(\theta)] = 2$.*

Proof. Clearly, $\alpha(\Gamma_G^\theta) \geq |G|/2$. On the other hand, by Corollary 4.6, $\alpha(\Gamma_G^\theta) \leq |G|/2$, from which it follows that $\alpha(\Gamma_G^\theta) = |G|/2$. Hence $G = I_1 \cup I_2$, where (I_1, I_2) is a bipartition of Γ_G^θ satisfying $|I_1| = |I_2| = |G|/2$. Assume $1 \in I_1$. Then, by Theorem 4.5, $I_1 = g_1 A_1$ for some $g_1 \in G$ in which $A_1 = \mathrm{Fix}(I_{g_1}\theta)$ is an abelian subgroup of G. Since $g_1 \in A_1$, it follows that $A_1 = \mathrm{Fix}(\theta)$. Clearly, $A_2 = A_1$ and $g_2 \in G \setminus A_1$. Now $\theta(a) = a^{g_1^{-1}} = a^{g_2^{-1}}$, for all $a \in A_1$, from which it follows that g_2 commutes with A_1. Thus G is abelian. The converse is obvious by Corollary 4.3. □

Theorem 5.4. *Let G be a finite group. Then $\chi(\Gamma_G^\theta) = 3$ if and only if either G is abelian and $[G : \mathrm{Fix}(\theta)] = 3$ or G is non-abelian and one of the following holds:*

(1) $G/Z(G) \cong C_2 \times C_2$ and θ is an inner automorphism;

(2) $[G : \mathrm{Fix}(\theta)] = [\mathrm{Fix}(\theta) : Z(G)] = 2$; or

(3) G has a characteristic subgroup A such that $[G : A] = [A : \mathrm{Fix}(\theta)] = 2$, $\mathrm{Fix}(\theta) = Z(G)$ and there exist elements $x \in A \setminus Z(G)$ and $y \in G \setminus A$ such that $\theta(x) = x^y$.

Proof. If G is abelian, then we are done by Corollary 4.3. Hence, we assume that G is non-abelian. Let $G = I_1 \cup I_2 \cup I_3$ be a tripartition of G in which $|I_1| \geq |I_2| \geq |I_3|$ and $I_i \subseteq g_i A_i$ $(i = 1, 2, 3)$ for some elements $g_i \in G$ and abelian subgroups A_i of $\mathrm{Fix}(I_{g_i}\theta)$. Without loss of generality, we may assume that $I_1 = g_1 A_1$. From Theorem 5.1, we know that $[G : A_1 \cap A_2 \cap A_3] \leq 3! = 6$. We distinguish two cases:

Case 1. $G = \langle A_1, A_2, A_3 \rangle$. Then $A_1 \cap A_2 \cap A_3 \subseteq Z(G)$ and we must have $G/Z(G) \cong C_2 \times C_2$ or S_3. Hence $A_1 \cap A_2 \cap A_3 = Z(G)$. One can verify that $2 = [G : A_1] \geq [G : A_2] \geq 3$ and $A_1 \neq A_2$. Since $G \neq A_1 \cup A_2$ and every element of $G/Z(G)$ has order 1, 2 or 3, one can always find an element $g \in G \setminus A_1 \cup A_2$ such that $gA_i = g_i A_i$, for $i = 1, 2$. Thus, for $a \in A_i$ we have $\theta(a) = a^{g_i^{-1}} = a^{g^{-1}}$ which implies that θ acts by conjugation via g^{-1} on $\langle A_1, A_2 \rangle = G$. Hence $\theta = I_{g^{-1}}$ is an inner automorphism. If $G/Z(G) \cong S_3$, then $\Gamma_G^\theta \cong \Gamma_G$ has a subgraph isomorphic to $\Gamma_{S_3} \cong K_5 \setminus K_2$ with chromatic number 4, a contradiction. Therefore $G/Z(G) \cong C_2 \times C_2$, which gives us part (1).

Case 2. $G \neq \langle A_1, A_2, A_3 \rangle$. Clearly, $A_2, A_3 \subseteq A_1$. Then $A_2, A_3 \subset A_1$ otherwise $A_1 = A_2$ and hence $\theta(a) = a^{g_1} = a^{g_2}$ for all $a \in A_1$. Since $g_1 A_1 \neq g_2 A_2$, it follows that $g_1 g_2^{-1} \in G \setminus A_1$ commutes with A_1 so that $G = A_1 \langle g_1 g_2^{-1} \rangle$ is abelian, a contradiction. Hence $A_2, A_3 \subset A_1$, from which together with Tomkinson's result we must have $[G : A_1] = 2$, $[A_1 : A_2] = 2$ and $A_2 = A_3$. Clearly, $g_2 A_2 \cup g_3 A_3 = g_1' A_1$ where $G = g_1 A_1 \cup g_1' A_1$. If $g_1 \in A_1$, then $A_1 = \mathrm{Fix}(\theta)$. As $A_2 \subseteq A_1$ we have $a = \theta(a) = a^{g_2^{-1}}$ for all $a \in A_2$, which implies that $A_2 = Z(G)$. Hence we obtain part (2). Next assume that $g_1 \notin A_1$. Then $g_1' \in A_1$ and consequently $g_2, g_3 \in g_1' A_1 = A_1$. This implies that $A_2 \subseteq \mathrm{Fix}(\theta)$. As $A_2 \subseteq A_1$, we have $a = \theta(a) = a^{g_1^{-1}}$ for all $a \in A_2$ showing that $A_2 = Z(G)$. Assuming $g_2 \in A_1 \setminus A_2$, we obtain $\theta(g_2) = g_2 z$ for some $z \in A_2$. Furthermore, $g_2 z = \theta(g_2) = g_2^{g_1^{-1}} = g_2^{g_1}$ as $g_2 \in A_1$. Thus $[g_2, g_1] = z$ and this yields part (3).

The converse is straightforward. □

We conclude this section with a characterization of complete multipartite-ness of the graphs Γ_G^θ.

Theorem 5.5. *The graph Γ_G^θ is a complete multipartite graph if and only if* $\mathrm{Fix}(I_g\theta)$ *is abelian for all $g \in G$.*

Proof. First assume that Γ_G^θ is a complete multipartite graph. From Theorem 4.5, it follows that all maximal abelian subgroups of $\mathrm{Fix}(I_g\theta)$ are disjoint so that $\mathrm{Fix}(I_g\theta)$ is abelian for all $g \in G$. Now assume that $\mathrm{Fix}(I_g\theta)$ is abelian for all $g \in G$. Let $x \in G$. By assumption, $x\mathrm{Fix}(I_x\theta)$ is an independent subset of Γ_G^θ. On the other hand, for $y \notin x\mathrm{Fix}(I_x\theta)$, we have $x^{-1}y \notin \mathrm{Fix}(I_x\theta)$ so that $\theta(x^{-1}y) = (x^{-1}y)^{x^{-1}}$. Hence y is adjacent to x. Since, by Theorem 4.5, the sets $g\mathrm{Fix}(I_g\theta)$ are maximal independent subsets of Γ_G^θ, it follows the sets $g\mathrm{Fix}(I_g\theta)$ partition G and hence Γ_G^θ is a complete multipartite graph, as required. □

6 Conclusion/Open problems

In this paper, we have generalized commutators of a group, in a compatible way, to θ-commutators with respect to a given automorphism θ. Accordingly, commutator identities as well as the corresponding centralizers and center are studied.

One may define θ-nilpotent and θ-solvable groups by means of θ-commutators in a natural way. So, we may ask:

Question. *How are the automorphism θ and the structure of θ-nilpotent and θ-solvable groups related?*

Next, we have defined the non-commuting graph Γ_G^θ associated to θ-commutators of a group G and established some connections between graph theoretical properties of Γ_G^θ and group theoretical properties of the automorphism θ. For instance, it is proved, among other results, that Γ_G^θ is complete as a graph if and only if θ is fixed-point-free and that Γ_G^θ receives minimum number of edges if and only if θ is class preserving.

Question. *Which other graph theoretical properties of Γ_G^θ can be interpreted (simply) in terms of group theoretical properties of θ (and vice versa)?*

The rest of paper is devoted to the study of independent sets in the graph Γ_G^θ. Firstly, a one-to-one correspondence between independent sets and abelian subgroups of $\mathrm{Fix}(I_g\theta)$ is established, where I_g denotes the inner automorphism of G induced by the element $g \in G$. This result provided us with another partial answer to the above question: the graph Γ_G^θ is empty if and only if G is abelian and θ is the identity automorphism. Secondly, a relationship between covers of G by subgroups and the chromatic number $\chi(\Gamma_G^\theta)$ of Γ_G^θ, i.e. the minimum number of independent sets to cover all vertices of Γ_G^θ, is revealed and a lower bound for $\chi(\Gamma_G^\theta)$ in terms of $\mathrm{Fix}(\theta)$ is deduced. Also, the structure of G or properties of θ when Γ_G^θ admits special colorings is obtained.

In contrast to our investigations on independent sets one may ask:

Question. *How can cliques of Γ_G^θ be described in terms of G and θ?*

Finally, a fundamental question to ask is:

Question. *Suppose θ_1 and θ_2 are automorphisms of groups G_1 and G_2, respectively.*

(1) Under which conditions on (G_1, θ_1) and (G_2, θ_2) are two graphs $\Gamma_{G_1}^{\theta_1}$ and $\Gamma_{G_2}^{\theta_2}$ isomorphic (in particular when $G_1 = G_2$)?

(2) How are the pairs (G_1, θ_1) and (G_2, θ_2) related, provided that $\Gamma_{G_1}^{\theta_1}$ and $\Gamma_{G_2}^{\theta_2}$ are isomorphic?

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