CONNECTIVITY AND PLANARITY OF g-NONCOMMUTING GRAPH OF FINITE GROUPS

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Abstract. Let G be a finite non-abelian group and $Z(G)$ be its center. For a fixed non-identity element g of G, the g-noncommuting graph of G, denoted by Δ_G^g, is a simple undirected graph whose vertices are $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if $[x, y] \neq g$ and g^{-1}. In this paper, we discuss about connectivity of Δ_G^g and determine all finite non-abelian groups such that their g-noncommuting graphs are 1-planar, toroidal or projective.

1. Introduction

Many sciences are concerned with groups. It has been shown that graphs can be interesting tools for the study of groups. We can investigate algebraic properties of groups in terms of properties of the associated graphs. There are different ways to associate a graph to a group G such that the vertices are families of elements or subsets of G and two vertices are adjacent if and only if they satisfy in a certain relation (for instance see [1, 2, 10]).

One of the important graphs associated to a group is non-commuting graph which was introduced by Neumann. The non-commuting graph of a finite group G is the graph with vertex set $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if $xy \neq yx$. Similarly, the commuting graph associated to a non-abelian group G is the complement of non-commuting graph and it is denoted by $\Delta(G)$ in this paper.

In [9], Tolue et al. assigned a simple undirected graph Γ_G^g to G and a fixed element $g \in G$ as follows: Take G as the vertex set and two distinct vertices x and y join whenever $[x, y] \neq g$ and g^{-1}. They investigated some graph theoretical properties of Γ_G^g such as regularity, clique and dominating numbers.

In this article, we consider an induced subgraph of Γ_G^g whose vertices are all non-central elements of G. It is called the g-noncommuting graph of G and is denoted by Δ_G^g. One can see that Δ_G^g is exactly known non-commuting graph whenever g is an identity element of G indicated by e. Moreover, the commuting graph is a spanning graph of Δ_G^g. For any group G, let $K(G) = \{[x, y] | x, y \in G\}$ be the set of all commutators of G and set $G' =< K(G) >$, where G' is the commutator subgroup of G. The g-noncommuting graph Δ_G^g is a complete graph if $g \notin K(G)$. Therefore, we always assume that g is a non-identity element, $g \in K(G)$ and G is a finite group.

It is known that the non-commuting graph is connected and its diameter is equal to 2 (see [1] for more details). But, the g-noncommuting graph is not always connected graph; for instance, $\Delta_{S_3}^{(123)}$ has one edge and three isolated vertices where S_3 is the symmetric group on 3 symboles. So, this is a reason for investigating

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connectivity of g-noncommuting graph for all groups. In section 2, we discuss about the connectivity of Δ_G^g for finite groups and prove that the diameter of g-noncommuting graph, of course when Δ_G^g is connected, is at most 4. Moreover, we indicate that the girth of graph is 3. In spite of the fact that we do believe $diam(\Delta_G^g) = 2$ for all groups, when Δ_G^g is connected it has not been evidently substantiated yet. Therefore, we decided to leave it as a conjecture at the end of section 2.

We remind that, 1-planar graph is a graph which can be drawn in the Euclidean plane in such a way that each edge contains at most one crossing. Also, a toroidal or a projective graph is a graph that can be embeded on a torus or projective plane, respectively.

The last section of the paper deals with the study of g-noncommuting graphs of groups with properties 1-plannar, toroidal and projective.

All over this paper, $\omega(G) = \{|x| : x \in G\}$, $exp(G) = lcm(\omega(G))$, $C_G(x)$ and x^G stand for the spectrum of G, the exponent of G, the centralizer and the conjugacy class of x, respectively. Here, our notations and terminologies are standard and one can refer to [4].

2. The connectivity of g-noncommuting graph

In this section, we investigate the connectivity of g-noncommuting graph of all finite groups. Let us start with the following two simple lemmas. The proofs are omitted here and we refer to [6].

Lemma 2.1 ([6]). *Let $x \in G \setminus Z(G)$.*

(i) *If $g^2 \neq e$, then $deg(x) = |G| - |Z(G)| - \epsilon|C_G(x)| - 1$, where $\epsilon = 1$ if x is conjugate to xg or xg^{-1}, but not both and $\epsilon = 2$ if x is conjugate to xg and xg^{-1}.*

(ii) *If $g^2 = e$ and $g \neq e$, then $deg(x) = |G| - |Z(G)| - |C_G(x)| - 1$, whenever xg is conjugate to x.*

(iii) *If xg and xg^{-1} are not conjugate to x, then $deg(x) = |G| - |Z(G)| - 1$.*

Lemma 2.2 ([6]). *If $K(G) = \{e, g\}$ or $\{e, g, g^{-1}\}$, then Δ_G^g is equal to commuting graph.*

The following lemma plays an important role in the proofs of some theorems in this section.

Lemma 2.3. *If x and y are two non-adjacent vertices such that $d(x, y) \geq 3$, then*

$$1 \leq \frac{t}{|x^G|} + \frac{s}{|y^G|} + \frac{1}{[G : Z(G)]},$$

where $t, s \in \{1, 2\}$.

Proof. We know that $deg(x) = |G| - |Z(G)| - t|C_G(x)| - 1$ and $deg(y) = |G| - |Z(G)| - s|C_G(y)| - 1$, where $t, s \in \{1, 2\}$ by Lemma 2.1. Since $d(x, y) \geq 3$, then $deg(x) + deg(y) + 2 \leq |V(\Delta_G^g)|$ and the proof is completed. □

In the following theorems, we determine connectivity and diameter of the g-noncommuting graph.

Theorem 2.4. *Let g be a non-central element of G and $|g| \neq 3$. Then $diam(\Delta_G^g) = 2$.*

Proof. If $|g| = 2$, then by Lemma 2.1, g is adjacent to all vertices and so $diam(\Delta_G^g) = 2$. Thus assume that, $|g| > 3$. Let x and y be two vertices of Δ_G^g which are not adjacent. Then $[x, y] = g$ or g^{-1} and we have the following cases:

Case 1. x and y are not adjacent to g i.e. $[x, g] = g^{-1}$ and $[y, g] = g^{-1}$. Then $[x, g^2] = [y, g^2] = g^{-2}$. Since $|g| > 3$, so $x \sim g^2 \sim y$, as required.

Case 2. x is not adjacent to g and $y \sim g$. If $yg \notin Z(G)$ then $x \sim yg \sim y$, because $[y, g] = g^{-1}$. Otherwise, $[x, g^2] = g^{-2}$ and $[y, g^2] = e$. Hence, $x \sim g^2 \sim y$.

Case 3. y is not adjacent to g and $x \sim g$. By replacement x with y in the case 2 and a similar argument, there exists one of the paths, $x \sim xg \sim y$ and $x \sim g^2 \sim y$. Thus the proof follows. □

Theorem 2.5. *Let g be a non-central element of order 3 of G. If $[C_G(g) : Z(G)] = 3$ and there exists a vertex x such that $d(x, g) \geq 3$, then Δ_G^g is disconnected. Moreover, $\frac{G}{Z(G)} \cong S_3$ and $\Delta_G^g = K_{2|Z(G)|} \cup 3K_{|Z(G)|}$. Otherwise Δ_G^g is connected and $diam(\Delta_G^g) \leq 4$.*

Proof. Assume that, $[C_G(g) : Z(G)] = 3$ and x is a vertex of Δ_G^g such that $d(x, g) \geq 3$. By Lemma 2.1, g and g^{-1} are conjugate in G and so $deg(g) = |G| - |Z(G)| - |C_G(g)| - 1$. Let $deg(x) = |G| - |Z(G)| - t|C_G(x)| - 1$ for some $t \in \{1, 2\}$. Then $[G : Z(G)] = [G : C_G(g)][C_G(g) : Z(G)] = 3|g^G|$ and Lemma 2.3 implies that

$$\frac{t}{|x^G|} \geq \frac{3|g^G| - 4}{3|g^G|} \geq \frac{1}{3}. \tag{1}$$

Thus $|x^G| \leq 3t$ for some $t \in \{1, 2\}$.
Let $t = 1$. If $|x^G| = 2$, then $[G : C_G(x)] = 2$ which is a contradiction, since $|g| = 3$ and $d(x, g) \geq 3$. Thus $|x^G| = 3$ and by (1) we should have $|g^G| = 2$. Therefore, $[G : Z(G)] = 6$ and $|C_G(x)| = 2|Z(G)|$. So $deg(x) = |G| - |Z(G)| - |C_G(x)| - 1 = 3|Z(G)| - 1$. Now, we show that x is adjacent to only non-central elements in $C_G(x)$. Suppose that $y \in G \setminus C_G(x)$. Since $G = C_G(x) \cup gC_G(x) \cup g^{-1}C_G(x)$ and $d(x, g) \geq 3$, so $[x, y] \in g^G = \{g, g^{-1}\}$ which implies that x and y are not adjacent. Thus $deg(x) = |C_G(x)| - |Z(G)| - 1 = |Z(G)| - 1$, that is a contradiction.
Therefore $t = 2$ and it is clear that $|x^G| \geq 3$. If $|x^G| \geq 4$ then by (1), $|g^G| = 2$ and so $[G : Z(G)] = 3|g^G| = 6$ which is impossible, because $|x^G| \mid [G : Z(G)]$. Hence $|x^G| = 3$ and $G = C_G(x) \cup gC_G(x) \cup g^{-1}C_G(x)$. Let $a \in G$, then $a = g^i s$ for some $s \in C_G(x)$ and $i \in \{0, 1, -1\}$, so $g^a = g^{g^i s} = g[g, s]$. Since $d(x, g) > 2$, so either $s \in Z(G)$ or $[g, s] = g$. Thus $g^a = \{g, g^{-1}\}$ and $[G : Z(G)] = 6$. It follows that $\frac{G}{Z(G)} \cong S_3$ and $deg(x) = |G| - 2|C_G(x)| - |Z(G)| - 1 = |Z(G)| - 1$. It means that x is adjacent to only non-central elements in $C_G(x)$. Also, one can easily see that for every two elements $a, b \in G$ we have $[a, b] \in \langle g \rangle$. Therefore $K(G) = G' = \langle g \rangle = \{e, g, g^{-1}\}$ and Δ_G^g coincides to the known commuting graph, by Lemma 2.2. In particular, $C_G(g) \setminus Z(G)$, $C_G(x) \setminus Z(G)$, $C_G(xg) \setminus Z(G)$ and $C_G(xg^{-1}) \setminus Z(G)$ are connected components of g-noncommuting graph and $\Delta_G^g = K_{2|Z(G)|} \cup 3K_{|Z(G)|}$.
If distance of every vertex of Δ_G^g to g is at most 2, then clearly Δ_G^g is connected and $diam(^g_G) \leq 4$.
Now, suppose that $[C_G(g) : Z(G)] \geq 4$ and x, g are not adjacent. Let $a \in C_G(g) \setminus (Z(G) \cup gZ(G) \cup g^{-1}Z(G))$. If $a \sim x$ then $d(x, g) = 2$, since $a \sim g$. If $[x, a] = g$ or g^{-1}, then we have two paths $x \sim ga \sim g$ or $x \sim g^{-1}a \sim g$, respectively and the proof is completed. □

Theorem 2.6. *Let g be a central element of G. If there are two vertices such that their distance is greater than 5, then Δ_G^g is disconnected and the following cases occur:*

(i) *If $|g| \geq 3$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Delta_G^g = 4K_{2|Z(G)|}$,*

(ii) *If $|g| = 2$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Delta_G^g = 3K_{|Z(G)|}$.*

Otherwise Δ_G^g is the connected graph and $diam(\Delta_G^g) = 2$.

Proof. Let x and y be two vertices of Δ_G^g such that $d(x,y) \geq 5$. Then $[x,y] = g$ or g^{-1}. Without loss of generality, we can assume that $[x,y] = g$.

First, assume $|g| \geq 3$. It is clear that $[x,y^2] = g^2$ and so $y^2 \notin Z(G)$. Since $d(x,y) \geq 5$, so $[x,y^2] = g^{-1}$ and it implies that $|g| = 3$ and $y^3 \in Z(G)$. Similarly, we can see that $x^3 \in Z(G)$. Let $a \in G \setminus Z(G)$, then $d(a,x) \geq 3$ or $d(a,y) \geq 3$. In any case by the same argument as before, $a^3 \in Z(G)$. It means that $exp(\frac{G}{Z(G)}) = 3$. Assume that $[G : Z(G)] = 3^\gamma$, $|x^G| = 3^\alpha$ and $|a^G| = 3^\beta$, where $\gamma > \alpha \geq \beta$. Then by Lemma 2.3,

$$1 \leq \frac{s}{3^\alpha} + \frac{t}{3^\beta} + \frac{1}{3^\gamma}, \quad s,t \in \{1,2\}. \tag{2}$$

Thus $3^{\gamma-\alpha}(3^\alpha - t3^{\alpha-\beta} - s) \leq 1$, where $s,t \in \{1,2\}$. If $\beta \geq 2$, then by the above inequality $3^\gamma \leq 1$, which is impossible. Therefore $\beta = 1$ and by a simple computation we arise that $\alpha = 1$. Thus $|a^G| = 3$ for every $a \in G \setminus Z(G)$. It can be proved that $[C_G(y) : Z(G)] = 3$ because if $b \in C_G(y) \setminus Z(G) \cup yZ(G) \cup y^2Z(G)$, then $[x, y^ib] = 1$ where $i \in \{1,2\}$. Since $d(x,y) \geq 5$, so $y^ib \in Z(G)$, which is a contradiction. Therefore $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $G = \langle x \rangle \langle y \rangle Z(G)$, since $[x,y] = g \in Z(G)$. Also $G' = K(G) = \{1, g, g^{-1}\}$ and Lemma 2.2 implies that $\Delta_G^g = 4K_{2|Z(G)|}$.

Now, suppose that $|g| = 2$. Then $y^2 \in Z(G)$, because $[x,y^2] = 1$ and $d(x,y) \geq 5$. By similarly argument as before, we have $\frac{G}{Z(G)} = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Delta_G^g = 3K_{|Z(G)|}$.

Finally, suppose distance between any two vertices is at most 4 in Δ_G^g. We want to show $diam(\Delta_G^g) = 2$. Let x and y be two vertices of Δ_G^g and there exists a path of length $n \geq 3$ between x and y as,

$$x \sim u_1 \sim u_2 \sim \ldots \sim u_{n-1} \sim y.$$

First, suppose $|g| \neq 3$. If $x^2 \notin Z(G)$, then x^2 is adjacent to x and y that is a contradiction. Therefore $x^2 \in Z(G)$ and since $[xu_1, y] \in \{e, g^2, g^{-2}\}$, it should be $xu_1 \in Z(G)$ otherwise $x \sim xu_1 \sim y$. Let $u_1 = xz$ for some $z \in Z(G)$. Then $[u_1, u_2] = [x, u_2] = g$ or g^{-1}, that is a contradiction again.

Now, we investigate the case $|g| = 3$. It is clear that $[x^3, y] = 1$ and since $d(x,y) \geq 3$, so $x^3 \in Z(G)$. It should be $x^2u_1 \in Z(G)$ otherwise x^2u_1 is common neighborhood of x and y. Thus $u_1 \in x^{-2}Z(G) = xZ(G)$ and again $[u_1, u_2] = g$ or g^{-1}, which is a contradiction. Hence $diam(\Delta_G^g) = 2$ for all possible cases and the proof is completed. □

Corollary 2.7. *If order of G is odd and Δ_G^g is connected, then $diam(\Delta_G^g) = 2$.*

Proof. We can see that $diam(\Delta_G^g) = 2$, when g is a central element by Theorem 2.6 or g is a non-central element and $|g| \neq 3$ by Theorem 2.4. So, assume that g is a non-central element and $|g| = 3$. If g is not conjugate to g^2, then g is adjacent to all vertices by Lemma 2.1. Otherwise $g^a = g^2$ for some $a \in G$, then $g^{a^2} = g$. So $a^2 \in C_G(g)$. Since the order of a is odd therefore $a \in C_G(g)$, that is a contradiction and the proof is completed. □

Now, let us state the following conjecture.
Conjecture. For any group G and any element $g \in K(G)$, $diam(\Delta_G^g) = 2$ when Δ_G^g is connected.

Theorem 2.8. *The girth of g-noncommuting graph is 3, unless $G \cong S_3, D_8$ or Q_8. Moreover, g-noncommuting graph of groups S_3, D_8 and Q_8 are forest.*

Proof. Let $x \in G \setminus Z(G)$. If $|Z(G)| \geq 3$, then $x \sim xz_1 \sim xz_2 \sim x$ is a cycle in Δ_G^g, where z_1 and z_2 are two distinct nontrivial elements of $Z(G)$.
So, assume that $|Z(G)| = 2$. If $|xZ(G)| \geq 3$, then the set of vertices $\{x, x^{-1}, xz\}$ forms a triangle in Δ_G^g for some $z \in Z(G)$. Otherwise, $\frac{G}{Z(G)}$ must be an elementary abelian 2-group. If $C_G(x) \setminus \{e, z, x, xz\} \neq \emptyset$, then $girth(\Delta_G^g) = 3$. Therefore $|C_G(x)| = 4$ and so $|G| = 8$. Hence $G \cong D_8$ or Q_8.
Now, suppose that $|Z(G)| = 1$. If $|x| \geq 4$, then $\{x, x^2, x^3\}$ forms a cycle of length 3 in Δ_G^g. Therefore $|G| = 2^\alpha 3^\beta$. It is obvious that, all Sylow 2-subgroups of G are abelian. Thus $girth(\Delta_G^g) = 3$, unless $\alpha = 1$. Let Q be a non-abelian Sylow 3-subgroup of G. Then we have a cycle as, $a \sim a^{-1} \sim ab \sim a$ for some $a \in Q \setminus Z(Q)$ and $b \in Z(Q)$. Therefore every Sylow 3-subgroup of G is abelian and so $girth(\Delta_G^g) = 3$, unless $\beta = 1$. It implies that $|G| = 6$ and $G \cong S_3$, as claimed. $\square$

3. 1-PLANAR, TOROIDAL AND PROJECTIVE CASES

In this section, all finite non-abelian groups whose their g-noncommuting graphs are 1-planar, toroidal or projective, are classified. First, we determine 1-planar g-noncommuting graphs. For this purpose, we recall the following two lemmas from [5].

Lemma 3.1 ([5]). *Let Γ be a 1-planar graph on n vertices and m edges. Then $m \leq 4n - 8$.*

Lemma 3.2 ([5]). *Every 1-planar graph contains a vertex of degree at most 7; the bound 7 is the best possible.*

Lemma 3.3. *Let $|Z(G)| \geq 4$ and Δ_G^g be 1-planar graph. Then $\frac{G}{Z(G)}$ is an elementary abelian 2-group.*

Proof. If $xZ \in \frac{G}{Z(G)}$ such that $|xZ| > 2$, then $xZ \cup x^{-1}Z$ induces a complete subgraph with at least 8 vertices, which is a contradiction. $\square$

Theorem 3.4. *Let G be a finite non-abelian group. Then Δ_G^g is 1-planar if and only if G is ismorphic to one of the following groups*

(i) S_3, D_8, Q_8, D_{10}, D_{12}, $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$,
$\mathbb{Z}_2 \times D_8$, $\mathbb{Z}_2 \times Q_8$, $\mathbb{Z}_3 \times S_3$, $\mathbb{Z}_3 \times D_8$, $\mathbb{Z}_3 \times Q_8$,
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong \langle a, b : a^4 = b^2 = (ab)^4 = [a^2, b] = 1\rangle$,
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong \langle a, b, c : a^2 = b^2 = c^4 = [a, c] = [b, c] = 1, [a, b] = c^2\rangle$,
$\mathbb{Z}_9 \rtimes \mathbb{Z}_3 \cong \langle a, b : a^3 = b^9 = 1, [b, a] = b^3\rangle$,
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3 \cong \langle a, b, c : a^3 = b^3 = c^3 = [b, c] = [a, c] = 1, [b, a] = c\rangle$,
for all $g \in G'$,

(ii) D_{16}, QD_{16} *and* Q_{16}, *for* $g \in G' \setminus Z(G)$.

Proof. Let x be a vertex of Δ_G^g of minimum degree δ. Then $\delta = |G| - |Z(G)| - \epsilon|C_G(x)| - 1$ for some $\epsilon \in \{1,2\}$ and by Lemma 3.2, $\delta \leq 7$ and,

$$|Z(G)|([G : Z(G)] - \epsilon[C_G(x) : Z(G)] - 1) \leq 8, \quad \epsilon \in \{1,2\}. \tag{3}$$

Set $l = [G : Z(G)] - \epsilon[C_G(x) : Z(G)] - 1$. Since the complete graph K_7 on seven vertices is not 1-planar therefore $|Z(G)| \leq 7$. We proceed the proof in some steps.

Step 1. $|Z(G)| \geq 4$. By (3), $[G : Z(G)] = \epsilon[C_G(x) : Z(G)] + (l+1)$ for some $l, \epsilon \in \{1,2\}$. Lemma 3.3 implies that $\frac{G}{Z(G)}$ is a 2-group, so $[G : Z(G)] = 2^\alpha$ and $[C_G(x) : Z(G)] = 2^\beta$. Thus $2^\alpha = \epsilon 2^\beta + (l+1)$. It can be seen that the only solution of the last equality is $(\alpha, \beta, \epsilon, l) = (2,1,1,1)$. Then $[G : Z(G)] = 4$ and $|G| = |C_G(x)| + 2|Z(G)|$. Since G is nilpotent, so $|Z(G)| \neq 5, 7$.
If $|Z(G)| = 6$, then $|G| = 24$ and $G \cong \mathbb{Z}_3 \times H$, where H is an extra special 2-group. Hence $G \cong \mathbb{Z}_3 \times D_8$ or $\mathbb{Z}_3 \times Q_8$. Then $|G'| = 2$ and $\Delta_G^g \cong \Delta(G) \cong 3K_6$ by Lemma 2.2.
If $|Z(G)| = 4$, then $|G| = 16$. We know that there exist only six non-abelian group of order 16 and $|Z(G)| = 4$. This groups have derived subgroup of order 2 and by lemma 2.2, $\Delta_G^g \cong \Delta(G)$. Therefore, Theorem 2.2 of [3] implies that Δ_G^g is 1-planar.

Step 2. $|Z(G)| = 3$. If there exists $y \in C_G(x) \setminus Z(G) \cup xZ(G) \cup x^{-1}Z(G)$, then $xZ(G) \cup yZ(G) \cup \{xy\}$ induces a subgraph isomorphic to K_7. Thus $C_G(x) = \langle x, Z(G) \rangle$ and $[C_G(x) : Z(G)] = 2$ or 3. Therefore 6 or 9 divide $|G|$ and (3) implies that $|G| = 18$ or 27.

Step 3. $|Z(G)| = 2$. If $yZ(G) \in \frac{G}{Z(G)}$ such that $|yZ(G)| \geq 5$, then $yZ(G) \cup y^2Z(G) \cup y^3Z(G) \cup y^4Z(G)$ is a commuting set which is a contradiction. Therefore, $\omega(\frac{G}{Z(G)}) = \{1,2,3,4\}$. Let $\frac{G}{Z(G)}$ be a 2-group. Then by the same computation as before, it can be seen that $|G| = 8$ or 16. Let Q be a Sylow 3-subgroup of G. Then by Lemma 2.1 in [3], $|Q|$ must be 3. Thus $\frac{G}{Z(G)}$ is not 3-group and so $|\frac{G}{Z(G)}| = 2^\alpha 3$. By (3), $[G : Z(G)] = \epsilon[C_G(x) : Z(G)] + (l+1)$ for some $l \in \{1,2,3,4\}$. If $\epsilon = 1$, then

$$\frac{|G|}{2} \leq |G| - |C_G(x)| = (l+1)|Z(G)| \leq 10.$$

and so $|G| = 12$, since $6 \big| |G|$. If $\epsilon = 2$, then $\frac{|G|}{3} \leq |G| - |C_G(x)| \leq 10$. Hence, $|G| = 12$ or 24.

Step 4. $|Z(G)| = 1$. Since Δ_G^g has no induced subgraph isomorphic to K_7, so $\omega(G) = \{1,2,3,4,5,6,7\}$. By (3), we should have $|G| - \epsilon|C_G(x)| \leq 9$. Hence, $|G| \leq 27$.

Finally, the degrees of vertices of the graph associated to the above groups are computed with group theory package GAP [8]. It can be seen that some graphs are not 1-planar by Lemmas 3.1 and 3.2. But the graph associated to groups (i) are same as the commuting graph which are the union of complete graphs with up to six vertices. Also, Δ_G^g is isomorphic to figure 1 when G is one of the groups (ii). The graph is 1-planar in any case and the proof is completed.
The converse is straightforward. $\square$

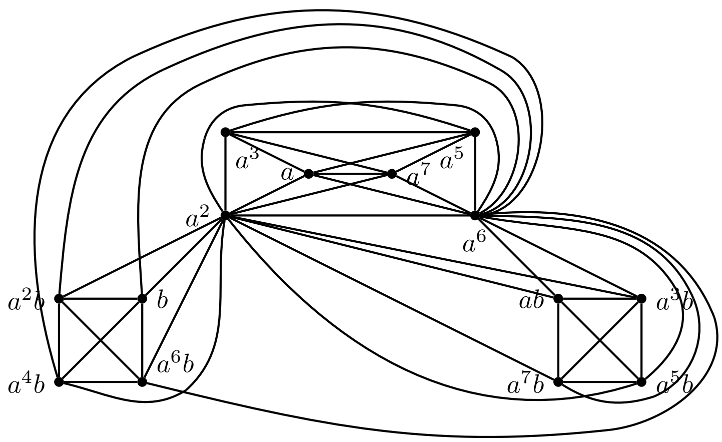

FIGURE 1. $\Delta_{D_{16}}^{a^2}$

The last theorem of the paper states the determination of toroidal and projective g-noncommuting graphs. Since the known commuting graph is a spanning subgraph of g-noncommuting graph and all groups which their commuting graph are toroidal or projective are classified in [3], so it is enough to investigate g-noncommuting graphs associated to those groups. First, we recall the following lemma which is used several times in the proof of our result.

Lemma 3.5 ([4]). *Let Γ be a simple connected graph that is embeddable on a surface S. Then*

$$m \leq 3(n - \chi)$$

where n, m and χ are the numbers of vertices, edges and Euler characteristic of S, respectively.

Theorem 3.6. *There is no toroidal and projective g-noncommuting graph.*

Proof. We know that $\chi = 0$ and 1 for torus and projective plane, respectively. The only groups that should be considered are D_{14}, D_{16}, QD_{16}, Q_{16}, $A_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ by [3]. By calculating the number of edges can be checked that $m > 3(n-\chi)$ and so the g-noncommuting graph can not be embedded on torus or projective plane. $\square$

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