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An approximate method for solving fractional TBVP with state delay by Bernstein polynomials

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Abstract
The current paper aims at investigating Fractional Hamiltonian Equations for a class of fractional optimal control problems with time delay. Furthermore, we introduce a method to solve the resulting two boundary values problem (TBVP) by extending Agrawal's fractional variational method in (Nonlinear Dyn. 38:323-337, 2004) and using Bernstein polynomials (BPs). In this paper we use the Caputo fractional derivative of order α where $0 < \alpha < 1$. Some numerical examples are included to demonstrate the validity of the present method.

Keywords: fractional optimal control problem with time delay; two boundary values problem (TBVP); Caputo fractional derivatives; Bernstein polynomials

1 Introduction

Fractional calculus is a branch of mathematics that generalizes the derivative and the integral of a function to a noninteger order [2]. Fractional calculus has received considerable attention in recent years and there is hardly a field in science and engineering that has remained untouched by this field. It has been shown that materials with memory and hereditary effects and dynamical processes including gas diffusion and heat conduction can be more adequately modeled by fractional differential equations (FDEs) than by integer-order differential equations [3, 4].

In fractional calculus the Caputo and Riemann-Liouville are two main kinds of derivatives where each presents some advantages and disadvantages (see, *e.g.*, [5]). The Caputo fractional derivative is commonly used in modeling physical phenomena but it is possible to assign a physical interpretation meaning for the Riemann-Liouville fractional derivative too. For instance, Heymans and Podlubny in [6] have illustrated some fractional differential equations with Riemann-Liouville fractional derivatives in the field of viscoelasticity.

A fractional optimal control problem (FOCP) is an optimal control problem in which the criterion and/or the differential equations governing the dynamics of the system contain at least one fractional derivative operator [7]. Most FOCPs do not have exact solutions, so in these cases approximation methods and numerical techniques must be used. Recently, several approximation methods to solve FOCPs have been introduced. Agrawal in [1] has introduced a general formula by making the TBVPs for some kinds of FOCP's. Tricaud and Chen have solved fractional order optimal control problems by means of rational approximation [3]. Moreover, the effectiveness of using Legendre and Bernstein polynomials for approximating the solution of FOCPs has been demonstrated in [8–15]). Real life phenomena have been described more precisely with delay differential equations, so the delay fractional optimal control problem (DFOCP) has become the focus of many researchers in the last decade. Baleanu *et al.* in [16, 17] analyzed the fractional variational principles for some kinds of DFOCPs within the Riemann-Liouville and Caputo fractional derivatives, respectively, and made their corresponding Euler-Lagrange equations.

In this paper, we extend the Agrawal method in [1] and use Bernstein polynomials (BPs) to solve linear DFOCPs with time-varying coefficients and quadratic objective function. In fact we consider the following DFOCP:

$$\min J\big[x(\cdot),u(\cdot)\big]=\frac{1}{2}\int_0^1\big[x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\big]\,dt, \tag{1}$$

subject to

$$ {}_0^cD_t^\alpha x_i(t)=A(t)x(t)+B(t)u(t)+A_d(t)x(t-d), \tag{2}$$

$$x(t)=x_0,\quad t\in[-d,0], \tag{3}$$

where $x(t)=[x_1(t)\ \cdots\ x_r(t)]^T$ and $u(t)=[u_1(t)\ \cdots\ u_s(t)]^T$. Also, $Q(t)\geq 0$ and $R(t)>0$ are, respectively, $r\times r$ and $s\times s$ time-varying matrices of the state and control coefficients in the cost function with continuous functions as their entries. Furthermore, $a_{i,j}(t)$, $(a_d)_{i,j}(t)$, and $b_{i,k}(t)$ are continuous functions which are, respectively, the coefficients of $x_j(t)$, $x_j(t-d)$ for $(1\leq j\leq r)$ and $u_k(t)$ for $(1\leq k\leq s)$ in the ith fractional differential equation (2), and $d>0$ is the given constant time delay. The state-control pair $p=(x(\cdot),u(\cdot))$ that satisfies (2) and (3) is called the state-control admissible pair. The target is to find the admissible pair $p^*=(x^*(\cdot),u^*(\cdot))$ that minimizes the cost function $J[x(\cdot),u(\cdot)]$ in (1).

The fractional derivative is defined in the Caputo sense, *i.e.*,

$$ {}_0^cD_t^\alpha x_i(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\dot{x}_i(\tau)\,d\tau, & 0<\alpha<1,\\ \dot{x}_i, & \alpha=1.\end{cases} \tag{4}$$

This paper is organized as follows. In Section 2 some preliminaries and definitions in the fractional calculus that used in this manuscript are reviewed. Section 3 gives a general introduction to Bernstein polynomials and their properties. In Section 4 the TBVP for a FOCP with time delay is analyzed and it is solved by using BPs. Section 5 contains some numerical examples. Conclusions are presented in Section 6.

2 Some preliminaries in fractional calculus

This section consists of some basic definitions and properties in fractional calculus [18, 19]. In the sequel, Γ represents the Gamma function.

Definition 2.1 The space $\mathrm{AC}^n([0,1])$ denotes the set of all functions $f(t)$ which have the continuous derivatives up to order $(n-1)$ on $[0,1]$ and $f^{(n-1)}(t)$ is absolutely continuous on

$[0,1]$; *i.e.* there exists (almost everywhere) a function $g \in \mathrm{L}^1([0,1])$ such that

$$f^{(n-1)}(t) = f^{(n-1)}(0) + \int_0^t g(\tau)\,d\tau.$$

Definition 2.2 Let $\varphi \in \mathrm{L}^1([0,1])$. The integrals

$$\begin{aligned} {}_0I_t^\alpha \varphi(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)\,d\tau, \quad 0 < t \le 1, \\ {}_tI_1^\alpha \varphi(t) &= \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau-t)^{\alpha-1}\varphi(\tau)\,d\tau, \quad 0 \le t < 1, \end{aligned} \tag{5}$$

where $\alpha > 0$, are, respectively, called the left-sided and right-sided Riemann-Liouville fractional integrals of order α. Also

$${}_0I_t^0 \varphi(t) = {}_tI_1^0 \varphi(t) = \varphi(t).$$

Definition 2.3 Let $n-1 \le \alpha < n$. The left-sided and right-sided Riemann-Liouville fractional derivatives of order α of the function $\varphi(t) \in \mathrm{AC}^n([0,1])$ are defined, respectively, as follows:

$$\begin{aligned} {}_0D_t^\alpha \varphi(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1}\varphi(\tau)\,d\tau, \\ {}_tD_1^\alpha \varphi(t) &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^1 (\tau-t)^{n-\alpha-1}\varphi(\tau)\,d\tau. \end{aligned} \tag{6}$$

Definition 2.4 Let $n-1 \le \alpha < n$. The left-sided and right-sided Caputo fractional derivatives of order α of the function $\varphi(t) \in \mathrm{C}^n([0,1])$ are defined, respectively, as follows:

$$\begin{aligned} {}_0^cD_t^\alpha \varphi(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n \varphi(\tau)\,d\tau, \\ {}_t^cD_1^\alpha \varphi(t) &= \frac{1}{\Gamma(n-\alpha)} \int_t^1 (\tau-t)^{n-\alpha-1} \left(-\frac{d}{dt}\right)^n \varphi(\tau)\,d\tau. \end{aligned} \tag{7}$$

The Riemann-Liouville fractional derivatives and the Caputo fractional derivatives are connected by the following relations:

$${}_0^cD_t^\alpha \varphi(t) = {}_0D_t^\alpha \varphi(t) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{\Gamma(k-\alpha+1)}(t-0)^{k-\alpha}, \tag{8}$$

$${}_t^cD_1^\alpha \varphi(t) = {}_tD_1^\alpha \varphi(t) - \sum_{k=0}^{n-1} \frac{(-1)^k\varphi^{(k)}(1)}{\Gamma(k-\alpha+1)}(1-t)^{k-\alpha}. \tag{9}$$

Moreover, the Caputo fractional derivative of a constant function is zero.

Definition 2.5 Let $\alpha > 0$. Then ${}_0I_t^\alpha(L^p([0,1]))$ denotes the space of all functions $f(t)$, represented by ${}_0I_t^\alpha \varphi$ where $\varphi \in L^p([0,1])$.

Lemma 2.1 ([18]) *Let $\alpha > 0$. The equality*

$$ {}_0D_t^\alpha {}_0I_t^\alpha \varphi(t) = \varphi(t) $$

is valid for any $\varphi \in L^1([0,1])$ while

$$ {}_0I_t^\alpha {}_0D_t^\alpha f(t) = f(t) $$

is satisfied for $f \in {}_0I_t{}^\alpha(L^1([0,1]))$. Furthermore, if $f \in L^1([0,1])$ and ${}_0I_t{}^{n-\alpha}f(t) \in \mathrm{AC}^n([0,1])$,

$$ {}_0I_t{}^\alpha {}_0D_t{}^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} \frac{t^{\alpha-i-1}}{\Gamma(\alpha-i)} \left(\frac{d}{dt}\right)^{n-i-1} \left({}_0I_t{}^{n-\alpha} f(t)\right), $$

where $n = [\alpha] + 1$ and $\mathrm{AC}^n([0,1])$ is defined in the sense of Definition 2.1.

3 Bernstein polynomials (BPs) and their properties

The Bernstein polynomial of degree n over the interval $[a,b]$ is defined as follows:

$$ B_{i,n}\left(\frac{t-a}{b-a}\right) = \binom{n}{i} \left(\frac{t-a}{b-a}\right)^i \left(\frac{b-t}{b-a}\right)^{n-i}, \quad i = 0,1,\dots,n. $$

So, within the interval $[0,1]$ we have

$$ B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}. $$

Define $\Phi_m(t) = [B_{0,m}(t)\ B_{1,m}(t)\ \cdots\ B_{m,m}(t)]^T$. To consider the vector $\Phi_m(t-d)$ (d is the given delay) in terms of $\Phi_m(t)$, we state the following lemmas.

Lemma 3.1 ([10, 14, 15]) *We can write $\Phi_m(t) = \Delta T_m(t)$, where $\Delta = (\Upsilon_{i,j})_{i,j=1}^{m+1}$ is an upper triangular $(m+1) \times (m+1)$ matrix, and*

$$ \Upsilon_{i+1,j+1} = \begin{cases} (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}, & i \le j, \\ 0, & i > j, \end{cases} $$

for $i,j = 0,1,\dots,m$ and $T_m(t) = [1\ t\ \cdots\ t^m]^T$.

Lemma 3.2 ([20]) *Let $L^2[0,1]$ be a Hilbert space with inner product $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$. If $y \in L^2[0,1]$, then y has a unique best approximation of order m as follows*:

$$ \sum_{i=0}^{m} c_i B_{i,m}(t) = C^T \Phi_m(t), \quad t \in [0,1], \tag{10} $$

where the unique vector C is defined as $C = [c_0\ c_1\ \cdots\ c_m]^T$.

In (10), $C = Q^{-1}\langle y, \Phi_m\rangle$ where

$$ \langle y, \Phi_m\rangle = \int_0^1 y(t)\Phi_m(t)\,dt = \left[\langle y, B_{0,m}\rangle \quad \langle y, B_{1,m}\rangle \quad \cdots \quad \langle y, B_{m,m}\rangle\right]^T, $$

and the entry of the matrix $Q = (Q_{i+1,j+1})_{i,j=0}^{m}$ is defined as follows:

$$Q_{i+1,j+1} = \int_0^1 B_{i,m}(t)B_{j,m}(t)\,dt = \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}. \tag{11}$$

Note that a polynomial of degree m can be expanded in terms of a linear combination of $B_{i,m}(t)$, $(i = 0,1,\ldots,m)$ as follows:

$$P(t) = \sum_{i=0}^{m} c_i B_{i,m}(t),$$

we recall that the set $\{B_{0,m}(t), B_{1,m}(t),\ldots,B_{m,m}(t)\}$ is a complete basis in Hilbert space $L^2[0,1]$.

Lemma 3.3 ([21]) *Derivatives of* $P_n(f) = \sum_{j=0}^{n} f(\frac{j}{n})B_{j,n}(t)$ *of any order converge to the corresponding derivatives of* f. *If* $f \in C^k[0,1]$, $k \geq 0$, *then*

$$\lim_{n\to\infty} \bigl(P_n(f)\bigr)^{(k)} = f^{(k)},$$

uniformly on $[0,1]$.

Lemma 3.4 ([14]) *For each given constant delay* $d > 0$, $\Phi_m(t-d) = \Omega\Phi_m(t)$, *where* Ω *is an* $(m+1)\times(m+1)$ *matrix in terms of* d.

It was shown in [14] that $\Omega = \Delta\Psi\Delta^{-1}$, where

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -d & 1 & 0 & \cdots & 0 \\ d^2 & -2d & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ (-d)^m & \binom{m}{m-1}(-d)^{m-1} & \cdots & & 1 \end{bmatrix},$$

and Δ is the matrix presented in Lemma 3.1.

4 Fractional optimal control problems with state delay

In this section, first we state some lemmas to investigate the variational method for DFOCP (1)-(3) and make the corresponding TBVP, then two operational matrices to approximate the left-sided and right-sided Caputo fractional derivatives of $\Phi_m(t)$ are introduced to numerically solve the TBVP.

Lemma 4.1 ([18]) *Let* $\alpha > 0$, $p,q \geq 1$ *and* $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ *and* $q \neq 1$ *in the case where* $\frac{1}{p}+\frac{1}{q} = 1+\alpha$). *If* $\psi \in \mathrm{L}^q([0,1])$ *and* $\varphi \in \mathrm{L}^p([0,1])$, *then*

$$\int_0^1 \varphi(t)\bigl({}_0I_t^{\alpha}\psi\bigr)(t)\,dt = \int_0^1 \psi(t)\bigl({}_tI_1^{\alpha}\varphi\bigr)(t)\,dt, \tag{12}$$

is valid and it is usually called the formula for fractional integration by parts.

Lemma 4.2 ([16]) *Let $\alpha > 0$, $p, q \geq 1$, $r \in (0,1)$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\psi \in \mathrm{L}^{\mathrm{q}}([0,1])$ and $\varphi \in \mathrm{L}^{\mathrm{p}}([0,1])$, then*

$$\int_r^1 \varphi(t)\big({}_0I_t^\alpha \psi\big)(t)\,dt = \int_r^1 \psi(t)\big({}_tI_1^\alpha \varphi\big)(t)\,dt + \frac{1}{\Gamma(\alpha)}\int_0^r \big({}_0I_t^\alpha \psi\big)(t)\bigg[\int_r^1 \varphi(s)(s-t)^{\alpha-1}\,ds\bigg]\,dt.$$

Lemma 4.3 *Let $0 < \alpha < 1$, $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $f, g \in \mathrm{C}([0,1])$ and $f(0) = 0$ and $g(1) = 0$, then*

$$\text{(a)} \qquad \int_0^1 g(t)\big({}_0^cD_t^\alpha f\big)(t)\,dt = \int_0^1 f(t)\big({}_t^cD_1^\alpha g\big)(t)\,dt \tag{13}$$

and

$$\text{(b)} \qquad \int_r^1 g(t)\big({}_0^cD_t^\alpha f\big)(t)\,dt = \int_r^1 f(t)\big({}_t^cD_1^\alpha g\big)(t)\,dt - \frac{1}{\Gamma(\alpha)}\int_0^r f(t){}_t^cD_r^\alpha\bigg\langle\int_r^1 \big({}_t^cD_1^\alpha g\big)(s)(s-t)^{\alpha-1}\,ds\bigg\rangle\,dt. \tag{14}$$

Proof Equations (13) and (14) are proved in [16] when the fractional derivatives assumed to be Riemann-Liouville. Now, by assuming $\varphi = {}_t^cD_1^\alpha g$ and $\psi = {}_0^cD_t^\alpha f$ and applying Lemmas 4.1 and 4.2, considering $f(0) = 0$ and $g(1) = 0$, finding the results is straightforward. □

Remark Consider the DFOCP (1)-(3) for $\alpha \in (0,1)$. Define the following corresponding unconstrained problem:

$$\min \bar{J}\big[x(\cdot), u(\cdot)\big] = \int_0^1 \bigg\{\frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) + \big[A(t)x(t) + B(t)u(t) + A_d(t)x(t-d) -{}_0^c D_t^\alpha x(t)\big]^T \lambda(t)\bigg\}\,dt, \tag{15}$$

with the initial condition

$$x(t) = x_0, \quad t \in [-d, 0]. \tag{16}$$

The problem (1)-(3) and the unconstraint problem (15)-(16) have the same optimal solution [22].

Theorem 4.1 *The necessary conditions for $\bar{J}[x(\cdot), u(\cdot)]$ to possess the extremum is that the triple $(x(t), u(t), \lambda(t))$ fulfills the following TBVP*:

$$R(t)u(t) + B^T(t)\lambda(t) = 0, \quad 0 \leq t \leq 1, \tag{17}$$

$${}_0^cD_t^\alpha x(t) = A(t)x(t) + B(t)u(t) + A_d(t)x(t-d), \quad 0 \leq t \leq 1, \tag{18}$$

$$Q(t)x(t) + \lambda(t)A(t) + \lambda(t+d)A(t+d) - \big({}_t^cD_{1-d}^\alpha \lambda\big)(t) + \frac{1}{\Gamma(\alpha)}{}_t^cD_{1-\eta}^\alpha\bigg\langle\int_{1-d}^1 \big({}_t^cD_1^\alpha \lambda\big)(s)(s-t)^{\alpha-1}\,ds\bigg\rangle = 0, \quad 0 \leq t \leq 1-\eta, \tag{19}$$

$$Q(t)x(t) + \lambda(t)A(t) - \left({}_t^cD_1^\alpha \Lambda\right)(t) = 0, \quad 1-\eta \le t \le 1, \tag{20}$$

$$x(0) = x_0, \qquad \lambda(1) = 0. \tag{21}$$

Proof Let the triple $(u^*(t), x^*(t), \lambda^*(t))$ be the optimal solution of (15)-(16). To find the optimal control, we follow the traditional approach by making variations in the optimal solution of the problem (15)-(16). Assume δx, δu, and $\delta\lambda$ are, respectively, the variations of $x^*(t)$, $u^*(t)$, and $\lambda^*(t)$, then a variation of the optimal control can be defined as follows:

$$x(t) = x^*(t) + \delta x, \qquad u(t) = u^*(t) + \delta u, \qquad \lambda(t) = \lambda^*(t) + \delta\lambda.$$

Now it is possible to make these changes on $\bar{J}$, so we have

$$\bar{J}\left[\left(x(\cdot), u(\cdot)\right)\right] = \bar{J}\left[\left(x^*(\cdot), u^*(\cdot)\right)\right] + \delta\bar{J}.$$

Since $\bar{J}$ reaches its minimum at $(x^*(\cdot), u^*(\cdot))$, it can be concluded that $\delta\bar{J} = 0$. Moreover,

$$\begin{aligned}\delta\bar{J} = \int_0^1 &\Big\{\delta x^T Q(t)x^*(t) + \delta u^T R(t)u^*(t) + \big[A(t)x^*(t) + B(t)u^*(t) + A_d(t)x^*(t-d)\\ &-{}_0^c D_t^\alpha x^*(t)\big]^T \delta\lambda + \big[A(t)\delta x + B(t)\delta u + A_d(t)\delta x(t-d) - \delta\left({}_0^cD_t^\alpha x^*\right)(t)\big]^T \lambda^*(t)\Big\}\,dt,\end{aligned}$$

or

$$\begin{aligned}\delta\bar{J} = \int_0^{1-d} &\Big\{\delta x^T\big(Q(t)x^*(t) + A^T(t)\lambda^*(t) + A_d^T(t+d)\lambda^*(t+d)\big) + \delta u^T\big(R(t)u^*(t)\\ &+ B(t)^T\lambda^*(t)\big) + \big[A(t)x^*(t) + B(t)u^*(t) + A_d(t)x^*(t-d) -{}_0^c D_t^\alpha x^*(t)\big]^T\delta\lambda\\ &- \delta\left({}_0^cD_t^\alpha x^*\right)^T(t)\lambda^*(t)\Big\}\,dt + \int_{1-d}^1 \Big\{\delta x^T\big(Q(t)x^*(t) + A^T(t)\lambda^*(t)\big) + \delta u^T\big(R(t)u^*(t)\\ &+ B^T(t)\lambda^*(t)\big) + \big[A(t)x^*(t) + B(t)u^*(t) + A_d(t)x^*(t-d) -{}_0^c D_t^\alpha x^*(t)\big]^T\delta\lambda\\ &- \delta\left({}_0^cD_t^\alpha x^*\right)^T(t)\lambda^*(t)\Big\}\,dt.\end{aligned}$$

Since $\delta({}_0^cD_t^\alpha x) = {}_0^cD_t^\alpha \delta x$ (this can be proved easily by applying the definition of first variation) and $\delta x(0) = 0$, by applying Lemma 4.3(a) and (b) one can obtain the following equivalent equation:

$$\begin{aligned}\delta\bar{J} = \int_0^{1-d} &\Big\{\delta x^T\big(Q(t)x^*(t) + A^T(t)\lambda^*(t) + A_d^T(t+d)\lambda^*(t+d)\big) + \delta u^T\big(R(t)u^*(t)\\ &+ B^T(t)\lambda^*(t)\big) + \big[A(t)x^*(t) + B(t)u^*(t) + A_d(t)x^*(t-d) -{}_0^c D_t^\alpha x^*(t)\big]^T\delta\lambda\\ &- \delta x^T\left({}_t^cD_{1-d}^\alpha \lambda^*\right)(t)\Big\}\,dt + \int_{1-d}^1 \Big\{\delta x^T\big(Q(t)x^*(t) + A(t)^T\lambda^*(t)\big) + \delta u^T\big(R(t)u^*(t)\\ &+ B^T(t)\lambda^*(t)\big) + \big[A(t)x^*(t) + B(t)u^*(t) + A_d(t)x^*(t-d) -{}_0^c D_t^\alpha x^*(t)\big]^T\delta\lambda\\ &- \delta x^T\left({}_t^cD_1^\alpha \lambda^*\right)(t)\Big\}\,dt\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{1-d}\left\{{}_t^cD_{1-d}^\alpha\left[\int_{1-d}^1 \delta x^T\left({}_t^cD_1^\alpha\lambda^*\right)(s)(s-t)^{\alpha-1}\,ds\right]\right\}dt.\end{aligned} \tag{22}$$

Note that the assumption $\lambda(1) = 0$ is necessary for applying the aforementioned lemmas. Now since $\delta\bar{J} = 0$, the proof will be completed by equalizing the coefficients of δx, δu, and $\delta\lambda$ in (22) with zero. □

According to Theorem 4.1 the necessary conditions for $(x^*(\cdot), u^*(\cdot))$ being the optimal solution of (1)-(3) are to satisfy in (17)-(21). These conditions are also sufficient because of the convexity of the quadratic form of the objective function. To solve the system of equations (17)-(21), first from equation (17) one can conclude that $u(t) = -R^{-1}(t)B^T(t)\lambda(t)$ (this is true since $R(t) > 0$), then using the characteristic functions $\chi_{[0,1-\eta]}(t)$ and $\chi_{[1-\eta,1]}(t)$ we incorporate equations (19) and (20) and apply the Agrawal method in [23]. In this work we use Bernstein polynomials to approximate the solution of (18)-(21) in which $u(t)$ is substituted by $-R^{-1}(t)B^T(t)\lambda(t)$. Furthermore, to simplify the relations, in the sequel, we assume the matrix functions $A(t)$, $B(t)$, $A_d(t)$, $Q(t)$ and $R(t)$ to be constant functions. Of course, when these functions are not constant the relations can be extended easily by approximating $A(t)$, $B(t)$, $A_d(t)$, $Q(t)$ and $R(t)$ in terms of BPs.

By Lemma 3.1, we have $\Phi_m(t) = \Delta T_m(t)$, so

$$ {}_0^cD_t^\alpha \Phi_m(t) = \Delta {}_0^cD_t^\alpha T_m(t) = \Delta \begin{bmatrix} {}_0^cD_t^\alpha 1 & {}_0^cD_t^\alpha t & \cdots & {}_0^cD_t^\alpha t^m \end{bmatrix}^T. \tag{23} $$

Furthermore, for any $0 < l \le 1$ there exists a $(m+1) \times (m+1)$ lower triangular matrix L where $T_m(l-t) = LT_m(t)$. So we also have

$$ \Phi_m(t) = \Delta L^{-1} T_m(l-t), $$

and as a result

$$ {}_t^cD_l^\alpha \Phi_m(t) = \Delta L^{-1} {}_t^cD_l^\alpha T_m(l-t) = \Delta L^{-1} \begin{bmatrix} {}_t^cD_l^\alpha 1 & {}_t^cD_l^\alpha (l-t) & \cdots & {}_t^cD_l^\alpha (l-t)^m \end{bmatrix}^T. \tag{24} $$

As a property of the left-sided and right-sided Caputo fractional derivative for $\alpha \in (0,1)$ we have

$$ {}_0^cD_t^\alpha t^j = \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} t^{j-\alpha}, \quad j = 1, \ldots, m, 0 < t < 1, $$

and for any $0 < l \le 1$

$$ {}_t^cD_l^\alpha (l-t)^j = \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (l-t)^{j-\alpha}, \quad j = 1, \ldots, m, 0 < t < l. $$

One may define

$$ {}_0^cD_t^\alpha T_m(t) = \tilde{\Sigma} \tilde{T}_1, \tag{25} $$

$$ {}_t^cD_l^\alpha T_m(l-t) = \tilde{\Sigma} \tilde{T}_2, \tag{26} $$

where $\tilde{\Sigma}$ is an $(m+1) \times (m+1)$ matrix and $\tilde{T}_1$ and $\tilde{T}_2$ are $(m+1) \times 1$ matrices, each one defined as follows:

$$ \tilde{\Sigma} = (\tilde{\Sigma}_{i+1,j+1}), \qquad \tilde{\Sigma}_{i+1,j+1} = \begin{cases} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}, & i,j = 1, \ldots, m, \text{ and } i = j, \\ 0, & i = j = 0, \end{cases} $$

and

$$\tilde{T}_1 = (\tilde{T}_{1,k+1}), \qquad \tilde{T}_{1,k+1} = \begin{cases} t^{k-\alpha}, & k = 1,\dots,m, \\ 0, & k = 0, \end{cases}$$

and

$$\tilde{T}_2 = (\tilde{T}_{2,k+1}), \qquad \tilde{T}_{2,k+1} = \begin{cases} (l-t)^{k-\alpha}, & k = 1,\dots,m, \\ 0, & k = 0. \end{cases}$$

Since $t^{k-\alpha}$ and $(l-t)^{k-\alpha}$ for $k = 1,\dots,m$ are continuous functions on $[0,1]$, one can apply the method in [10] to find approximated vectors $P_{1,k}$ and $P_{2,k}$ such that

$$t^{k-\alpha} \approx P_{1,k}^T \Phi_m(t), \quad k = 1,\dots,m, 0 < t < 1, \tag{27}$$

and

$$(l-t)^{k-\alpha} \approx P_{2,k}^T \Phi_m(t), \quad k = 1,\dots,m, 0 < t < l, \tag{28}$$

where $P_{1,k} = Q^{-1}\langle t^{k-\alpha}, \Phi_m(t)\rangle$ and $P_{2,k} = Q^{-1}\langle (l - l \times t)^{k-\alpha}, \Phi_m(t)\rangle$ for $k = 1,\dots,m$ while Q is defined in (11). Now if P_1 and P_2 be $(m+1)\times(m+1)$ matrices with zero vector in their first column and have $P_{1,i}$ and $P_{2,i}$, respectively, as their $(i+1)$th column for $i = 1,\dots,m$, then

$${}_0^cD_t^\alpha \Phi_m(t) \approx D_\alpha \Phi_m(t) \tag{29}$$

and

$${}_t^cD_l^\alpha \Phi_m(t) \approx {}_\alpha D \Phi_m(t), \tag{30}$$

where $D_\alpha = \Delta\tilde{\Sigma}P_1^T$ and ${}_\alpha D = \Delta L^{-1}\tilde{\Sigma}P_2^T$, are called the operational matrices of ${}_0^cD_t^\alpha \Phi_m(t)$ and ${}_t^cD_d^\alpha \Phi_m(t)$, respectively.

Now assume that

$$\begin{aligned} x_i(t) &\approx X_i^T \Phi_m(t), \\ \lambda_i(t) &\approx \Lambda_i^T \Phi_m(t), \end{aligned} \tag{31}$$

where the entries of $X_i = [X_i(0) \ \cdots \ X_i(m)]^T$ and $\Lambda_i = [\Lambda_i(0) \ \cdots \ \Lambda_i(m)]^T$ are, respectively, the Bernstein coefficients in the approximation of $x_i(t)$ and $\lambda_i(t)$ for $i = 1,\dots,r$ and $0 \le t \le 1$. Therefore,

$${}_0^cD_t^\alpha x_i(t) \approx X_i^T \, {}_0^cD_t^\alpha \Phi_m(t) \approx X_i^T D_\alpha \Phi_m(t) \tag{32}$$

and

$${}_t^cD_l^\alpha \lambda_i(t) \approx \Lambda_i^T \, {}_t^cD_l^\alpha \Phi_m(t) \approx \Lambda_i^T \, {}_\alpha D \Phi_m(t). \tag{33}$$

As a result, the TBVP (18)-(21) can be approximated as follows:

$$\bigl[X_i^T D_\alpha - \bigl(AX_i^T - BR^{-1}B^T\Lambda_i^T + A_d X_i^T \Omega_1\bigr)\bigr]\Phi_m(t) = 0, \tag{34}$$

$$\Big[QX_i^T + A^T\Lambda_i^T + A_d^T\Lambda_i^T\Omega_2 - \Lambda_i^T{}_\alpha D^1 + \frac{1}{\Gamma(\alpha)}\Lambda_i^T{}_\alpha D^2 * H^T{}_\alpha D^1\Big]\Phi_m(t) = 0, \quad 0 \le t \le 1-d, \tag{35}$$

$$\big[QX_i^T + A^T\Lambda_i^T - \Lambda_i^T{}_\alpha D^2 = 0\big]\Phi_m(t) = 0, \quad 1-d \le t \le 1. \tag{36}$$

In the above equations $H = [H_0\, H_1\, \cdots\, H_m]$ is an $(m+1)\times(m+1)$ matrix where H_i (its $(i+1)$th column of H) is the coefficients vector in approximating function $h(t) = \int_{1-d} B_{i,m}(s)(s-t)^{\alpha-1}\,ds$ with BPs. Indeed ${}_\alpha D^1$ and ${}_\alpha D^2$ can be computed by substituting $1-d$ and 1 in equation (33) instead of l (note that to achieve ${}_\alpha D^2$ a suitable change of variable is needed to transform the time interval $[1-d,1]$ to $[0,1]$ before approximation). Also Ω_1, Ω_2 can be calculated by applying Lemma 3.4 to $\Phi_m(t-\eta)$ and $\Phi_m(t+\eta)$, respectively.

Also, we need to recall that the initial conditions $x_i(0) = x_{i,0}$ and $\lambda_i(1) = 0$ in (21) can be written in term of the Bernstein basis as follows:

$$x_{i,0} = \begin{bmatrix} X_i(0) & \cdots & X_i(m)\end{bmatrix}\Phi_m(0) = \begin{bmatrix} X_i(0) & \cdots & X_i(m)\end{bmatrix}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$$

and

$$0 = \begin{bmatrix} \Lambda_i(0) & \cdots & \Lambda_i(m)\end{bmatrix}\Phi_m(1) = \begin{bmatrix} \Lambda_i(0) & \cdots & \Lambda_i(m)\end{bmatrix}\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix},$$

or

$$X_i(0) = x_{i,0}, \quad 1 \le i \le r, \tag{37}$$

and

$$\Lambda_i(m) = 0, \quad 1 \le i \le r. \tag{38}$$

In order to solve the approximated system of equations (34)-(38) we apply the Agrawal method in [23]. First, using the characteristic functions $\chi_{[0,1-d]}(t)$ and $\chi_{[1-d,1]}(t)$ to incorporate equations (35) and (36),

$$\chi_{[0,1-d]}(t)\Big[QX_i^T + A^T\Lambda_i^T + A_d^T\Lambda_i^T\Omega_2 - \Lambda_i^T{}_\alpha D^1 + \frac{1}{\Gamma(\alpha)}\Lambda_i^T{}_\alpha D^2 * H^T{}_\alpha D^1\Big]\Phi_m(t) + \chi_{[1-d,1]}(t)\big[QX_i^T + A^T\Lambda_i^T - \Lambda_i^T{}_\alpha D^2 = 0\big]\Phi_m(t) = 0, \quad i = 1,\ldots,r.$$

Then, by defining arbitrary virtual variations in the state and costate variables $\delta x \approx (\Delta x)^T\Phi_m(t)$ and $\delta\lambda \approx (\Delta\lambda)^T\Phi_m(t)$ and setting the coefficients of Δx and $\Delta\lambda$ to zero one can obtain the following algebraic system with $(m+1)$ linear equations and $(m+1)$ unknowns:

$$\int_0^1 \{X_i^T D_\alpha - (AX_i^T - BR^{-1}B^T\Lambda_i^T + A_d X_i^T \Omega_1)\}\Phi_m(t)B_{j,m}(t)\,dt + \mu B_{j,m}(0) = 0, \tag{39}$$

$$\begin{aligned}&\int_0^{1-\eta}\left\{QX_i^T + A^T\Lambda_i^T + A_d^T\Lambda_i^T - \Lambda_i^T{}_\alpha D^1 + \frac{1}{\Gamma(\alpha)}\Lambda_i^T{}_\alpha D^2 * H^T{}_\alpha D^1\right\}\Phi_m(t)B_{j,m}(t)\,dt \\ &\quad + \int_{1-\eta}^1 \{QX_i^T + A^T\Lambda_i^T - \Lambda_i^T{}_\alpha D^2\}\Phi_m(t)B_{j,m}(t)\,dt + \nu B_{j,m}(1) = 0,\end{aligned} \tag{40}$$

for every $j = 0,1,\ldots,m$ and $i = 1,\ldots,r$ with two boundary conditions $X(0) = x_0$ and $\Lambda(m) = 0$.

5 Numerical examples

In this section we give some numerical examples and apply the method presented in Section 4 for solving them. Our examples are solved using *Matlab2011a* on an *Intel Core i5-430M* processor with 4 GB of *DDR3* Memory. These test problems demonstrate the validity and efficiency of this technique.

Example 1 Consider the following two-dimensional FDOCP in which $0 < \alpha \leq 1$ (see [24]):

$$\min \frac{1}{2}\int_0^1 \left[\left(x_1(t) + x_2(t)\right)^2 + u^2(t)\right] dt,$$

s.t.

$$\begin{aligned}&{}_0^cD_t^\alpha x_1(t) = x_1(t) + x_2\left(t - \frac{1}{4}\right), \quad 0 \leq t \leq 1,\\ &{}_0^cD_t^\alpha x_2(t) = -5x_1\left(t - \frac{1}{4}\right) + x_2(t) - x_2\left(t - \frac{1}{4}\right) + u(t),\\ &x_1(t) = 1, \quad -\frac{1}{4} \leq t \leq 0,\\ &x_2(t) = 1, \quad -\frac{1}{4} \leq t \leq 0.\end{aligned}$$

This problem for $\alpha = 1$ has been studied in [24], where the obtained approximate cost function is $I = 2.7930$. Using the presented method for $\alpha = 1$ and $m = 6$, gives the approximate cost function as $J^* = 1.9493$. So we achieved satisfactory numerical results in comparison with what have been obtained in [24] for $\alpha = 1$. In the case $\alpha = 1$ the approximate trajectories and control functions for $t \in [0,1]$ are

$$\begin{aligned}&x_1(t) \simeq -6.1198t^6 + 23.071t^5 - 34.697t^4 + 22.424t^3 - 5.1686t^2 + 2.1208t + 1,\\ &x_2(t) \simeq 56.779t^6 - 196.94t^5 + 263.83t^4 - 169.23t^3 + 43.179t^2 - 5.9202t + 1,\\ &u(t) \simeq -(t-1)\left(9.9706t^5 - 6.2514t^4 - 1.7909t^3 + 0.77024t^2 + 1.2626t + 0.30261\right).\end{aligned}$$

Also by varying the value of α the obtained trajectories and control functions are shown, respectively, in Figure 1 and Figure 2. Moreover, the optimal objective value and the end points of the optimal trajectories for these values of α are shown in Table 1.

Figure 1 Approximate solution of $x_1(\cdot)$ and $x_2(\cdot)$ for $\alpha = 1, 0.9, 0.8$ in Example 1.

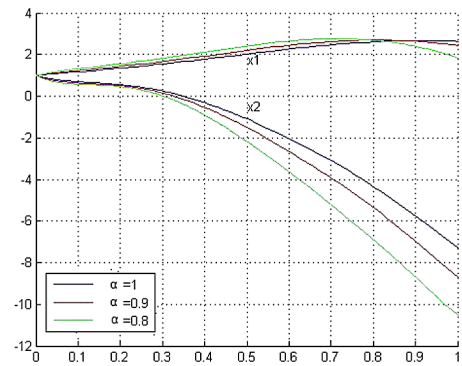


Figure 2 Approximate solution of $u(\cdot)$ for $\alpha = 1$, 0.9, 0.8 in Example 1.

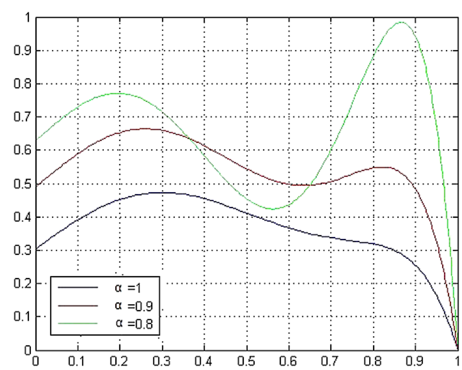


Table 1 The objective value and the end point of trajectory for $\alpha = 1, 0.9, 0.8$ in Example 1

α	Objective value	End points
1	1.9493	2.632, – 7.3009
0.9	3.1472	2.4246, – 8.7117
0.8	5.8783	1.8422, – 10.5162

Example 2 Consider the following FDOCP in which $0 < \alpha \leq 1$ (see [25]):

$$\min \frac{1}{2} \int_0^2 \left[x^2(t) + u^2(t)\right] dt,$$

s.t.

$${}_0^c D_t^\alpha x(t) = x(t-1) + u(t), \quad 0 \leq t \leq 2,$$

$$x(t) = 1, \quad -1 \leq t \leq 0.$$

Since our method is described for $t \in [0,1]$, first, the interval $[0,2]$ must be mapped into $[0,1]$ by using the transformation function $\theta = \frac{t}{2}$. So by letting

$$x(t) = x(2\theta) = y(\theta), \qquad u(t) = u(2\theta) = v(\theta),$$

and using

$$x(t-1) = x(2\theta - 1) = x\left\langle 2\left(\theta - \frac{1}{2}\right)\right\rangle = y\left(\theta - \frac{1}{2}\right)$$

and

$$\begin{aligned}
{}_0^cD_t^\alpha x(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\frac{dx}{dt}(\tau)}{(t-\tau)^\alpha}\,d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^{2\theta} \frac{\frac{1}{2}\frac{dy}{d\theta}(\frac{\tau}{2})}{(2\theta-\tau)^\alpha}\,d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^{\theta} \frac{\frac{1}{2}\frac{dy}{d\theta}(\rho)}{(2\theta-2\rho)^\alpha}(2\,d\rho) \\
&= \frac{1}{\Gamma(1-\alpha)}2^{-\alpha}\int_0^{\theta} \frac{\frac{dy}{d\theta}(\rho)}{(\theta-\rho)^\alpha}(d\rho) \\
&= 2^{-\alpha}\times {}_0^cD_\theta^\alpha y(\theta),
\end{aligned}$$

the above problem changes thus:

$$\min \frac{1}{2}\int_0^1 \bigl[2y^2(\theta)+2v^2(\theta)\bigr]\,d\theta,$$

s.t.

$$\begin{aligned}
&{}_0^cD_\theta^\alpha y(\theta) = 2^\alpha y\left(\theta-\frac{1}{2}\right)+2^\alpha v(\theta), \quad 0\le\theta\le 1, \\
&y(\theta)=1, \quad -\frac{1}{2}\le\theta\le 0.
\end{aligned}$$

Note that in this example

$$Q(t)=R(t)=2, \qquad A(t)=0, \qquad A_d(t)=B(t)=2^\alpha.$$

For $\alpha = 1$, this problem has been numerically solved by applying Bezier curves in [25] and the objective value $I = 1.5936$ has been achieved. In the presented method the solution has the objective value $J^* = 1.0447$ for $\alpha = 1$. Thus, our results with $m = 6$ are in good agreement with the results demonstrated in [25] for $\alpha = 1$. In addition by varying the value of α we can obtain the optimal trajectory $x(\cdot)$ and the control function $u(\cdot)$ which are shown, respectively, for some values of α in Figure 3 and Figure 4. Moreover, the optimal objective value and the end point of the optimal trajectory for these values of α are shown in Table 2.

Figure 3 Approximate solution of $x(\cdot)$ for $\alpha = 1$, 0.9, 0.8 in Example 2.

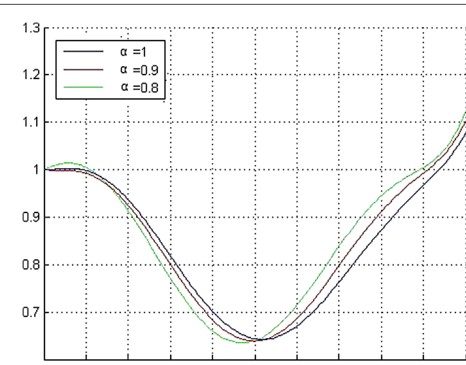


Figure 4 Approximate solution of $u(\cdot)$ for $\alpha = 1, 0.9, 0.8$ in Example 2.

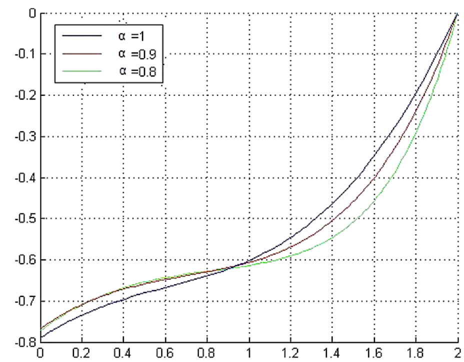
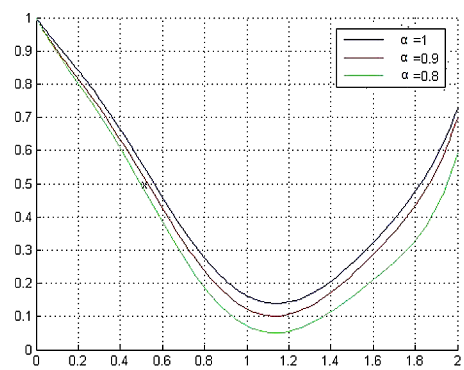


Table 2 The objective value and the end point of trajectory for $\alpha = 1, 0.9, 0.8$ in Example 2

α	Objective value	End point
1	1.0447	1.0772
0.9	1.0574	1.1032
0.8	1.0864	1.1240

Figure 5 Approximate solution of $x(\cdot)$ for $\alpha = 1, 0.9, 0.8$ in Example 3.



Example 3 Consider the following time-varying FDOCP in which $0 < \alpha \leq 1$ (see [26, 27]):

$$\min \int_0^2 \left[x^2(t) + u^2(t)\right] dt,$$

s.t.

$$ {}_0^cD_t^\alpha x(t) = tx(t) + x(t-1) + u(t), \quad 0 \leq t \leq 2,$$

$$x(t) = 1, \quad -1 \leq t \leq 0.$$

This problem for $\alpha = 1$ has been studied in [27] and [26] where the obtained approximate cost functions are, respectively, $I = 5.1713$ and $I = 4.7407$. Using the presented method for $\alpha = 1$ and $m = 6$, we find the approximate cost function as $J^* = 2.7384$. Also by varying the value of α the obtained trajectories and control functions are shown, respectively, in Figure 5 and Figure 6. Moreover, the optimal objective value and the end points of the optimal trajectories for these values of α are shown in Table 3.

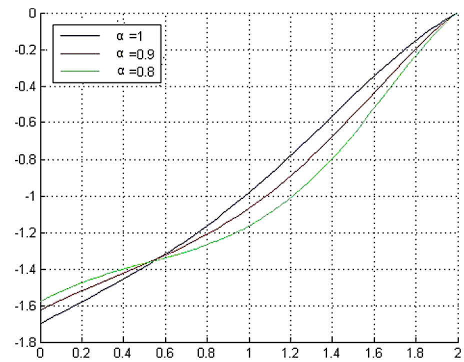


Figure 6 Approximate solution of $u(\cdot)$ for $\alpha = 1, 0.9, 0.8$ in Example 3.

Table 3 The objective value and the end point of trajectory for $\alpha = 1, 0.9, 0.8$ in Example 3

α	Objective value	End point
1	2.7384	0.7261
0.9	2.7504	0.6985
0.8	2.8108	0.5908

6 Conclusion

In this paper, we introduce the TBVP for the fractional optimal control problem with constant delay on trajectory. In order to solve the TBVP we have extended the method used in [1], then using Bernstein polynomials to approximate the solution. Since we use polynomials to approximate state and control, the approximating results are smooth and therefore no fitting curves are needed. We need to mention that in the case that the objective function is convex, the Hamiltonian condition would be necessary and sufficient. Thus by increasing the degree of Bernstein polynomials, the convergence should occur. Finally, some test problems are included to show the efficiency of this method.

Competing interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' contributions
The authors ES and MHF contributed equally to the writing of the paper. The authors read and approved the final manuscript.

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