SYNCHRONIZATION OF MULTIPLE CHAOTIC SYSTEMS USING A NONLINEAR GROUPING FEEDBACK FUNCTION METHOD

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Abstract

Multi-agent chaotic synchronization arises in numerous applications, both in natural and in artificial systems. Many coupling methods have been introduced for synchronization of chaotic systems in the literature. In this article, we have proposed three expanded nonlinear coupling feedback function methods for synchronization of multi-agent chaotic systems. Complete and phase synchronizations of multi-agent chaotic systems were studied together with their stability analysis.

Key Words

Multi-agent synchronization, chaotic systems, nonlinear coupling feedback function method

1. Introduction

Synchronization topic has received a large amount of attentions due to its applications. Since Pecora and Carroll who raised the issue of coupling method for synchronization of two chaotic systems [1], researchers have applied many other methods such as active control [2], adaptive control [3]–[6], and sliding mode control [7] to couple and synchronize two chaotic systems. In most of these methods, the basic configuration for synchronization is a master–slave pattern where the slave system has to track the trajectory of master system, that is, "*uni-directional* synchronization". For example, Shieh considered synchronization of master–slave chaotic systems with uncertainties [8].

Nonlinear coupling feedback function method which was first introduced by Ali and Fang [9] is a bi-directional coupling method. In this method, the trajectory of each system changes with other system, and each system tracks the other system. Erjaee Momani [10] used this method to synchronize two fractional order chaotic systems. This method has also been used for phase synchronization of two coupled integer order chaotic systems [11].

In many applications such as multi-agent systems or systems with interconnected dynamics, it is vital to synchronize a network of dynamical systems. Lu *et al.* employed event-triggered coupling configurations to realize synchronization for networks of linearly coupled dynamical systems [12]. Wu studied networks of coupled dynamical systems where an external forcing control signal is applied to the network to align the state of all the individual systems to the forcing signal [13]. Kazemy considered the problem of synchronization for complex dynamic networks with state and coupling time delays [14]. Cai *et al.* presented necessary and sufficient conditions for asymptotic swarm stability of general LTI systems and certain nonlinear dynamical multi-agent systems [15]. Lixia *et al.* used a combination of the master–slave method to derive the multi-robot formation architecture [16].

To synchronize two or more systems, it is not necessary to have one system as a master and others as slaves. Furthermore, the trajectories of each system may change due to variation of other systems. We expand the nonlinear feedback function coupling method to a grouping method for synchronization of more than two systems. We applied the expanded method in three various ways and expressed the results.

Therefore, our paper is organized as follows. We shortly discuss the nonlinear coupling feedback function method in Section 2. The proposed method is discussed in Section 3, for this aim, three methods are considered. Stability analysis of proposed methods is given in Section 4. Finally, some examples are illustrated in Section 5.

2. Nonlinear Coupling Feedback Function Method

To apply a nonlinear coupling feedback function method, introduced by Ali and Fang, we assume vector-valued function $\mathbf{F}(\mathbf{x}(t))$ in a chaotic system

$$\dot{\mathbf{x}}(\boldsymbol{t}) = \mathbf{F}(\mathbf{x}(t)),\ \ \mathbf{x}(\boldsymbol{t_0}) = \mathbf{x_0} \tag{1}$$

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is decomposed into linear, $\mathbf{A}\mathbf{x}(t)$, and nonlinear, $\mathbf{H}(\mathbf{x}(t))$, components as follows:

$$\mathbf{F}(\mathbf{x}(t)) = \mathbf{A}\mathbf{x}(t) - \mathrm{H}(\mathbf{x}(t)) \tag{2}$$

where $\mathbf{H}(\mathbf{x}(t))$ is differentiable with respect to states. Now, consider two chaotic systems, such that their associated vector functions are decomposed as above. Then, using nonlinear components of their vector functions, they are coupled as follows:

$$\begin{aligned}\dot{\mathbf{x}}_1(t) &= \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t)) + s[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))]\\ \dot{\mathbf{x}}_2(t) &= \mathbf{A}\mathbf{x}_2(t) - \mathbf{H}(\mathbf{x}_2(t)) + s[\mathbf{H}(\mathbf{x}_2(t)) - \mathbf{H}(\mathbf{x}_1(t))]\end{aligned}$$

where s is the strength of coupling. In the case of chaotic synchronization for these coupled systems, the stability analysis can be studied by considering the following linearized difference equation:

$$\dot{\mathbf{e}}(\mathbf{t}) = \left[\mathbf{A} + (\mathbf{2s}-\mathbf{1})\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e} \tag{3}$$

where $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$.

Hereafter, we expand this coupling method and its stability analysis for multi-agent models.

3. Weighed Extended Nonlinear Coupling Feedback Function Method

Recently, in the study of interaction between agents, interconnection in a large-scale network has become an important topic of research. In the past decade, such systems have received great attentions, due to the existence of actual networks composed of a large number of smaller components. It is notable that in such networks all devices should work simultaneously to produce the best output. In another word, in such networks all the interconnected systems may have the same or different significance and so, a single-master multi-slave synchronization may not be useful or applicable. Therefore, a bidirectional synchronization method such as the nonlinear coupling feedback function method could be more applicable. Here, we have attempted to propose three extensions of the nonlinear coupling feedback function method to group multiple systems. Amongst these three methods the third one does not bring the systems to a synchrony mode unless a *suitable coupling strength* is chosen for coupling.

The network structure of agents can be represented as a graph. For each proposed method, we have illustrated the weighed directed graph for the structure of proposed network.

To extend the nonlinear coupling feedback function method, consider a network containing n chaotic systems

$$\begin{cases}\dot{\mathbf{x}}_1(t) = \mathbf{F}(t, \mathbf{x}_1(t)), & \mathbf{x}_1(t_0) = \mathbf{x}_{1_0}\\ \dot{\mathbf{x}}_2(t) = \mathbf{F}(t, \mathbf{x}_2(t)), & \mathbf{x}_2(t_0) = \mathbf{x}_{2_0}\\ \quad\vdots & \\ \dot{\mathbf{x}}_n(t) = \mathbf{F}(t, \mathbf{x}_n(t)), & \mathbf{x}_n(t_0) = \mathbf{x}_{n_0}\end{cases} \tag{4}$$

Here, we have proposed methods for synchronizing n systems in (4).

3.1 First Method

In the first method, we decompose the right side of each n system in (4) to the linear, $\mathbf{A}\mathbf{x}(t)$, and nonlinear, $\mathbf{H}(\mathbf{x}(t))$ parts and then couple them with the last statement as in the following systems

$$\begin{cases}\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t)) + s_1[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))]\\ \dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) - \mathbf{H}(\mathbf{x}_2(t)) + s_2[\mathbf{H}(\mathbf{x}_2(t)) - \mathbf{H}(\mathbf{x}_1(t))]\\ \dot{\mathbf{x}}_3(t) = \mathbf{A}\mathbf{x}_3(t) - \mathbf{H}(\mathbf{x}_3(t)) + s_3[\mathbf{H}(\mathbf{x}_3(t)) - \mathbf{H}(\mathbf{x}_1(t))]\\ \quad\vdots\\ \dot{\mathbf{x}}_n(t) = \mathbf{A}\mathbf{x}_n(t) - \mathbf{H}(\mathbf{x}_n(t)) + s_n[\mathbf{H}(\mathbf{x}_n(t)) - \mathbf{H}(\mathbf{x}_1(t))]\end{cases} \tag{5}$$

We may illustrate this method as a weighed directed graph shown in Fig. 1.

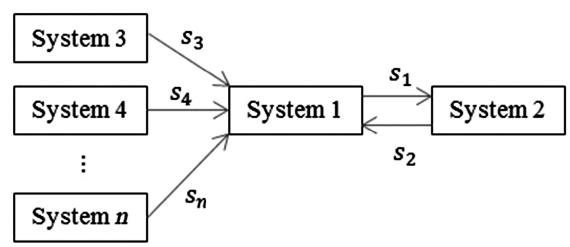

Figure 1. The graph for the first proposed method.

3.2 Second Method

In this method, similar to the first method, we start to decompose n systems in (4) to linear and nonlinear parts, however, couple them with different terms as in n following systems:

$$\begin{cases}\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t)) + s_1[+\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t)) - \cdots - \mathbf{H}(\mathbf{x}_n(t))]\\ \dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) - \mathbf{H}(\mathbf{x}_2(t)) + s_2[-\mathbf{H}(\mathbf{x}_1(t)) + \mathbf{H}(\mathbf{x}_2(t)) - \cdots - \mathbf{H}(\mathbf{x}_n(t))]\\ \quad\vdots\\ \dot{\mathbf{x}}_n(t) = \boldsymbol{A}\boldsymbol{x}_n(t) - \mathbf{H}(\mathbf{x}_n(t)) + s_n[-\mathbf{H}(\mathbf{x}_1(t)) - \cdots - \mathbf{H}(\mathbf{x}_{n-1}(t)) + \mathbf{H}(x_n(t))]\end{cases} \tag{6}$$

The scheme of this method is a weighted directed complete graph as in Fig. 2.

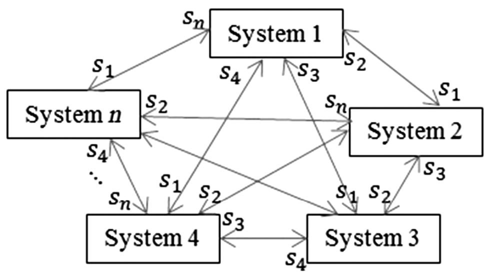

Figure 2. The graph for the second proposed method.

3.3 Third Method

In this method, again we decompose the n systems in (4) as before, but with different coupling terms as follows

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t)) + s_1[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))] \\ \dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) - \mathbf{H}(\mathbf{x}_2(t)) + s_2[\mathbf{H}(\mathbf{x}_2(t)) - \mathbf{H}(\mathbf{x}_3(t))] \\ \vdots \\ \dot{\mathbf{x}}_{n-1}(t) = \mathbf{A}\mathbf{x}_{n-1}(t) - \mathbf{H}(\mathbf{x}_{n-1}(t)) + s_{n-1}[\mathbf{H}(\mathbf{x}_{n-1}(t)) - \mathbf{H}(\mathbf{x}_n(t))] \\ \dot{\mathbf{x}}_n(t) = \mathbf{A}\mathbf{x}_n(t) - \mathbf{H}(\mathbf{x}_n(t)) \end{cases} \tag{7}$$

As is shown in Fig. 3, the scheme of this method is an "open ring directed graph".

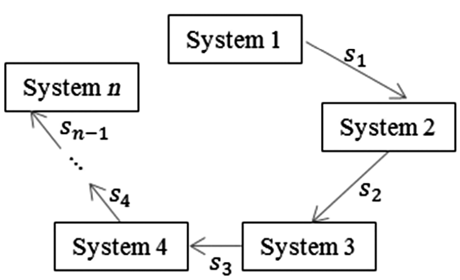

Figure 3. The graph for the third proposed method.

4. Stability Analysis

To analyse the stability of the proposed methods, we consider difference equations

$$\dot{\mathbf{e}}_i(t) = \dot{\mathbf{x}}_1(t) - \dot{\mathbf{x}}_{i+1}(t), \quad i = 1, 2, \ldots, n-1 \tag{8}$$

for the first and the second method, and difference equations

$$\dot{\mathbf{e}}_i(t) = \dot{\mathbf{x}}_i(t) - \dot{\mathbf{x}}_{i+1}(t), \quad i = 1, 2, \ldots, n-1 \tag{9}$$

to analyse the stability of the third method.

4.1 Stability of the First Method

Considering (8), from the first and the second system in (5), we obtain $\dot{\mathbf{e}}_1(t) = \dot{\mathbf{x}}_1(t) - \dot{\mathbf{x}}_2(t) = [\mathbf{A} + (s_1 + s_2 - 1)\partial\mathbf{H}/\partial\mathbf{x}]\mathbf{e}_1$. In particular for $s_1 = s_2 = s$, we obtain $\dot{\mathbf{e}}_1(t) = [\mathbf{A} + (2s-1)\partial\mathbf{H}/\partial\mathbf{x}]\mathbf{e}_1$. In the case of $s = 0.5$, we obtain $\dot{\mathbf{e}}_1(t) = \mathbf{A}\mathbf{e}_1$. Based on the linear stability theorem, if all eigenvalues of matrix $\mathbf{A}$ have negative real parts, then systems $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ synchronize. Similarly, for systems $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$, we can write

$$\begin{aligned} \dot{\mathbf{e}}_2(t) &= \dot{\mathbf{x}}_1(t) - \dot{\mathbf{x}}_3(t) \\ &= \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t)) + s_1[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))] \\ &\quad - \mathbf{A}\mathbf{x}_3(t) + \mathbf{H}(\mathbf{x}_3(t)) - s_3[\mathbf{H}(\mathbf{x}_3(t)) - \mathbf{H}(\mathbf{x}_1(t))] \\ &= \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_3) + (s_3 - 1)(\mathbf{H}(\mathbf{x}_1) - \mathbf{H}(\mathbf{x}_3)) \\ &\quad + s_1[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))] \\ &= \left[\mathbf{A} + (s_3 - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_2 + s_1 \frac{\partial \mathbf{H}}{\partial \mathbf{x}}\mathbf{e}_1 \end{aligned}$$

Now, if systems $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ synchronize then $\mathbf{e}_1 \to 0$, and so

$$\dot{\mathbf{e}}_2(t) = \left[\mathbf{A} + (s_3 - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_2$$

With the same argument as above, $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$ can be synchronized. Therefore, in general form we obtain

$$\dot{\mathbf{e}}_i(t) = \left[\mathbf{A} + (s_{i+1} - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_i \quad i = 2, 3, \ldots, n-1$$

That is for synchronization of $\mathbf{x}_1(t)$ and $\mathbf{x}_i(t)$ with $i = 3, 4, \ldots, n$, we need synchronization of the first and the second system in (4). Consequently, synchronization of all $n > 2$ systems in (4) is exactly similar to the synchronization of just two systems.

4.2 Stability of the Second Method

Let $\dot{\mathbf{e}}_i(t) = \dot{\mathbf{x}}_1(t) - \dot{\mathbf{x}}_{i+1}(t)$. Then

$$\begin{aligned} \dot{\mathbf{e}}_i(t) &= \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t)) + s_1[+\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t)) \\ &\quad - \cdots - \mathbf{H}(\mathbf{x}_n(t))] - \mathbf{A}\mathbf{x}_{i+1}(t) + \mathbf{H}(\mathbf{x}_{i+1}(t)) \\ &\quad - s_{i+1}[-\mathbf{H}(\mathbf{x}_1(t)) - \cdots - \mathbf{H}(\mathbf{x}_i(t)) + \mathbf{H}(\mathbf{x}_{i+1}(t)) \\ &\quad - \cdots - \mathbf{H}(\mathbf{x}_n(t))] = \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_{i+1})) \\ &\quad + (s_1 + s_{i+1} - 1)[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_{i+1}(t))] \\ &\quad + (s_{i+1} - s_1)\psi \end{aligned}$$

where $\psi = -\sum_{\substack{j=2 \\ j \neq i+1}}^{n} \mathbf{H}(\mathbf{x}_j(t))$.

By linear approximation, we obtain $\dot{\mathbf{e}}_i(t) = [\mathbf{A} + (s_1 + s_{i+1} - 1)\partial\mathbf{H}/\partial\mathbf{x}]\mathbf{e}_i + (s_{i+1} - s_1)\psi$. Now, if $s_1 = s_2 = \cdots = s_n = s$, then $\dot{\mathbf{e}}_i(t) = [\mathbf{A} + (2s-1)\partial\mathbf{H}/\partial\mathbf{x}]\mathbf{e}_i$. Therefore, as in case one, the stability analysis in the second method will be similar to the stability criterion of two-system synchronization.

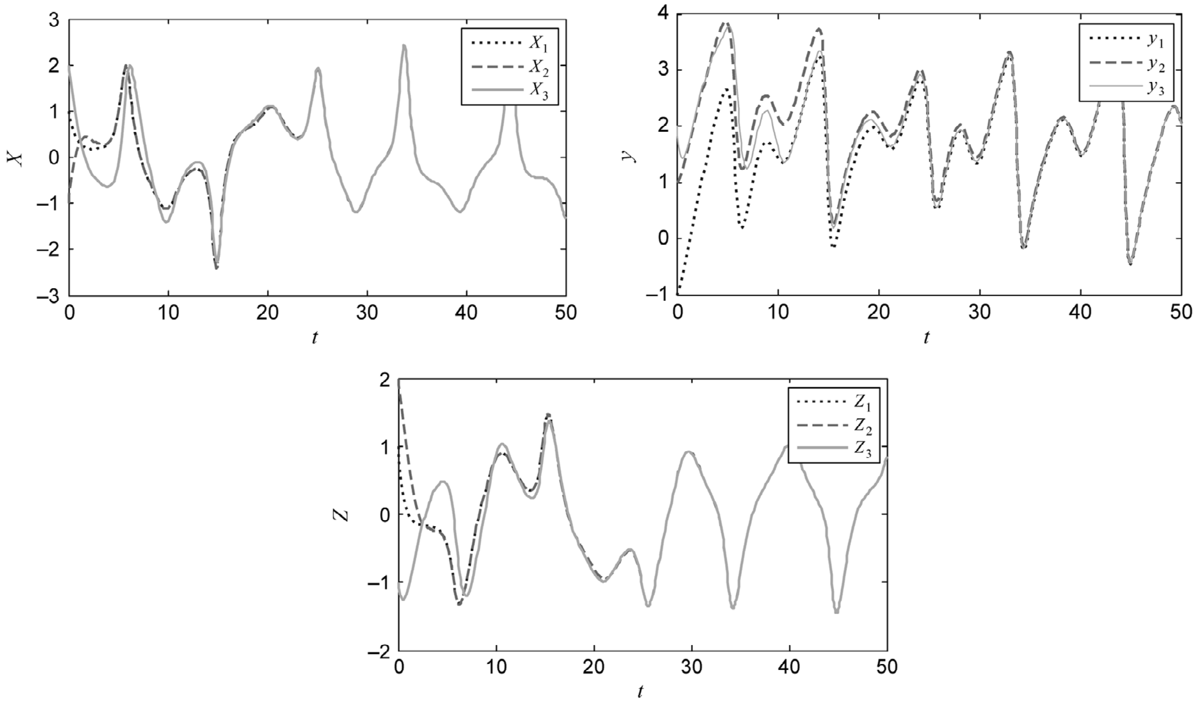

Figure 4. Synchronization of three financial systems, using proposed method 1.

4.3 Stability of the Third Method

For this method, we define

$$\begin{aligned}
\mathbf{e}_1(t) &= \mathbf{x}_1(t) - \mathbf{x}_2(t)\\
\mathbf{e}_2(t) &= \mathbf{x}_2(t) - \mathbf{x}_3(t)\\
&\vdots\\
\mathbf{e}_{n-1}(t) &= \mathbf{x}_{n-1}(t) - \mathbf{x}_n(t)
\end{aligned}$$

Then

$$\begin{aligned}
\dot{\mathbf{e}}_1(t) \;=\; & \dot{\mathbf{x}}_1(t) - \mathbf{x}_2(t) = \mathbf{A}\mathbf{x}_1(t) - \mathbf{H}(\mathbf{x}_1(t))\\
& + s_1[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))] - \mathbf{A}\mathbf{x}_2(t) + \mathbf{H}(\mathbf{x}_2(t))\\
& - s_2[\mathbf{H}(\mathbf{x}_2(t)) - \mathbf{H}(\mathbf{x}_3(t))] = \mathbf{A}\mathbf{e}_1\\
& +(s_1 - 1)[\mathbf{H}(\mathbf{x}_1(t)) - \mathbf{H}(\mathbf{x}_2(t))]\\
& -s_2[\mathbf{H}(\mathbf{x}_2(t)) - \mathbf{H}(\mathbf{x}_3(t))]
\end{aligned}$$

and by linear approximation, we have

$$\dot{\mathbf{e}}_1(t) = \left[\mathbf{A} + (s_1 - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_1 - s_2\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\mathbf{e}_2$$

In general case for $i = 1, 2, \ldots, n-2$, we have

$$\dot{\mathbf{e}}_i(t) = \left[\mathbf{A} + (s_i - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_i - s_{i+1}\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\mathbf{e}_{i+1} \text{ and for } i = n-1$$

$$\begin{aligned}
\dot{\mathbf{e}}_{n-1}(t) \;&=\; \dot{\mathbf{x}}_{n-1}(t) - \dot{\mathbf{x}}_n(t)\\
&= \mathbf{A}\mathbf{x}_{n-1}(t) - \mathbf{H}(\mathbf{x}_{n-1}(t))\\
&\quad + s_{n-1}[\mathbf{H}(\mathbf{x}_{n-1}(t)) - \mathbf{H}(\mathbf{x}_n(t))]\\
&\quad - \mathbf{A}\mathbf{x}_n(t) + \mathbf{H}(\mathbf{x}_n(t))\\
&= \left[\mathbf{A} + (s_{n-1} - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_{n-1}
\end{aligned}$$

Consequently, we can see that for $i = 1, 2, \ldots, n-2$, variations of $\mathbf{e}_i$ depends on $\mathbf{e}_{i+1}$. Based on the linear stability theorem, if all eigenvalues of matrix $\left[\mathbf{A} + (s_{n-1} - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]$ have negative real parts, then systems $\mathbf{x}_{n-1}(t)$ and $\mathbf{x}_n(t)$ will have synchrony and $\mathbf{e}_{n-1} \to 0$. Therefore, $\dot{\mathbf{e}}_{n-2}(t) = \left[\mathbf{A} + (s_{n-2} - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]\mathbf{e}_{n-2}$, and similarly, if all eigenvalues of matrix $\left[\mathbf{A} + (s_{n-2} - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]$ and in general all eigenvalues of matrix $\left[\mathbf{A} + (s_i - 1)\frac{\partial \mathbf{H}}{\partial \mathbf{x}}\right]$ for $i = 1, 2, \ldots, n-2$, have negative real parts, then systems $\mathbf{x}_i(t)$ and $\mathbf{x}_{i+1}(t)$ will have synchrony.

Remark. *In all of the proposed methods, stability type of the zero equilibrium in difference equations (8) and (9) reflects the stability type of synchronization between the systems. Obviously, synchronization of multiple systems using each proposed method is complete, if for each $i = 1, \ldots, n-1$, $\mathbf{e}_i \to 0$, and in the case of phase synchronization, $\mathbf{e}_i \to c_i$, where c_i is a constant real number.*

5. Numerical Results

As an example, consider a nonlinear financial model which is presented as follows [17], [18]:

$$\begin{cases}
\dot{x} = z + (y - a)x\\
\dot{y} = 1 - by - x^2\\
\dot{z} = -x - cz
\end{cases}$$

where x, y and z denote the interest rate, investment demand and price index, respectively. The positive constants a, b and c are the saving amount, cost per investment and demand elasticity of commercial markets, respectively.

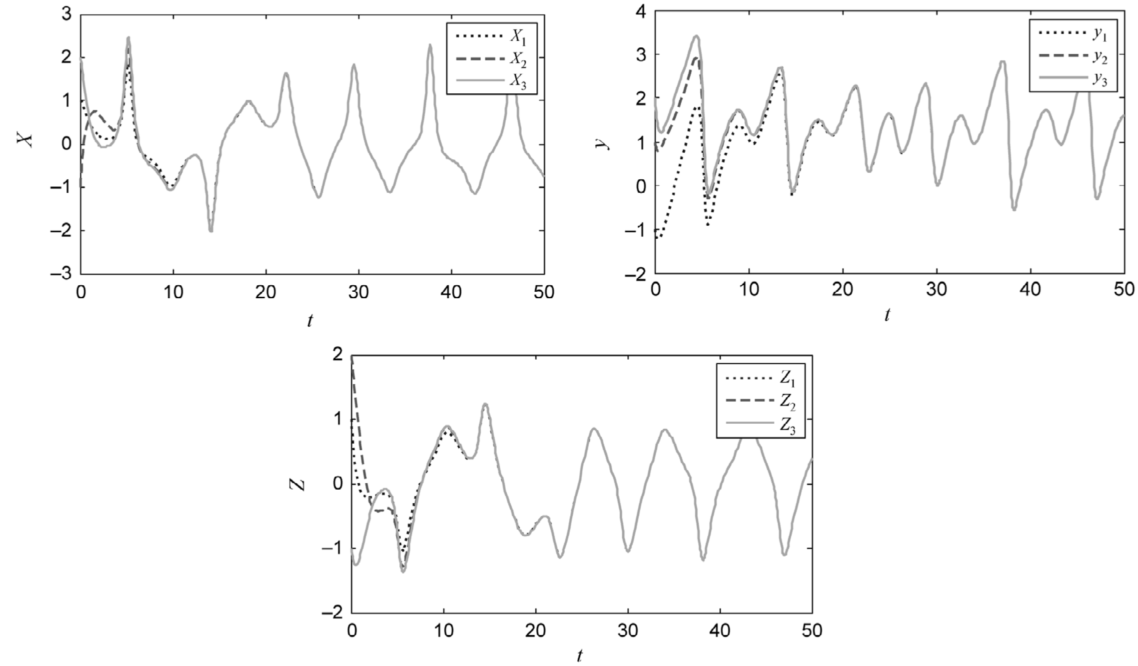

Figure 5. Synchronization of three financial systems, using proposed method 2.

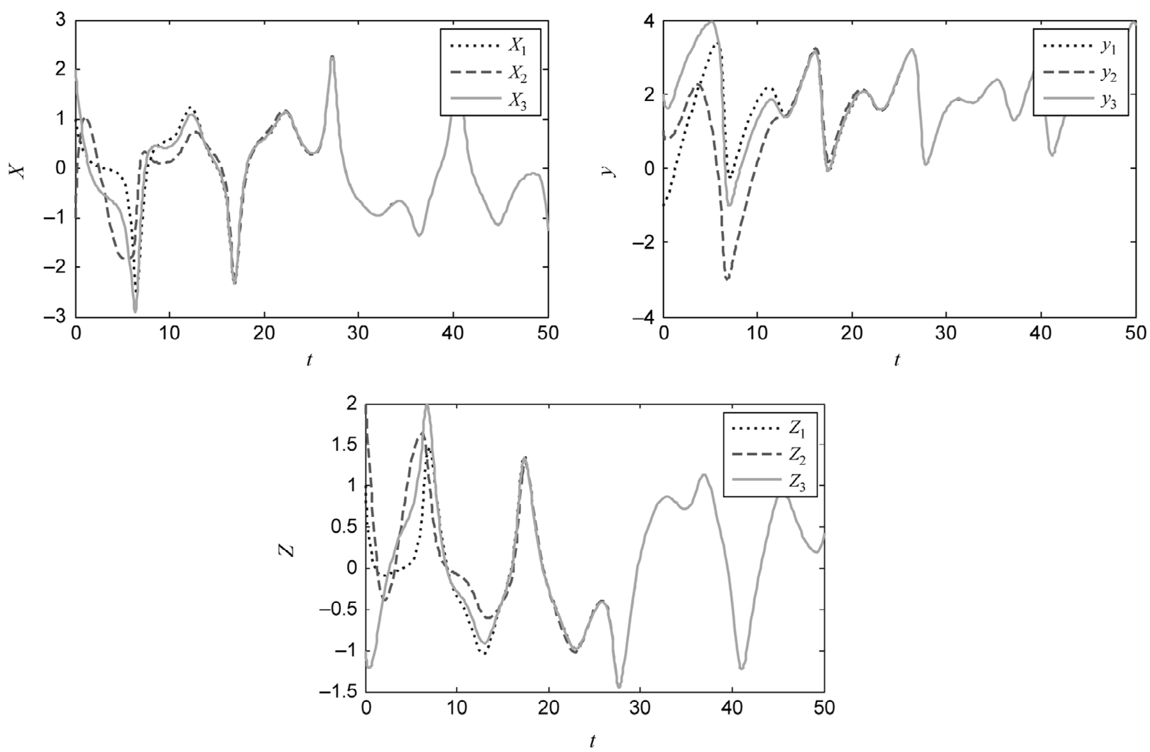

Figure 6. Synchronization of three financial systems, using proposed method 3.

Now, consider three chaotic financial systems

$$
\begin{cases}
\dot{x}_1 = z_1 + (y_1 - a)x_1 \\
\dot{y}_1 = 1 - by_1 - x_1^2 \\
\dot{z}_1 = -x_1 - cz_1
\end{cases},
\quad
\begin{cases}
\dot{x}_2 = z_2 + (y_2 - a)x_2 \\
\dot{y}_2 = 1 - by_2 - x_2^2 \\
\dot{z}_2 = -x_2 - cz_2
\end{cases},
$$

$$
\begin{cases}
\dot{x}_3 = z_3 + (y_3 - a)x_3 \\
\dot{y}_3 = 1 - by_3 - x_3^2 \\
\dot{z}_3 = -x_3 - cz_3
\end{cases}
\tag{10}
$$

with different initial points [1;−1;1], [−1;1;2], [2;2;−1], respectively, and $a=1$, $b=0.1$, $c=1$. Then, synchronization of the states of three systems, using proposed Methods 1, 2 and 3 are shown in Figs. 4, 5 and 6, respectively.

6. Conclusion

In this article, we expanded the Ali and Fang method for synchronization of $n > 2$ systems. Three methods are proposed and numerical results are illustrated. We also studied the stability criterion of these proposed methods. The simulation results verify the accuracy and significance of the mentioned methods.

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