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On a goodness of fit test for normality based on Lin-Wong divergence with type-I censored data

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Abstract

Normality testing in goodness of fit test field has long been an interesting issue in statistical inferences. In this paper, we use the Lin-Wong divergence as our proposed measure of distance between distributions and compare its performance with that of other well known distance measures using the empirical distribution function and Shannon entropy concept such as Kullback-Leibler, Cramer von Misses, Anderson-Darling and Kolmogorov-Smirnov. Finally, the use of the proposed test is shown in a illustrative example.

Keywords: Anderson-Darling, Goodness of fit test, Lin Wong divergence, Normal distribution, Type I censored scheme.

1 Introduction

A divergence measure shows that how one probability distribution diverges from an alternative probability distribution. In other words, it is an instrument for measuring the similarity between two probability distribution patterns. Thus the divergence family unifies commonly used goodness of fit (GoF) test.
The Lin-Wong (LW) divergence measure which was introduced by Lin and Wong [7] yields

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a quasi-distance value related to well known Kullback-Leibler (KL) divergence measure which was introduced by Kullback and Leibler [6] and named skew divergence [3]. The LW measure is defined as

$$D_{\mathrm{LW}}(f\|f_0) = \int_{-\infty}^{\infty} f(x)\, \ln \frac{2f(x)}{f(x)+f_0(x)}\, \mathrm{d}x,$$

in which $f(x)$ and $f_0(x)$ represent the probability density function (PDF) for the data in practical and estimated cases, respectively.
The PDF for Type-I censored data represented by a random variable $\min(X, C)$ with C as the censor point (a fixed value) would be as the following:

$$f_C(x) = \begin{cases} f(x) & \text{if } \ x < C \\ \bar{F}(C) & \text{if } \ x \geq C, \end{cases} \tag{1}$$

where, $\bar{F}(x) = 1 - F(x)$, represent survival functions. Here, we construct a GoF test based on LW for normality which can be observed in Type-I censored scheme. This can be often seen in cases which there is not a large enough time window to make observations over all the events while we are not totally inattentive toward events which happen after the censor point. An example can be when we conduct monitoring or follow-up studies on chronic or epidemiological illnesses such as long term cancers or addiction to narcotics.
The LW divergence measure on Type-I censored data is defined as

$$D_{\mathrm{LW}}^{C1}(f\|f_0) = \int_{-\infty}^{C} f(x)\, \ln \frac{2f(x)}{f(x)+f_0(x)}\, \mathrm{d}x + \bar{F}(C) \ln \frac{2\bar{F}(C)}{\bar{F}(C)+\bar{F}_0(C)}. \tag{2}$$

Equivalently,

$$D_{\mathrm{LW}}^{C1}(f\|f_0) = -H^{C1}(f) - \int_{-\infty}^{C} f(x)\, \ln \frac{f(x)+f_0(x)}{2}\, \mathrm{d}x - \bar{F}(C) \ln \frac{\bar{F}(C)+\bar{F}_0(C)}{2}, \tag{3}$$

where $H^{C1}(f) = -\int_{-\infty}^{C} f(x)\, \ln f(x)\, \mathrm{d}x - \bar{F}(C) \ln \bar{F}(C)$, is called as Shannon entropy on Type-I censored data.
The GoF hypothesis test is expressed here as the following: $H_0 : f(x;\theta) = f_0(x;\theta)$, while the alternative hypothesis can be written as $H_1 : f(x;\theta) \neq f_0(x;\theta)$.
GoF tests are categorized in different ways. The statistical instruments based on empirical distribution function (EDF) and the methods based on Shannon entropy are two well known examples of various types of those tests. The most common measures used usually along with EDF method in tests are: Anderson-Darling (AD), Cramer von Misses (CM) and Kolmogorov-Smirnov (KS). Pettitt and Stephens [9] examined the AD, CM and KS measures when applied to censored data.
Assuming $u_{(i)} = F_n(x_i;\hat{\theta})$, and $u_{(n_0+1)} = F_n(C;\hat{\theta})$, the revised forms of AD and CM statistics on Type-I censored data are as the following:

$$\begin{aligned} AD = &-\frac{1}{n}\sum_{i=1}^{n_0+1}(2i-1)\{\ln u_{(i)} - \ln(1-u_{(i)})\} - 2\sum_{i=1}^{n_0+1}\ln(1-u_{(i)}) \\ &-\frac{1}{n}\{(n_0-n+1)^2\ln(1-u_{(n_0+1)}) - (n_0+1)^2\ln u_{(n_0+1)} + n^2, u_{(n_0+1)}\}. \end{aligned} \tag{4}$$

$$CM = \sum_{i=1}^{n_0+1} \big(u_{(i)} - \frac{2i-1}{2n}\big)^2 + \frac{n_0+1}{12n^2} + \frac{n}{3}\big(u_{(n_0+1)} - \frac{n_0+1}{n}\big)^3. \tag{5}$$

Where n_0 is the number of data value less than the fixed value censor point C within the relevant sample.
Also, the revised form of KS statistic applied on Type-I censored data is defined as the following according to [5]:

$$KS = max_{1\leq i\leq n_0}\{\frac{i}{n} - u_{(i)}, u_{(i)} - \frac{i-1}{n}, u_{(n_0+1)} - \frac{n_0}{n}\}, \tag{6}$$

Recently many GoF tests have been reported based on normal probability distribution. For instance, Vasicek in (1976), Arizonoand and Ohta (1989), Esteban et al. (2001), Balakrishnan et al. (2007), and Alizadeh (2010) Zamanzade and Arghami (2012) , Bitaraf et al (2017) as well Evren and Ustaoglu (2017) have tested for normal distribution using some distance measures based on Shannon entropy. Furthermore, procedures of GoF tests for exponential and normal probability distributions can be found which are conducted using censored data. Some cases are reported by Pettitt (1976), Park (2005), Balakrishnan et al. (2007), Lim and Park (2007), Habibi Rad et al. (2011), and Park and Shin (2014) as well as Baratpour and Habibi Rad (2016). However, relatively fewer studies have examined the Goodness of Fit tests when applied to Type-I censored data. Park (2013) and Pakyari (2015) can be mentioned as two examples.
This paper is organized as the following: In Section 2 we introduced LW divergence estimators based on Vasicek [10] and Bitaraf et al.[2]. Then, we express the convergence and stability characteristics of those proposed estimators as the new probe of GoF tests on Type-I censored data in order to normal distribution. The critical values and distinction power of AD, CM, KS, KL and two proposed estimators for LW distance measure are presented in section 3. In addition, the results obtained using Monte Carlo (MC) simulation methods along with those mentioned tests are analyzed over two sets of real data samples. Finally, some concluding remarks are presented in section 4.

2 Test statistics

We use the LW divergence measure given in (3) to facilitate the decision-making process when choosing the hypothesis for a GoF test. Obviously, the numerical value obtained for LW measure should be non-negative and it is clearly equal to zero if and only if $f(x;\theta) = f_0(x;\theta)$ a.e.
Considering $(X_{(i-m)}, X_{(i+m)})$ as an interval with a continuous non-negative PDF $f(x;\theta)$, we would have $x_i \in (x_{(i-m)}, x_{(i+m)})$ in which

$$f(x_i) = \frac{F(x_{(i+m)}) - F(x_{(i-m)})}{x_{(i+m)} - x_{(i-m)}}.$$

Since the EDF converges to the CDF, we can write:

$$\frac{2m/n_0}{x_{(i+m)} - x_{(i-m)}} \xrightarrow{pr.} f(x_i) \quad as\ n_0 \to \infty, \quad m \to \infty, \quad \frac{m}{n_0} \to \infty, \tag{7}$$

where the window size m is a positive integer smaller than $\frac{n_0}{2}$.
Hence, if we assume $\hat{\theta}$ to be a convergent estimator for the parameter θ, and assume that $n_0 \to \infty,\ m \to \infty$, and $\frac{m}{n_0} \to \infty$ so it can be inferred that:

$$\frac{1}{n}\sum_{i=1}^{n_0} \ln(1 + \frac{f_0(x;\hat{\theta})(x_{(i+m)} - x_{(i-m)})}{2m/n_0}) \overset{pr.}{\to} E_f(\ln(1 + \frac{f_0(X;\theta)}{f(X;\theta)})). \tag{8}$$

Characteristics of (7) have been studied by [10] in an attempt to find an estimator for entropy. On the other hand, using the definition T_i provided by [2], we can get $T_i \overset{pr.}{\to} f(x_i)$, in which $T_i = \frac{T_{i_1} + T_{i_2}}{2}$, and $T_{i_j} = \frac{n_0}{W_j(m-j)}(X_{(i+m-j)} - X_{(i+m-j)}), \quad j = 1, 2$, so that W_j satisfies the following relation:

$$W_j = \begin{cases} 1, & 1 \le i \le m - j, \\ 2, & m - j + 1 \le i \le n_0 - m + j, \\ 1, & n_0 - m + j + 1 \le i \le n_0. \end{cases}$$

So, as $n_0 \to \infty, \quad m \to \infty, \quad \frac{m}{n_0} \to 0$, we would have

$$\frac{1}{n}\sum_{i=1}^{n_0} \ln(1 + f_0(x;\hat{\theta})T_i) \overset{pr.}{\to} E_f(\ln(1 + \frac{f_0(X;\theta)}{f(X;\theta)})). \tag{9}$$

Letting $n_0 \ge 4$, and $X_{(1)} \le X_{(2)} \le ... \le X_{(n_0)}$ as the order statistics satisfies the condition $X_{(i)} = X_{(1)}$, if $i < 1$, and $X_{(i)} = X_{(n_0)}$, if $i > n_0$. Then we can use the Vasicek estimator for the entropy and formulate an estimator based on Shannon entropy for Type-I censored data as the following [10]:

$$H_V^{C1} = \frac{1}{n}\sum_{i=1}^{n_0} \ln \frac{n_0}{2m}(X_{(i+m)} - X_{(i-m)}) - \frac{n - n_0}{n} \ln \frac{n - n_0}{n}. \tag{10}$$

Besides, by taking $\frac{i-1}{n_0} < p \le \frac{i}{n_0}$, $i = m + 1, m + 2, ..., n_0 - m$, the estimator proposed by Bitaraf et al. for Shannon entropy is defined as the following [2]:

$$H_B^{C1} = \frac{1}{n}\sum_{i=1}^{n_0} \ln T_i - \frac{n - n_0}{n} \ln \frac{n - n_0}{n}. \tag{11}$$

Therefore, the Vasicek estimator for LW divergence measure when applied on Type-I censored data would be given as:

$$LW_V^{C1} = -\frac{1}{n}\sum_{i=1}^{n_0} \ln(\frac{1}{2} + \frac{n(X_{(i+m)} - X_{(i-m)})f_0(x;\hat{\theta})}{4m}) - \frac{n - n_0}{n} \ln(\frac{n - n_0 + n\bar{F}_0(C;\hat{\theta})}{2n(n - n_0)}), \tag{12}$$

where $\bar{F}_0(C;\hat{\theta})$ is the estimated survival function.
The revised form of the KL divergence measure on Type-I censored data is given as:

$$KL_V^{C1} = -\frac{1}{n}\sum_{i=1}^{n_0} \ln(\frac{n(X_{(i+m)} - X_{(i-m)})f_0(x;\hat{\theta})}{2m}) - \frac{n - n_0}{n} \ln(\frac{n\bar{F}_0(C;\hat{\theta})}{n - n_0}). \tag{13}$$

Also, we can introduce another estimator for LW from (11) as:

$$LW_B^{C1} = -\frac{1}{n}\sum_{i=1}^{n_0}\ln(\frac{1}{2}+\frac{f_0(x;\hat{\theta})T_i}{2}) - \frac{n-n_0}{n}\ln(\frac{n-n_0+n\bar{F}_0(C;\hat{\theta})}{2n(n-n_0)}). \tag{14}$$

Here, we will show that above mentioned tests produce consistent results.
Considering (7) and the definition of Shannon entropy, we would have,

$$\frac{1}{n}\sum_{i=1}^{n_0}\ln\frac{n_0}{2m}(X_{i+m}-X_{i-m}) \overset{pr.}{\to} -\int_0^C f(x)\ln f(x)\,dx.$$

Furthermore, if $n_0 \to \infty$, then $\frac{n-n_0}{n}\ln\frac{n-n_0}{n} \overset{pr.}{\to} \bar{F}(C)\ln\bar{F}(C)$. So we can write:

$$H_V^{C1} \overset{pr.}{\to} H_f^{C1}(X) \quad as \quad n_0 \to \infty\,, m \to \infty\,, \frac{m}{n_0} \to 0.$$

Also, from the relation (8), the actual observed distribution function (EDF) converges to CDF:

$$LW_V^{C1} \overset{pr.}{\to} D_{LW}^{C1}(f||f_0) \quad as \quad n_0 \to \infty\,, m \to \infty\,, \frac{m}{n_0} \to 0.$$

Similarly, given the convergence of the expression T_i which is proved by Bitaraf et al. (2017) we can write [2]:

$$LW_B^{C1} \overset{pr.}{\to} D_{LW}^{C1}(f||f_0) \quad as \quad n_0 \to \infty\,, m \to \infty\,, \frac{m}{n_0} \to 0.$$

3 Simulation study

In this section, we present the GoF tests and critical values within results; meanwhile we study the power of those tests to recognize normal distribution through test hypothesis. We assess the performance of the proposed tests using simulated Type-I censored data in sample sizes of 10 and 20, as well as the significance levels of $\alpha = 0.10$ and 0.05. The results of analysis are provided here. We also studied the characteristics of divergence measures within the structure of Shannon entropy. For example LW divergence measure based on Vasicek and Bitaraf estimators besides, KL based on Vasicek estimator in comparison to AD, CM and KS. Since lifetime data distributions are within the alternatives, we would be able to use non-negative data within the sample set and get to a normal standard distribution. Therefore, the censor point C can be chosen as equivalent to quantiles rather than the median point. In this case, the censor point C refers to the quantiles 0.7 and 0.8. We determine the critical points using MC simulation with 10000 replicates. All simulations were carried out in R software package. We use the formula $m = [\sqrt{n}+0.5]$, in order to chose the value of m. This formula was used by Wieczorkowski and Grzegorzewski in 1999.
To facilitate the comparison between distinction powers of the tests, we selected various alternatives such as Beta (B), Standard Cauchy (SC), Gamma (G), Gumbel (Gum), Laplace (Lap), Log-Normal (LN), Logistic (Log), T student (T) and Uniform (U) distributions which can be classified into four groups shown in Table 1. Those mentioned data

Table 1: Alternative distributions used in each power study

Non-negative random variable	
Group I: $X \in (0,1)$	Group II: $X \in (0,\infty)$
U(0,1)	G(2)
B(2,2)	G(0.5)
B(2,0.5)	LN(0,0.5)
B(0.5, 0.5)	LN(0,1)
Real-valued random variable	
Group III: Symmetric distributions	Group IV: Asymmetric distributions
SC	Gum(2)
T(3)	Gum(0.5)
Log(0,1)	Log(0.5)
Lap(0,1)	Log(2)

distributions have been used by [4], [1] and [2].
From Tables 2 to 5, LW_B^{C1} statistic is the preferred one when compared to other examined statistics. It seems to have better performance in groups I and II than groups III and IV. When we compared the results in groups III and IV, the LW_B^{C1} statistic has a much superior power for group IV respect to group III. Our results show that it is better to use AD statistic for group III; while the statistics such as KL_V^{C1}, LW_V^{C1} and LW_B^{C1} which are based on Shannon entropy and covered by the present paper should be preferred with group IV. For group II of distributions, LW_B^{C1} statistic seems to outperform the other statistics. As we expected before, the results of tables 4 and 5 are almost the same as tables 2 and 3. Also by comparing tables 2 and 3 to tables 4 and 5 it is clear that the same results are obtained. It can be observed that our proposed test is preferred at $\alpha = 0.10$, over the case of $\alpha = 0.05$. Besides obviously, when n number increases LW_B^{C1} test has more power. Also for group I, LW_B^{C1} statistic performs quite better than other statistics.

Table 2: Power comparisons of $\alpha = 0.05$ and $F(C) = 0.7$ for normal distribution versus alternatives in Table 1

	$n = 10$						$n = 20$					
	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD
	Group I											
U(0,1)	**0.1412**	0.0947	0.1119	0.0434	0.0314	0.0321	**0.2323**	0.1528	0.1602	0.0603	0.0485	0.0668
B(2,2)	**0.0909**	0.0554	0.0666	0.0245	0.0173	0.0164	**0.1230**	0.0717	0.0629	0.0257	0.0174	0.0209
B(2,0.5)	**0.0697**	0.0193	0.0416	0.0006	0.0000	0.0000	**0.1314**	0.0015	0.0154	0.0000	0.0000	0.0000
B(0.5,0.5)	0.2732	0.2160	**0.2822**	0.1113	0.0974	0.1092	**0.5618**	0.4178	0.5245	0.2107	0.1919	0.2823
	Group II											
G(2)	**0.0870**	0.0260	0.0549	0.0015	0.0000	0.0000	**0.1802**	0.0006	0.0241	0.0000	0.0000	0.0000
G(0.5)	0.3144	0.2864	**0.3351**	0.1528	0.1454	0.1551	**0.6175**	0.5441	0.6042	0.3087	0.3170	0.4030
LN(0,0.5)	**0.0742**	0.0187	0.0419	0.0009	0.0000	0.0000	**0.1429**	0.0009	0.0192	0.0000	0.0000	0.0001
LN(0,1)	**0.1434**	0.0494	0.0870	0.0055	0.0017	0.0017	**0.2405**	0.0350	0.0601	0.0046	0.0013	0.0028
	Group III											
SC	0.0328	0.0261	0.1726	**0.3015**	0.2715	0.2876	0.0290	0.0090	0.4377	0.5643	0.565	**0.5997**
t(3)	0.0473	0.0400	0.0782	0.1323	0.1224	**0.1346**	0.0236	0.0142	0.1463	0.2474	0.2640	**0.2936**
Log(0,1)	0.0612	0.0568	0.0714	0.0950	0.0944	**0.1061**	0.0298	0.019	0.0846	0.1488	0.168	**0.1864**
Lap(0,1)	**0.0593**	0.0419	0.0511	0.0458	0.0368	0.0399	0.041	0.0253	0.0335	0.0464	0.0453	**0.0511**
	Group IV											
Gum(2)	**0.1006**	0.0429	0.0579	0.0170	0.0092	0.0094	**0.1248**	0.0522	0.0497	0.0169	0.0061	0.0088
Gum(0.5)	0.1250	**0.1348**	0.1273	0.0696	0.0653	0.0651	0.1276	**0.1501**	0.1305	0.0835	0.0903	0.0867
Log(0.5)	**0.6253**	0.5839	0.6206	0.6029	0.5901	0.5960	0.6568	0.6383	**0.6473**	0.6401	0.6304	0.6321
Log(2)	0.0857	0.0943	**0.0946**	0.0668	0.0733	0.0803	0.0355	0.0370	0.0538	0.0825	**0.0984**	**0.0984**

Table 3: Power comparisons of $\alpha = 0.05$ and $F(C) = 0.8$ for normal distribution versus alternatives in Table 1

	$n = 10$						$n = 20$					
	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD
						Group I						
U(0,1)	0.2251	0.2348	**0.2381**	0.0721	0.0855	0.0898	0.2054	0.1690	**0.2350**	0.1109	0.1505	0.1765
B(2,2)	0.5135	**0.5228**	0.5213	0.0719	0.0919	0.1078	0.3132	0.2924	**0.3234**	0.0835	0.1219	0.147
B(2,0.5)	**0.0558**	0.0123	0.022	0.0062	0.0015	0.0015	**0.0647**	0.0096	0.0113	0.007	0.0015	0.002
B(0.5,0.5)	0.2002	0.1953	**0.2348**	0.1239	0.1664	0.1708	0.4898	0.3955	**0.5432**	0.2575	0.3136	0.3868
						Group II						
G(2)	**0.1160**	0.0184	0.0451	0.0002	0.0001	0.0001	**0.2381**	0.0072	0.0283	0.0007	0.0001	0.0001
G(0.5)	0.3926	0.3830	**0.4249**	0.2269	0.2501	0.2497	0.7530	0.7265	**0.7797**	0.4429	0.5255	0.5876
LN(0,0.5)	**0.1250**	0.0325	0.0576	0.0048	0.0014	0.0014	**0.1742**	0.0354	0.0368	0.0052	0.0006	0.0011
LN(0,1)	**0.2206**	0.0838	0.1189	0.0189	0.0088	0.0074	**0.3258**	0.1424	0.1330	0.0324	0.0157	0.0215
						Group III						
SC	0.0671	0.0507	0.2118	**0.2974**	0.2789	0.2874	0.0604	0.0237	0.4809	0.5826	0.5865	**0.6116**
t(3)	0.0936	0.0836	0.1289	0.1321	0.1328	**0.1390**	0.0559	0.046	0.057	0.0679	0.0621	**0.0721**
Logc(0,1)	0.1339	0.1275	**0.1424**	0.0846	0.0916	0.0987	0.0464	0.0388	0.0890	0.1494	0.1674	**0.1833**
Lap(0,1)	**0.0798**	0.0618	0.0663	0.0378	0.0355	0.0357	0.0436	0.0299	0.0340	**0.0482**	0.0407	0.0462
						Group IV						
Gum(2)	**0.1159**	0.0385	0.0542	0.0121	0.0062	0.0057	**0.1497**	0.0660	0.0490	0.0176	0.0071	0.0074
Gum(0.5)	0.2592	**0.2773**	0.2705	0.0745	0.0808	0.0782	0.1781	**0.2167**	0.1826	0.1140	0.1332	0.1294
Log(0.5)	**0.6319**	0.5843	0.6182	0.5940	0.5813	0.5847	**0.6292**	0.6135	0.6158	0.6123	0.6086	0.6096
Log(2)	0.2248	**0.2319**	0.2260	0.0670	0.0659	0.0701	0.08	0.0796	0.0899	0.0754	0.0910	**0.1001**

Table 4: Power comparisons of $\alpha = 0.10$ and $F(C) = 0.7$ for normal distribution versus alternatives in Table 1

	$n = 10$						$n = 20$					
	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD
						Group I						
U(0,1)	**0.2432**	0.1772	0.1893	0.0897	0.0667	0.0695	**0.3903**	0.2643	0.2607	0.1200	0.0963	0.1262
B(2,2)	**0.1737**	0.1111	0.1169	0.0599	0.0457	0.0442	**0.2211**	0.1372	0.1138	0.0638	0.0406	0.0469
B(2,0.5)	**0.1337**	0.0572	0.0724	0.0840	0.0001	0.0011	**0.2528**	0.0090	0.0338	0.0002	0.0000	0.0000
B(0.5,0.5)	**0.4099**	0.3309	0.3834	0.1800	0.148	0.1780	**0.7174**	0.5563	0.6302	0.3056	0.2947	0.3920
						Group II						
G(2)	**0.1771**	0.0752	0.0986	0.0110	0.0000	0.0010	**0.3098**	0.0136	0.0458	0.0002	0.0000	0.0000
G(0.5)	**0.4424**	0.4114	0.4422	0.2460	0.2312	0.2540	**0.7623**	0.6792	0.7162	0.4220	0.4291	0.5209
LN(0,0.5)	**0.1415**	0.0586	0.0766	0.0096	0.0000	0.0013	**0.2557**	0.0098	0.0342	0.0000	0.0000	0.0000
LN(0,1)	**0.2368**	0.1152	0.1431	0.232	0.0360	0.0054	**0.3987**	0.0754	0.0993	0.0710	0.0160	0.0041
						Group III						
SC	0.0652	0.0417	0.2387	**0.3593**	0.3427	0.3573	0.0678	0.0169	0.4924	0.6257	0.6241	**0.6515**
t(3)	0.0807	0.0673	0.1297	0.1897	0.1856	**0.1970**	0.0484	0.0291	0.2058	0.3312	0.3332	**0.3646**
Log(0,1)	0.0967	0.0870	0.1223	0.1537	0.1615	**0.1684**	0.0629	0.0420	0.1397	0.2320	0.2494	**0.2682**
Lap(0,1)	**0.1009**	0.0743	0.0877	0.0856	0.0772	0.0813	0.0885	0.0576	0.0694	**0.0929**	0.0818	0.0921
						Group IV						
Gum(2)	**0.1752**	0.0874	0.1014	0.0371	0.0202	0.0209	**0.2210**	0.1003	0.0818	0.0347	0.0181	0.0219
Gum(0.5)	0.1965	**0.2196**	0.2088	0.1249	0.1294	0.1239	0.1997	**0.2473**	0.2026	0.1559	0.1672	0.1560
Log(0.5)	**0.6566**	0.6204	0.6419	0.6255	0.6088	0.6164	**0.6592**	0.6284	0.6373	0.6288	0.6149	0.6166
Log(2)	0.1222	0.1443	**0.1456**	0.1205	0.1327	0.1364	0.0753	0.0870	0.1072	0.1289	0.1477	**0.1530**

3.1 Illustrative real examples

A real data set is analyzed in this section. We present the proposed methods in application in order to analyze the results. Example 3.1 represents the times of breakdown of insulation fluid samples (in minutes) tested at 34 kVs ([8], Table 1.1, page 105).

Example 3.1. *Suppose that a decision had been made to terminate the life-testing after* 12.06 *minute. So* 5 *out of* 19 *observations would have been censored while* 14 *data points remained for study (see Table 6). Note that* 12.06*, datum is equivalent to the quantile*

Table 5: Power comparisons of $\alpha = 0.10$ and $F(C) = 0.8$ for normal distribution versus alternatives in Table 1

	$n = 10$						$n = 20$					
	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD
						Group I						
U(0,1)	0.2821	0.3055	**0.3133**	0.1454	0.1698	0.1808	0.3224	0.2763	**0.3393**	0.2089	0.2545	0.2905
B(2,2)	0.5501	0.5674	**0.5684**	0.1448	0.1769	0.2070	0.3753	0.3441	**0.3836**	0.1580	0.2111	0.2492
B(2,0.5)	**0.1125**	0.0304	0.0392	0.0176	0.0072	0.0077	**0.1112**	0.0188	0.0186	0.0156	0.0045	0.0064
B(0.5,0.5)	0.3235	0.3040	**0.3506**	0.2326	0.2502	0.2698	**0.6474**	0.5593	0.6548	0.3810	0.4381	0.5155
						Group II						
G(2)	**0.2244**	0.0592	0.0853	0.0060	0.0002	0.0002	**0.3631**	0.0260	0.0512	0.0013	0.0000	0.0003
G(0.5)	0.5428	0.5295	**0.5645**	0.3350	0.3593	0.3691	0.8597	0.8443	**0.8600**	0.5915	0.6593	0.7238
LN(0,0.5)	**0.2241**	0.0712	0.0963	0.0138	0.0029	0.0026	**0.3001**	0.0756	0.0632	0.0140	0.0035	0.0050
LN(0,1)	**0.3497**	0.1563	0.1868	0.0490	0.0223	0.0205	**0.4718**	0.2310	0.2051	0.0729	0.0298	0.0440
						Group III						
SC	0.1094	0.0743	0.2732	**0.3753**	0.3525	0.3605	0.1078	0.0393	0.5219	0.6444	0.6344	**0.6586**
t(3)	0.1342	0.1122	0.1728	0.2126	0.2065	**0.2145**	0.0642	0.0459	0.2279	0.3328	0.3368	**0.3614**
Log(0,1)	0.1690	0.1563	**0.1775**	0.1541	0.1571	0.1631	0.0788	0.0685	0.1470	0.2292	0.2400	**0.2554**
Lap(0,1)	**0.1348**	0.0932	0.0995	0.0773	0.0667	0.0692	0.0874	0.0611	0.0729	**0.096**	0.0773	0.0830
						Group IV						
Gum(2)	**0.2048**	0.0847	0.0946	0.0363	0.0207	0.0190	**0.2387**	0.1143	0.0882	0.0462	0.0170	0.0194
Gum(0.5)	0.3237	**0.3568**	0.3381	0.1537	0.1663	0.1650	0.2759	**0.3298**	0.2834	0.2114	0.2327	0.2341
Log(0.5)	**0.6686**	0.6156	0.6424	0.6199	0.5993	0.6026	**0.6619**	0.6325	0.6330	0.6299	0.6194	0.6206
Log(2)	0.2611	0.2756	**0.2817**	0.1316	0.1367	0.1422	0.1126	0.1202	0.1291	0.1449	0.1589	**0.1678**

0.7 *in the sample set and censored observations are marked by an asterisk. In this case,* $n = 19$, $n_0 = 14$ *and* $C = 12.06$ *(equivalently the* $F(12.06) = 0.7$*). We are interested to assess whether the times to breakdown follow a normal distribution. Table 7 shows the values of the test statistics and the corresponding p-values for the proposed tests on Type-I censored data.*

Table 6: Times to breakdown in minutes of an insulating fluid at 34 kV voltage level in Example 3.1.

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.5
7.35	8.01	8.27	12.06	~~31.75~~	~~32.52~~	~~33.91~~	~~36.71~~	~~72.89~~	

Table 7: Test statistics values and the p-values for normal distribution in Example 3.1.

The null hypothesis of normality						
Test	LW_B^{C1}	LW_V^{C1}	KL_V^{C1}	KS	CM	AD
Value (Example 1)	0.0353	0.1112	0.3344	0.2820	1.0655	5.667
P-value	0.0048	0.0006	0.0018	0.0428	0.0047	0.0063

As it can be seen in Table 7, the p-values obtained based on above mentioned tests are in support of the null hypothesis of normal distribution. The values of the statistics and corresponding p-values in Table 7 are quite large, strongly reject the alternative hypothesis. The LW_V^{C1} is stronger than LW_B^{C1} through the corresponding p-values.

4 Conclusions

In this paper, we have developed some GoF test for the normal distribution based on Type-I censored samples. The LW divergence measure which is proposed as the preferred

information measure in this study can be evaluated using the comparison of various distance measures such as AD, CM, KS and KL.
We have selected the alternatives from four different families based on random variables. The test statistics representing the capability of the tests for measuring the distance between the subject and reference distributions are compared here. Results suggest that GoF test for normal distribution would have more powerful when the distance is evaluated using the LW divergence measure. So that the LW as an estimator can lead to better results in measuring the magnitude of difference between two distribution patterns with higher precision. Therefore, we recommend the LW measure to be used with Type-I censored data.

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