R^4 terms in supergravities via T-duality constraint

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(Received 17 January 2018; published 17 May 2018)

It has been speculated in the literature that the effective actions of string theories at any order of α' should be invariant under the Buscher rules plus their higher covariant-derivative corrections. This may be used as a constraint to find effective actions at any order of α', in particular, the metric, the B-field, and the dilaton couplings in supergravities at order α'^3 up to an overall factor. For the simple case of zero B-field and diagonal metric in which we have done the calculations explicitly, we have found that the constraint fixes almost all of the seven independent Riemann curvature couplings. There is only one term which is not fixed, because when metric is diagonal, the reduction of two R^4 terms becomes identical. The Riemann curvature couplings that the T-duality constraint produces for both type II and heterotic theories are fully consistent with the existing couplings in the literature which have been found by the S-matrix and by the sigma-model approaches.

DOI: 10.1103/PhysRevD.97.106013

I. INTRODUCTION

String theory is a candidate for the quantum gravity which produces the classical supergravity at low energy. The stringy signature of the quantum gravity appears in the higher-derivative corrections to the supergravity. There are various techniques in string theory for extracting these higher-derivative corrections, including the scattering amplitude approach [1,2], sigma-model approach [3–5], supersymmetry approach [6–9], Double Field Theory (DFT) approach [10–12], and duality approach [13–15]. In the duality approach, the consistency of the effective actions with duality transformations is imposed to find the higher-derivative couplings [15]. In particular, it has been speculated that the consistency of the effective actions at any order of α' with the T-duality transformations may fix both the effective actions and the T-duality transformations [16].

The T-duality in string theory is realized by studying the spectrum of the closed string on a tours. The spectrum is invariant under the transformation in which the Kaluza-Kelin modes and the winding modes are interchanged, and at the same time, the set of scalar fields parametrizing the tours transforms to another set of scalar fields parametrizing the dual tours. The transformations on the scalar fields have been extended to curved spacetime with background fields by Buscher [17,18]. It has been observed that the effective actions at the leading order of α' are invariant under the Buscher rules [19,20]. The effective actions at the higher order of α' are also expected to be invariant under the T-duality transformations which are the Buscher rules and their appropriate α' corrections. These corrections at order α' have been found in Refs. [21,22].

In type II superstring theory, the higher-derivative corrections to the supergravity begin at order α'^3. As a result, the corrections to the Buscher rules also begin at order α'^3. Hence, one expects the effective actions of the O-plane in the type II superstring theory at order α'^2 to be invariant under the Buscher rules. This may be used as a constraint to find the O-plane effective actions. The Neveu Schwarz-Neveu Schwarz (NS-NS) couplings on the world volume of the O-plane at order α'^2 have been found in Refs. [23,24] by this constraint.

The T-duality constraint has been used in Ref. [16] for the bosonic theory to find the effective actions at order α', α'^2 and their corresponding T-duality transformations when the B-field is zero. Even though the constraint does not completely fix the corrections to the Buscher rules, it does, however, fix the effective actions which are exactly the same as the effective actions that have been found by the S-matrix and sigma-model approaches up to an overall factor. In this paper, we are going to examine the T-duality constraint at order α'^3. The bosonic, the heterotic, and the type II theories all have corrections at order α'^3. However, we are interested in the heterotic and the type II theories in this paper. In the type II theory, we will find that the T-duality constraint almost fixes the effective action up to an overall factor, whereas there may

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remain residual T-duality parameters in the T-duality transformations. In the heterotic theory, we will again find that the constraint almost fixes the effective action while leaving many parameters in the T-duality transformations at orders α', α'^2, α'^3. In this case, however, there are gravity couplings which result from the Green-Schwarz mechanism [25]. Constraining these couplings to be invariant under the T-duality transformations may fix the residual parameters in the T-duality transformations. We will find, by explicit calculations, that this constraint fixes the residual parameters at order α'.

The outline of the paper is as follows. In Sec. II, we explain our strategy for implementing the T-duality constraint on the effective actions and discuss our speculation that the T-duality constraint at any order of α' may be used only on the specific Riemann curvature couplings at that order which are invariant under field redefinitions. In Sec III, we write all independent Riemann curvature couplings at order α'^3 and show that they all are invariant under the field redefinitions in type II theory, whereas two of them are not invariant under the field redefinitions in the heterotic theory. In Sec. III. A, we impose the T-duality constraint on the couplings in the type II theory and show that, even though the T-duality at order α'^3 cannot fix all parameters in the T-duality transformations, it can almost fix all the Riemann curvature couplings up to an overall factor. Since the reduction of two ten-dimensional independent Riemann curvature couplings produces identical nine-dimensional couplings, the T-duality constraint can fix the coefficient of the sum of these two terms. For one particular choice for one of these unfixed coefficients, we show that the couplings that the T-duality constraint produces are exactly the same as the couplings that the S-matrix and sigma-model approaches produce, up to an overall factor.

In Sec. III. B, we impose the T-duality constraint on the couplings in the heterotic theory and show that the constraint almost fixes the Riemann curvature couplings that are invariant under the field redefinitions. The T-duality constraint relates the coefficient of the two Riemann curvatures couplings that are not invariant under the field redefinitions to the T-duality invariant couplings at order α'. For the particular couplings at order α' which do not change the graviton and dilaton propagators, we find that the couplings that the T-duality constraint produces are the same as the couplings in the literature, up to an overall factor. In this section, we also show that the gravity couplings which result from the Green-Schwarz mechanism are invariant under the T-duality transformations. We show that this constraint fixes the residual T-duality parameters at order α'.

II. STRATEGY

The higher-derivative couplings involving the graviton and dilaton in the effective action at order α'^n can be classified as

$$S_n = S_n^{(1)} + S_n^{(2)}, \tag{1}$$

where $S_n^{(1)}$ contains the couplings which are unambiguous and $S_n^{(2)}$ contains the couplings which are ambiguous as their coefficients are changed under field redefinitions. In general, $S_n^{(1)}$ contains Riemann curvature couplings with some specific contraction of indices, whereas $S_n^{(2)}$ contains Riemann curvature couplings with some other contraction of indices and contains Ricci and scalar curvatures as well as a dilaton. Using field redefinitions, one can rearrange the couplings in $S_n^{(2)}$ into two parts. One part contains the couplings which are invariant under the field redefinitions, and the second part contains the couplings which are arbitrarily changed under the field redefinitions. These latter couplings may or may not be zero depending on the field variables. For example, at order α', it has been shown in Ref. [26] that there are eight ambiguous coefficients, and they satisfy one relation which is invariant under the field redefinitions. So, one can set all the ambiguous coefficients to zero except one of them. So, $S_n^{(2)}$ in this case can be simplified to have only one coupling. At order α'^2, there are 42 ambiguous coefficients. They satisfy five relations which are invariant under the field redefinitions [27–30]. So, one can fix all ambiguous coefficients to zero except five of them. As a result, $S_n^{(2)}$ in this case can be written in terms of only five couplings. It is similar for couplings at higher order of α'. Therefore, using field redefinitions, one can write $S_n^{(2)}$ as

$$S_n^{(2)} = \sum_{i=1}^{m} \zeta_i f_i + \cdots, \tag{2}$$

where $f_1, f_2, \cdots$, in the first part are the couplings for which their coefficients $\zeta_1, \zeta_2, \cdots$ are invariant under the field redefinitions. The dots in the above equation represent the second part which contains the couplings which can be set to zero for specific field variables. The coefficients $\zeta_1, \zeta_2, \cdots$, may be fixed by S-matrix calculations.

We now show that the consistency of the effective actions with T-duality constrains the coefficients $\zeta_1, \zeta_2, \cdots$, to be zero. If one dimensionality reduces the D-dimensional effective action $S_n^{(2)}$ to the d-dimensional effective action $\mathbf{S}_n^{(2)}$ where $D = d + 1$, the functions $f_1, f_2, \cdots$, each can have terms with an odd number of σ where $\sigma = (\ln G_{yy})/2$. We call them f_i^{odd}. There are also terms with an even number of σ which we call f_i^{even}. Under the Buscher rules, i.e.,

$$\begin{aligned} \sigma &\to -\sigma \\ P &\to P \\ g_{ab} &\to g_{ab}, \end{aligned} \tag{3}$$

where P and g_{ab} are the d-dimensional dilaton and metric, respectively, $f_i^{\rm even}$ is invariant, and $f_i^{\rm odd}$ changes its sign. Then, the transformation of $\boldsymbol{S}_n^{(2)}$ under the Buscher rules becomes

$$\delta \boldsymbol{S}_n^{(2)} = 2\sum_{i=1}^{m} \zeta_i f_i^{\rm odd} + \cdots \quad (4)$$

Since we have already used the D-dimensional field redefinition to write the action S_2 as in (2) in which the coefficients are invariant under the field redefinitions, the d-dimensional field redefinition does not change the coefficients $\zeta_1, \zeta_2, \cdots$. Now using the observation that the d-dimensional effective action must be invariant under the Buscher rules up to d-dimensional field redefinitions, one concludes that

$$\zeta_1 = \zeta_2 = \cdots = \zeta_m = 0. \quad (5)$$

As a result, the T-duality constraint on the effective action fixes the $S_n^{(2)}$ part of the effective action to be zero up to field redefinitions. Explicit calculations at orders α' and α'^2 confirm the above conclusion [16].

Now, let us consider the $S_n^{(1)}$ part of the effective action (1). Unlike the $S_n^{(2)}$ part, the coefficients of the couplings in $S_n^{(1)}$ are not changed under the D-dimensional field redefinitions; however, after reducing them to the d-dimension effective action $\boldsymbol{S}_n^{(1)}$, the coefficients of the d-dimensional couplings are changed under the d-dimensional field redefinitions. Under the dimensional reduction, one can write $\boldsymbol{S}_n^{(1)} = \boldsymbol{S}_n^{(1)\rm odd} + \boldsymbol{S}_n^{(1)\rm even}$ where $\boldsymbol{S}_n^{(1)\rm odd}$ contains the terms with an odd number of σ and $\boldsymbol{S}_n^{(1)\rm even}$ contains the terms with an even number of σ. Under the Buscher rules,

$$\begin{aligned} \boldsymbol{S}_n^{(1)\rm even} &\to \boldsymbol{S}_n^{(1)\rm even} \\ \boldsymbol{S}_n^{(1)\rm odd} &\to -\boldsymbol{S}_n^{(1)\rm odd}. \end{aligned} \quad (6)$$

Then, the transformation of $\boldsymbol{S}_n^{(1)}$ under the Buscher rules becomes

$$\delta \boldsymbol{S}_n^{(1)} = 2\boldsymbol{S}_n^{(1)\rm odd}. \quad (7)$$

It must be zero up to the d-dimensional field redefinitions. The d-dimensional field redefinitions can be interpreted as higher-derivative corrections to the Buscher rules. Since $\delta \boldsymbol{S}_n^{(1)}$ contains only the terms with an odd number of σ, the appropriate field redefinition should produce also terms with an odd number of σ. If one does not use the d-dimensional field redefinitions, then one would find that $S_n^{(1)}$ is zero, which is not correct. The d-dimensional field redefinitions add some extra terms to the above equation, which makes $S_n^{(1)}$ not to be zero. In fact, the resulting constraint may fix the effective action $S_n^{(1)}$ up to an overall factor. In the rare cases that some of the D-dimensional independent couplings produce identical d-dimensional couplings in $\boldsymbol{S}_n^{(1)\rm odd}$, the above constraint can fix only the sum of their corresponding coefficients. It has been shown in Ref. [16] that $S_n^{(1)}$ at orders α' and α'^2 are fixed up to an overall factor by the above constraint. In this paper, we are going to use the above strategy to find $S_n^{(1)}$ at order α'^3 in the type II superstring and in the heterotic string theories.

III. RIEMANN CURVATURE COUPLINGS AT ORDER α'^3

It is known that the three-point functions at two-momentum level in the superstring and heterotic string theories are reproduced by their corresponding supergravities which have the following graviton and dilaton couplings:

$$S_0 = -\frac{2}{\kappa^2}\int d^{d+1}x e^{-2\Phi}\sqrt{-G}(R + 4\nabla_\alpha\Phi\nabla^\alpha\Phi). \quad (8)$$

This action is invariant under the Buscher rules. There are no higher-momentum corrections to the three-point functions in type II superstring theories; however, there are four momentum corrections to the three-point functions in the heterotic string theory which are reproduced by the following effective action when there is no B-field,

$$\begin{aligned} S_1 = \frac{-2}{\kappa}\alpha'\int d^{d+1}x e^{-2\Phi}\sqrt{-G}(& b_1 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + b_2 R_{\alpha\beta}R^{\alpha\beta} \\ & + b_3 R^2 + b_4 R_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + b_5 R\nabla_\alpha\Phi\nabla^\alpha\Phi \\ & + b_6 R\nabla_\alpha\nabla^\alpha\Phi + b_7\nabla_\alpha\nabla^\alpha\Phi\nabla_\beta\nabla^\beta\Phi \\ & + b_8\nabla_\alpha\Phi\nabla^\alpha\Phi\nabla_\beta\nabla^\beta\Phi \\ & - 2(8b_3 - 2b_5 - 4b_6 + 2b_7 + b_8)\nabla_\alpha\Phi\nabla^\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi), \end{aligned} \quad (9)$$

where $b_1 = 1/8$ for the heterotic theory and $b_1 = 0$ for the type II superstring theories [31]. The couplings with coefficients $b_2, \ldots, b_8$ which belong to the $S_n^{(2)}$ part are not fixed by the S-matrix elements. They are changed under field redefinitions. The form of effective action at the higher orders of α' depends on the form of these couplings, so we keep these terms in the effective action. It has been shown in Ref. [16] that the above action is invariant under T-duality at order α'.

There are no six-momentum nor higher corrections to the three-point functions in either type II or heterotic theories; hence, the higher-derivative corrections to the above actions belonging to $S_n^{(1)}$ part of the effective action must have at least four curvatures. Using the cyclic symmetry for the Riemann curvature, one finds there are only seven such independent couplings, i.e.,

$$\begin{aligned} S_3 = -\frac{2}{\kappa^2}\int d^{d+1}x e^{-2\Phi}\sqrt{-G}[& d_1 R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\zeta}{}^{\theta\iota}R_{\delta\eta\theta\iota} \\ & + d_2 R_{\alpha}{}^{\zeta}{}_{\gamma}{}^{\eta}R^{\alpha\beta\gamma\delta}R_{\beta}{}^{\theta}{}_{\zeta}{}^{\iota}R_{\delta\theta\eta\iota} + d_3 R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\theta}{}_{\zeta}{}^{\iota}R_{\delta\theta\eta\iota} \\ & + d_4 R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\theta\iota}R_{\zeta\eta\theta\iota} + d_5 R_{\alpha\beta\gamma}{}^{\zeta}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\eta\theta\iota}R_{\zeta\eta\theta\iota} \\ & + d_6 R_{\alpha}{}^{\zeta}{}_{\gamma}{}^{\eta}R^{\alpha\beta\gamma\delta}R_{\beta}{}^{\theta}{}_{\delta}{}^{\iota}R_{\zeta\theta\eta\iota} + d_7 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}R_{\zeta\eta\theta\iota}R^{\zeta\eta\theta\iota}], \end{aligned} \tag{10}$$

where $d_1, d_2, \ldots, d_7$ are some unknown coefficients that we are going to find by the T-duality constraint.

Examining the structure of the terms with coefficients d_5 and d_7, one realizes that they can be produced by the variation of $\sqrt{-G}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ at order α'^2, i.e.,

$$\begin{aligned} & \delta(\alpha'\sqrt{-G}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}) \\ & = \alpha'^3\sqrt{-G}\bigg(4R^{\gamma\alpha}R^{\delta}{}_{\alpha} + \frac{1}{2}G^{\gamma\delta}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R^{\alpha\beta}R^{\gamma}{}_{\alpha}{}^{\delta}{}_{\beta} \\ & \quad - 2R^{\gamma\alpha\beta\mu}R^{\delta}{}_{\alpha\beta\mu} - 4\nabla_{\alpha}\nabla^{\alpha}R^{\gamma\delta} + 2\nabla^{\delta}\nabla^{\gamma}R\bigg)\delta G^{(2)}_{\gamma\delta}. \end{aligned} \tag{11}$$

For appropriate variation $\delta G^{(2)}_{\gamma\delta}$, the second and the fourth terms produce the terms with the coefficients d_5 and d_7. So, the coefficients d_5 and d_7 in (10) are changed under field redefinitions in the heterotic theory, whereas these terms do not change under the field redefinitions in type II superstring theory because this theory does not have the Riemann squared coupling. Hence, these terms in the heterotic theory belong to the $S_n^{(2)}$ part, which can be set to zero for specific field variables, whereas in type II theory, they belong to the $S_n^{(1)}$ part which should be fixed by the T-duality constraint. Since we are going to compare the couplings that the T-duality constraint produces with the couplings that the S-matrix method produces for which the field variables are not those that correspond to zero d_5 and d_7 couplings, we keep these terms in both type II and heterotic theories and let the T-duality fix them.

To study the T-duality transformation of the couplings (10), one should first reduce the ten-dimensional action to the nine-dimensional action. For the case that the B-field is zero, the T-duality transformations are consistent for the diagonal metric. So, we consider the reduction of the metric as $G_{\mu\nu} = \mathrm{diag}(g_{ab}, e^{2\sigma})$, where g_{ab} is the d-dimensional metric. This reduction of the metric produces the following reductions for the different components of the Riemann curvature,

$$\begin{aligned} R_{abcd} &= \tilde{R}_{abcd} \\ R_{abcy} &= 0 \\ R_{ayby} &= e^{2\sigma}(-\tilde{\nabla}_a\sigma\tilde{\nabla}_b\sigma - \tilde{\nabla}_b\tilde{\nabla}_a\sigma), \end{aligned} \tag{12}$$

where we have assumed that the fields are independent of the killing coordinate y. The tilde sign over the covariant derivatives and curvature means the metric in them is the d-dimensional metric g_{ab}. Using the *Mathematica* package xAct [32], one can separate the indices in (10) to the d-dimensional indices $a, b, c, \cdots$, and the killing y-index. Then, using the reduction (12), one finds the following reduction for the action (10),

$$\begin{aligned} S_3 = -\frac{2}{\kappa^2}\int d^d x e^{-2P}\sqrt{-g}[& d_1\tilde{R}_{ab}{}^{ei}\tilde{R}^{abcd}\tilde{R}_{ce}{}^{jk}\tilde{R}_{dijk} + d_2\tilde{R}_{a}{}^{e}{}_{c}{}^{i}\tilde{R}^{abcd}\tilde{R}_{b}{}^{j}{}_{e}{}^{k}\tilde{R}_{djik} + d_3\tilde{R}_{ab}{}^{ei}\tilde{R}^{abcd}\tilde{R}_{c}{}^{j}{}_{e}{}^{k}\tilde{R}_{djik} \\ & + d_4\tilde{R}_{ab}{}^{ei}\tilde{R}^{abcd}\tilde{R}_{cd}{}^{jk}\tilde{R}_{eijk} + d_5\tilde{R}_{abc}{}^{e}\tilde{R}^{abcd}\tilde{R}_{d}{}^{ijk}\tilde{R}_{eijk} + d_6\tilde{R}_{a}{}^{e}{}_{c}{}^{i}\tilde{R}^{abcd}\tilde{R}_{b}{}^{j}{}_{d}{}^{k}\tilde{R}_{ejik} + d_7\tilde{R}_{abcd}\tilde{R}^{abcd}\tilde{R}_{eijk}\tilde{R}^{eijk} \\ & + 8d_7\tilde{R}_{cdei}\tilde{R}^{cdei}\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma + 16d_7\tilde{R}_{cdei}\tilde{R}^{cdei}\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma + 8d_7\tilde{R}_{cdei}\tilde{R}^{cdei}\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\tilde{\nabla}^a\sigma \\ & + \frac{8}{3}d_5\tilde{R}_{b}{}^{dei}\tilde{R}_{cdei}\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma + \frac{8}{3}d_5\tilde{R}_{b}{}^{dei}\tilde{R}_{cedi}\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma + \frac{16}{3}d_5\tilde{R}_{b}{}^{dei}\tilde{R}_{cdei}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma \\ & + \frac{16}{3}d_5\tilde{R}_{b}{}^{dei}\tilde{R}_{cedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma + \frac{8}{3}d_5\tilde{R}_{b}{}^{dei}\tilde{R}_{cdei}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma + \frac{8}{3}d_5\tilde{R}_{b}{}^{dei}\tilde{R}_{cedi}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma \\ & + 2(d_2 + 2d_6)\tilde{R}_{a}{}^{e}{}_{b}{}^{i}\tilde{R}_{cedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}^d\sigma - 4(d_2 + d_3)\tilde{R}_{bdce}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\tilde{\nabla}^e\tilde{\nabla}^d\sigma \\ & + 2(4d_1 + d_2 + 8d_4 + 4d_5 + 2d_6 + 8d_7)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}_c\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}_d\sigma\tilde{\nabla}^d\sigma \\ & + 8(4d_1 + d_2 + 8d_4 + 4d_5 + 2d_6 + 8d_7)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\sigma \\ & + 2(8d_1 + 3d_2 + 16d_4 + 12d_5 + 6d_6 + 32d_7)\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\sigma \\ & + 4(8d_1 + d_2 + 2(8d_4 + 2d_5 + d_6))\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}_b\sigma \\ & + 4(8d_1 + d_2 + 2(8d_4 + 2d_5 + d_6))\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}_b\sigma \end{aligned}$$

$$
\begin{aligned}
&+(8d_1+d_2+2(8d_4+2d_5+d_6))\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}_b\sigma\\
&+\frac{4}{9}(d_2+2d_3)\tilde{R}_{ac}{}^{ei}\tilde{R}_{bdei}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+\frac{8}{9}(d_2+2d_3)\tilde{R}_{ac}{}^{ei}\tilde{R}_{bedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+\frac{4}{9}(5d_2+d_3)\tilde{R}_a{}^e{}_c{}^i\tilde{R}_{bedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+\frac{4}{9}(4d_2-d_3)\tilde{R}_a{}^e{}_c{}^i\tilde{R}_{bide}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+8d_6\tilde{R}_a{}^e{}_b{}^i\tilde{R}_{cedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+\frac{2}{9}(d_2+2d_3)\tilde{R}_{ac}{}^{ei}\tilde{R}_{bdei}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+\frac{4}{9}(d_2+2d_3)\tilde{R}_{ac}{}^{ei}\tilde{R}_{bedi}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+\frac{2}{9}(5d_2+d_3)\tilde{R}_a{}^e{}_c{}^i\tilde{R}_{bedi}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+\frac{2}{9}(4d_2-d_3)\tilde{R}_a{}^e{}_c{}^i\tilde{R}_{bide}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+4d_6\tilde{R}_a{}^e{}_b{}^i\tilde{R}_{cedi}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+2(d_2+2(2d_5+d_6+8d_7))\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+4(d_2+2(2d_5+d_6+8d_7))\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+(d_2+2(2d_5+d_6+8d_7))\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&-4(d_2+d_3)\tilde{R}_{adbe}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\tilde{\nabla}^e\tilde{\nabla}_c\sigma-4(d_2+d_3)\tilde{R}_{bdce}\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}^e\tilde{\nabla}^d\sigma\\
&-8(d_2+d_3)\tilde{R}_{bdce}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\tilde{\nabla}^e\tilde{\nabla}^d\sigma], \qquad (13)
\end{aligned}
$$

where the d-dimensional dilaton is $P=\Phi-\sigma/2$. The transformation of $\boldsymbol{S}_3$ under the Buscher rules is constrained to be zero, i.e., $\delta\boldsymbol{S}_3=0$, up to the d-dimensional field redefinitions. Under the Buscher rules, the terms in $\boldsymbol{S}_3$ with an odd number of σ survive, i.e.,

$$
\begin{aligned}
\delta\boldsymbol{S}_3=-\frac{2}{\kappa^2}\int d^dxe^{-2P}\sqrt{-g}\Big[&32d_7\tilde{R}_{cdei}\tilde{R}^{cdei}\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma+\frac{32}{3}d_5\tilde{R}_b{}^{dei}\tilde{R}_{cdei}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\\
&+\frac{32}{3}d_5\tilde{R}_b{}^{dei}\tilde{R}_{cedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma-8(d_2+d_3)\tilde{R}_{bdce}\tilde{\nabla}^b\tilde{\nabla}^a\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\tilde{\nabla}^e\tilde{\nabla}^d\sigma\\
&+16(4(d_1+2d_4)+d_2+4d_5+2d_6+8d_7)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\sigma\\
&+8(8(d_1+2d_4)+d_2+2(2d_5+d_6))\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\tilde{\nabla}_a\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}_b\sigma\\
&+\frac{8}{9}(d_2+2d_3)\tilde{R}_{ac}{}^{ei}\tilde{R}_{bdei}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+\frac{16}{9}(d_2+2d_3)\tilde{R}_{ac}{}^{ei}\tilde{R}_{bedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+\frac{8}{9}(5d_2+d_3)\tilde{R}_a{}^e{}_c{}^i\tilde{R}_{bedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma+\frac{8}{9}(4d_2-d_3)\tilde{R}_a{}^e{}_c{}^i\tilde{R}_{bide}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\\
&+16d_6\tilde{R}_a{}^e{}_b{}^i\tilde{R}_{cedi}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma-8(d_2+d_3)\tilde{R}_{bdce}\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}^c\sigma\tilde{\nabla}^e\tilde{\nabla}^d\sigma\\
&+8(d_2+2(2d_5+d_6+8d_7))\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma\tilde{\nabla}_d\tilde{\nabla}_c\sigma\tilde{\nabla}^d\tilde{\nabla}^c\sigma\Big]=0. \qquad (14)
\end{aligned}
$$

As can be seen, the coefficients d_1 and d_4 appear only through the combination d_1+2d_4. This results from the reduction (12) which has the simple form for the case that metric is diagonal. So, the T-duality for the case that the metric is diagonal cannot fix the coefficients d_1 and d_4 separately. However, all other coefficients appear in different forms in different terms. So, we expect the T-duality to fix them separately.

Since the constrain (14) is on the action, one is free to add to the Lagrangian all total covariant-derivative terms at order α'^3 which have an odd number of σ. Using xAct, it is very simple to construct all such total-derivative terms. One should first write all contractions of curvature, covariant derivatives of σ, and covariant derivatives of P which have an odd number of σ, at seven-derivative order with one free index. We choose the coefficient of each term to be arbitrary. Then, we multiply them with the d-dimensional dilaton factor e^{-2P}. We call the resulting vector J^a. Then, taking a covariant derivative on J^a, i.e., ∇_aJ^a, one finds all d-dimensional total-derivative terms. If one adds to the constraint (14) all the d-dimensional total-derivative terms, one would find the wrong result that $d_1+2d_4=d_2=d_3=d_5=d_6=d_7=0$. Therefore, we have to take into account the d-dimensional field redefinitions as well.

To construct the d-dimensional field redefinitions, one should first reduce the lower α'-order D-dimensional actions (8) and (9) to the d dimensions. Then, one should consider the transformation of the resulting actions under the following field redefinitions:

$$\begin{aligned}\sigma &\to -\sigma + \delta\sigma\\ P &\to P + \delta P\\ g_{ab} &\to g_{ab} + \delta g_{ab}.\end{aligned} \tag{15}$$

The corrections to the Buscher rules, i.e., $\delta\sigma$, δP, and δg_{ab}, for the type II theory begin at order α'^3 because there are no effective actions at orders α' and α'^2. In the heterotic theory, the corrections begin at order α'. So, let use consider each case separately.

A. Couplings in type II supergravity

It is known that the type IIA theory transforms to the type IIB theory under the T-duality transformation [33,34]. The effective actions of these theories, however, are identical in the NS-NS sector. As a result, the NS-NS couplings at any order of α' must be invariant under the T-duality transformation. The effective actions of these theories at the leading order of α' are invariant under the Buscher rules [20]; however, the α'^3 corrections to these couplings are not invariant under the Buscher rules unless one extends them by some α'^3 corrections, i.e.,

$$\begin{aligned}\sigma &\to -\sigma + \alpha'^3\delta\sigma^{(3)}\\ P &\to P + \alpha'^3\delta P^{(3)}\\ g_{ab} &\to g_{ab} + \alpha'^3\delta g^{(3)}_{ab}.\end{aligned} \tag{16}$$

One should replace (16) in the reduction of (8), which is

$$\boldsymbol{S}_0 = -\frac{2}{\kappa^2}\int d^dx e^{-2P}\sqrt{-g}(\tilde{R} + 4\tilde{\nabla}_aP\tilde{\nabla}^aP - \tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma), \tag{17}$$

and keep terms linear in the variations. Up to some total-derivative terms, the variations $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$ produce the following variation for $\boldsymbol{S}_0$,

$$\begin{aligned}\delta\boldsymbol{S}_0 &= \boldsymbol{S}_0(-\sigma + \alpha'^3\delta\sigma^{(3)}, P + \alpha'^3\delta P^{(3)}, g_{ab} + \alpha'^3\delta g^{(3)}_{ab}) - \boldsymbol{S}_0(\sigma, P, g_{ab})\\ &= \frac{2\alpha'^3}{\kappa^2}\int d^dx e^{-2P}\sqrt{-g}\Big[2(\tilde{\nabla}_a\tilde{\nabla}^a\sigma - 2\tilde{\nabla}_a\sigma\tilde{\nabla}^aP)\delta\sigma^{(3)} + (\tilde{R}^{ab} + 2\tilde{\nabla}^a\tilde{\nabla}^bP - \tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma\\ &\quad - \frac{1}{2}g^{ab}(\tilde{R} + 4\tilde{\nabla}_c\tilde{\nabla}^cP - 4\tilde{\nabla}_cP\tilde{\nabla}^cP - \tilde{\nabla}_c\sigma\tilde{\nabla}^c\sigma))\delta g^{(3)}_{ab} + 2(\tilde{R} + 4\tilde{\nabla}_a\tilde{\nabla}^aP - 4\tilde{\nabla}_aP\tilde{\nabla}^aP - \tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma)\delta P^{(3)}\Big] + \cdots,\end{aligned} \tag{18}$$

where dots represent terms at higher orders of α' in which we are not interested. In order to produce couplings at order α'^3, the variations $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$ should all be contractions of the d-dimensional fields at six-derivative level with unknown coefficients. To produce the field redefinitions with an odd number of σ as in (14), one should consider terms in $\delta\sigma^{(3)}$ that have an even number of σ and terms in $\delta P^{(3)}$ and $\delta g^{(3)}_{ab}$ that have an odd number of σ. Adding these field redefinition terms as well as all total-derivative terms to the constraint (14), and writing them in terms of independent couplings, one finds many algebraic equations involving the d-coefficients, the coefficients of the total-derivative terms, and the coefficients of the variations $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$. Solving these equations, one finds many coefficients in the corrections $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$ are not fixed, and the remaining coefficients are fixed in terms of the unfixed coefficients and the d-coefficients. The equations for zero $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$ fix the effective action to be zero; hence, the nonzero effective action forces the Buscher rules to receive α'^3 corrections. The equations for nonzero $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$, however, fix the d-coefficients in the effective action (10) up to an overall factor, i.e.,

$$d_1 + 2d_4 = -\frac{d_2}{4}; \qquad d_3 = d_5 = d_6 = d_7 = 0. \tag{19}$$

The only unknown coefficient is d_2. Note that, since the coefficients d_5 and d_7 are not changed under the field redefinitions in type II theory, they are fixed by the T-duality constraint.

As we have pointed out before, the coefficients d_1 and d_4 appear as one coefficient $d_1 + 2d_4$. We expect the coefficient d_4 to be fixed by the T-duality if one extends the present calculations in which there is no B-field to the calculations in the presence of B-field, which we leave for future work. If we choose it to be zero, the effective action (10) then becomes

$$\begin{aligned}S_3 = -\frac{2d_1}{\kappa^2}\int d^{10}xe^{-2\Phi}\sqrt{-G}[&R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\zeta}{}^{\theta\iota}R_{\delta\eta\theta\iota}\\ &- 4R_\alpha{}^\zeta{}_\gamma{}^\eta R^{\alpha\beta\gamma\delta}R_\beta{}^\theta{}_\zeta{}^\iota R_{\delta\theta\eta\iota}].\end{aligned} \tag{20}$$

In type II theory, using the Kawai-Lewellen-Tye relation between the scattering amplitudes of the closed strings and the scattering amplitudes of open strings [35], one expects the closed string couplings to be written as a product of two open string couplings. Using tensor t_8 which is defined in Ref. [36] to contract four arbitrary antisymmetric tensors $M^1, \cdots, M^4$ as

$$t^{\alpha\beta\gamma\delta\mu\nu\rho\sigma}M^1_{\alpha\beta}M^2_{\gamma\delta}M^3_{\mu\nu}M^4_{\rho\sigma} = 8(\mathrm{tr}M^1M^2M^3M^4 + \mathrm{tr}M^1M^3M^2M^4 + \mathrm{tr}M^1M^3M^4M^2) - 2(\mathrm{tr}M^1M^2\mathrm{tr}M^3M^4 + \mathrm{tr}M^1M^3\mathrm{tr}M^2M^4 + \mathrm{tr}M^1M^4\mathrm{tr}M^2M^3), \tag{21}$$

and the Levi-Cività tensor ϵ_{10}, the couplings (20) can be written as the following expression:

$$S_3 = -\frac{2d_1}{3.2^7\kappa^2}\int d^{10}x e^{-2\Phi}\sqrt{-G}\left(t_8t_8R^4 + \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4\right). \tag{22}$$

For $d_1 = \alpha'^3\zeta(3)/2^7$, this is exactly the R^4 correction to the supergravity that was first found from the sphere-level four-graviton scattering amplitude [36,37] as well as from the σ-model beta function approach [38,39]. The Riemann curvature couplings given by $t_8t_8R^4$, i.e.,

$$\begin{aligned} t_8t_8R^4 \equiv{}& t^{\mu_1\cdots\mu_8}t^{\nu_1\cdots\nu_8}R_{\mu_1\mu_2\nu_1\nu_2}R_{\mu_3\mu_4\nu_3\nu_4}R_{\mu_5\mu_6\nu_5\nu_6}R_{\mu_7\mu_8\nu_7\nu_8} \\ ={}& 3\times 2^7\Big[R_{\alpha\beta\gamma\delta}R_{\beta\mu\delta\rho}R_{\mu\nu\sigma\gamma}R_{\nu\alpha\rho\sigma} + \frac{1}{2}R_{\alpha\beta\gamma\delta}R_{\beta\mu\delta\rho}R_{\mu\nu\rho\sigma}R_{\nu\alpha\sigma\gamma} - \frac{1}{2}R_{\alpha\beta\gamma\delta}R_{\beta\mu\gamma\delta}R_{\mu\nu\rho\sigma}R_{\nu\alpha\rho\sigma} - \frac{1}{4}R_{\alpha\beta\gamma\delta}R_{\beta\mu\rho\sigma}R_{\mu\nu_\gamma\delta}R_{\nu\alpha\rho\sigma} \\ &+ \frac{1}{16}R_{\alpha\beta\gamma\delta}R_{\beta\alpha\rho\sigma}R_{\mu\nu\gamma\delta}R_{\nu\mu\rho\sigma} + \frac{1}{32}R_{\alpha\beta\gamma\delta}R_{\beta\alpha\gamma\delta}R_{\mu\nu\rho\sigma}R_{\nu\mu\rho\sigma}\Big], \end{aligned} \tag{23}$$

have nonzero contribution at the four-graviton level, so they were found from the sphere-level S-matrix element of four-graviton vertex operators [36,37], whereas the couplings given by $\epsilon_{10}\epsilon_{10}R^4$ of which the Riemann curvature couplings are

$$\begin{aligned} \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4 ={}& 3\times 2^7\Big[-R_{\alpha\beta\gamma\delta}R_{\rho\sigma\beta\mu}R_{\delta\mu\sigma\nu}R_{\gamma\nu\alpha\rho} + R_{\alpha\beta\gamma\delta}R_{\rho\sigma\alpha\beta}R_{\delta\mu\sigma\nu}R_{\gamma\nu\rho\mu} + \frac{1}{2}R_{\alpha\beta\gamma\delta}R_{\rho\sigma\mu\nu}R_{\gamma\mu\rho\sigma}R_{\delta\nu\alpha\beta} - \frac{1}{2}R_{\alpha\beta\gamma\delta}R_{\rho\sigma\mu\nu}R_{\gamma\mu\alpha\rho}R_{\delta\nu\beta\sigma} \\ &- \frac{1}{16}R_{\alpha\beta\gamma\delta}R_{\gamma\delta\rho\sigma}R_{\rho\sigma\mu\nu}R_{\mu\nu\alpha\beta} - \frac{1}{32}R_{\alpha\beta\gamma\delta}R_{\gamma\delta\alpha\beta}R_{\rho\sigma\mu\nu}R_{\mu\nu\rho\sigma}\Big] \end{aligned} \tag{24}$$

have nonzero contribution at the five-graviton level [40]. However, the presence of this term in the tree-level effective action was first dictated by the σ-model beta function approach [38,39]. It has been shown in Ref. [41] that the sphere-level scattering amplitude of five gravitons confirms the presence of $\epsilon_{10}\epsilon_{10}R^4$ in the tree-level effective action. It is interesting that the T-duality constrain could fix the presence of both terms in the effective action.

B. Couplings in heterotic supergravity

In the heterotic theory, the corrections to the Buscher rules begin at order α', i.e.,

$$\begin{aligned} \sigma &\to -\sigma + \alpha'\delta\sigma^{(1)} \\ P &\to P + \alpha'\delta P^{(1)} \\ g_{ab} &\to g_{ab} + \alpha'\delta g^{(1)}_{ab}, \end{aligned} \tag{25}$$

where the corrections are parametrized by nine parameters:

$$\begin{aligned} \delta\sigma^{(1)} &= A_1\tilde{R} + A_2\tilde{\nabla}_a\tilde{\nabla}^aP + A_3\tilde{\nabla}_aP\tilde{\nabla}^aP + A_4\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma \\ \delta P^{(1)} &= A_5\tilde{\nabla}_a\tilde{\nabla}^a\sigma + A_6\tilde{\nabla}_a\sigma\tilde{\nabla}^aP \\ \delta g^{(1)}_{ab} &= A_7\left(\frac{1}{2}\tilde{\nabla}_a\sigma\tilde{\nabla}_bP + \frac{1}{2}\tilde{\nabla}_aP\tilde{\nabla}_b\sigma\right) + g_{ab}(A_8\tilde{\nabla}_c\tilde{\nabla}^c\sigma + A_9\tilde{\nabla}_c\sigma\tilde{\nabla}^cP). \end{aligned} \tag{26}$$

We have excluded the parameter corresponding to the d-dimensional coordinate transformations. These corrections are required to make the d-dimensional reduction of the couplings (9), i.e.,

$$
\begin{aligned}
\boldsymbol{S}_1 = -\frac{2}{\kappa^2}\alpha' \int d^d x e^{-2P}\sqrt{-g}\Big(& b_1 \tilde{R}_{abcd}\tilde{R}^{abcd} + b_2 \tilde{R}_{ab}\tilde{R}^{ab} + b_3 \tilde{R}^2 + b_6 \tilde{R}\tilde{\nabla}_a\tilde{\nabla}^a P + b_5 \tilde{R}\tilde{\nabla}_a P\tilde{\nabla}^a P + (b_5+b_6)\tilde{R}\tilde{\nabla}_a\sigma\tilde{\nabla}^a P \\
& + \frac{1}{4}(-16b_3 + b_5 + 2b_6)\tilde{R}\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma - 2b_2\tilde{R}^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\sigma + b_7\tilde{\nabla}_a\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}^b P + b_8\tilde{\nabla}_a P\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}^b P \\
& + (2b_7 + b_8)\tilde{\nabla}_a\sigma\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}^b P + \left(-2b_6 + b_7 + \frac{1}{4}b_8\right)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}^b P + (-2b_6 + b_7)\tilde{\nabla}_a\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}^b\sigma \\
& + \left(b_2 + 4b_3 - b_6 + \frac{1}{4}b_7\right)\tilde{\nabla}_a\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}^b\sigma + \frac{1}{2}(-4b_5 + b_8)\tilde{\nabla}_a P\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}^b\sigma \\
& + \left(-2b_5 - 2b_6 + b_7 + \frac{1}{2}b_8\right)\tilde{\nabla}_a\sigma\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}^b\sigma + \frac{1}{8}(16b_2 + 64b_3 - 4b_5 - 16b_6 + 4b_7 + b_8)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}^b\sigma \\
& + b_4\tilde{R}_{ab}\tilde{\nabla}^a P\tilde{\nabla}^b P - (2b_8 + 4b_7 - 8b_6 - 4b_5 + 16b_3)\tilde{\nabla}_a P\tilde{\nabla}^a P\tilde{\nabla}_b P\tilde{\nabla}^b P + b_4\tilde{R}_{ab}\tilde{\nabla}^a P\tilde{\nabla}^b\sigma \\
& + (-b_4 - 3b_3 - b_8 + 8b_6 + 4b_5 - 16b_3)\tilde{\nabla}_a\sigma\tilde{\nabla}^a P\tilde{\nabla}_b\sigma\tilde{\nabla}^b P - b_4\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b P + \frac{1}{4}(-8b_2 + b_4)\tilde{R}_{ab}\tilde{\nabla}^a\sigma\tilde{\nabla}^b\sigma \\
& + \frac{1}{2}(-b_8 - 4b_7 + 8b_6 - 16b_3)\tilde{\nabla}_a P\tilde{\nabla}^a P\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma + \left(-b_4 + 2b_6 - b_7 - \frac{1}{4}b_8\right)\tilde{\nabla}_a\sigma\tilde{\nabla}^a P\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma \\
& + \left(8b_1 + 2b_2 - \frac{1}{4}b_4\right)\tilde{\nabla}^a\sigma\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma + \frac{1}{16}(64b_1 + 32b_2 + 48b_3 - 4b_4 - 4b_5 - 8b_6)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\tilde{\nabla}_b\sigma\tilde{\nabla}^b\sigma \\
& - b_4\tilde{\nabla}^a P\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\sigma - (3b_8 + 8b_7 - 16b_6 - 8b_5 + 32b_3)\tilde{\nabla}_a P\tilde{\nabla}^a P\tilde{\nabla}_b\sigma\tilde{\nabla}^b P \\
& + \frac{1}{2}(-8b_3 + b_6)\tilde{R}\tilde{\nabla}_a\tilde{\nabla}^a\sigma + (4b_1 + b_2)\tilde{\nabla}_b\tilde{\nabla}_a\sigma\tilde{\nabla}^b\tilde{\nabla}^a\sigma\Big), \qquad (27)
\end{aligned}
$$

to be invariant under the T-duality [16]. That is, when applying these corrections on the leading-order d-dimensional couplings in $\boldsymbol{S}_0$, the resulting field redefinition terms guarantee that the couplings at order α' in $\boldsymbol{S}_1$ are invariant under the Buscher rules, i.e.,

$$
\boldsymbol{S}_0(-\sigma + \alpha'\delta\sigma^{(1)}, P + \alpha'\delta P^{(1)}, g + \alpha'\delta g^{(1)}) - \boldsymbol{S}_0(\sigma, P, g) + \boldsymbol{S}_1(-\sigma, P, g) - \boldsymbol{S}_1(\sigma, P, g) = 0. \qquad (28)
$$

In the perturbation of the first term, one must ignore the terms at orders α'^2 and higher. The corrections to the Buscher rules at order α', i.e., Eq. (26), should satisfy the above constraint. In solving this constraint, one must add all total-derivative terms at order α' to the above constraint. The result is [16]

$$
\begin{aligned}
A_1 &= \frac{1}{8}(4A_6 - A_9(D-3) + 2b_4 + 4b_5 + 4b_6), \\
A_2 &= \frac{1}{2}(4A_6 - A_9(D-2) - 8b_2 + 3b_4 + 2b_7 + b_8), \\
A_3 &= \frac{1}{2}(-4A_6 + A_9(D-1) + 16b_2 - 32b_3 - 5b_4 + 8b_5 + 16b_6 - 8b_7 - 3b_8), \\
A_4 &= \frac{1}{8}(-4A_6 + A_9(D-3) + 32b_1 - 32b_3 - 3b_4 + 8b_6 - 4b_7 - b_8), \\
A_5 &= \frac{1}{8}(-4A_6 + 8b_2 + 32(D-2)b_3 + (D-5)b_4 - 4(D-2)b_5 - 4(3D-7)b_6 + 4(D-3)b_7 + (D-3)b_8), \\
A_7 &= 8b_2 - 2b_4, \\
A_8 &= \frac{1}{2}(-A_9 + 32b_3 + b_4 - 4b_5 - 12b_6 + 4b_7 + b_8). \qquad (29)
\end{aligned}
$$

The residual parameters A_6 and A_9 are not fixed by the calculations at order α' and α'^2 that have been done in Ref. [16]. In the heterotic theory, we will see that these parameters as well as the parameters b_4, b_5, and b_8 will be fixed by requiring the couplings at order α'^2 which are produced by the Green-Schwarz mechanism [25], to be invariant under the T-duality at order α'.

Applying the variations (29) to the couplings $\boldsymbol{S}_1$, one finds some couplings at order α'^2. On the other hand, it is known that there is no curvature couplings at order α'^2 in the heterotic theory; hence, there must be corrections to the Buscher rules at order α'^2 as well. The effect of applying these corrections to the couplings $\boldsymbol{S}_0$ must be canceled by the effect of applying the corrections at order α' on the couplings in $\boldsymbol{S}_1$. Therefore, the corrections to the Buscher rules at orders α' and α'^2, i.e.,

$$\begin{aligned}\sigma &\to -\sigma + \alpha'\delta\sigma^{(1)} + \alpha'^2\delta\sigma^{(2)}\\ P &\to P + \alpha'\delta P^{(1)} + \alpha'^2\delta P^{(2)}\\ g_{ab} &\to g_{ab} + \alpha'\delta g^{(1)}_{ab} + \alpha'^2\delta g^{(2)}_{ab},\end{aligned}\tag{30}$$

must satisfy the following constraint:

$$\begin{aligned}&\boldsymbol{S}_0(-\sigma + \alpha'\delta\sigma^{(1)} + \alpha'^2\delta\sigma^{(2)}, P + \alpha'\delta P^{(1)}\\ &\quad + \alpha'^2\delta P^{(2)}, g + \alpha'\delta g^{(1)} + \alpha'^2\delta g^{(2)})\\ &\quad - \boldsymbol{S}_0(\sigma, P, g) + \boldsymbol{S}_1(-\sigma + \alpha'\delta\sigma^{(1)},\\ &P + \alpha'\delta P^{(1)}, g + \alpha'\delta g^{(1)}) - \boldsymbol{S}_1(\sigma, P, g) = 0.\end{aligned}\tag{31}$$

In the perturbation of the first and the third terms, one must ignore the terms at order α'^3 and higher. In solving the above constraint, one must add to it all total-derivative terms at order α'^2. Using the fact that the T-duality transformations must be a $\mathbb{Z}_2$-group, one finds that there are 98 coefficients in the variations $\delta\sigma^{(2)}$, $\delta P^{(2)}$, and $\delta g^{(2)}_{ab}$. The above constraint fixes 61 coefficients in terms of other 37 terms and in terms of the b-coefficients [16].

In order to study the couplings at order α'^3 under the T-duality, one must consider corrections to the Buscher rules at order α'^3 as well, i.e.,

$$\begin{aligned}\sigma &\to -\sigma + \alpha'\delta\sigma^{(1)} + \alpha'^2\delta\sigma^{(2)} + \alpha'^3\delta\sigma^{(3)}\\ P &\to P + \alpha'\delta P^{(1)} + \alpha'^2\delta P^{(2)} + \alpha'^3\delta P^{(3)}\\ g_{ab} &\to g_{ab} + \alpha'\delta g^{(1)}_{ab} + \alpha'^2\delta g^{(2)}_{ab} + \alpha'^3\delta g^{(3)}_{ab}.\end{aligned}\tag{32}$$

A straightforward extension of the constraint (31) to order α'^3 is given by the following constraint,

$$\begin{aligned}&\boldsymbol{S}_0(-\sigma + \alpha'\delta\sigma^{(1)} + \alpha'^2\delta\sigma^{(2)} + \alpha'^3\delta\sigma^{(3)}, P + \alpha'\delta P^{(1)}\\ &\quad + \alpha'^2\delta P^{(2)} + \alpha'^3\delta P^{(3)}, g + \alpha'\delta g^{(1)} + \alpha'^2\delta g^{(2)} + \alpha'^3\delta g^{(3)})\\ &\quad - \boldsymbol{S}_0(\sigma, P, g) + \boldsymbol{S}_1(-\sigma + \alpha'\delta\sigma^{(1)} + \alpha'^2\delta\sigma^{(2)}, P + \alpha'\delta P^{(1)}\\ &\quad + \alpha'^2\delta P^{(2)}, g + \alpha'\delta g^{(1)} + \alpha'^2\delta g^{(2)})\\ &\quad - \boldsymbol{S}_1(\sigma, P, g) + \boldsymbol{S}_3(-\sigma, P, g) - \boldsymbol{S}_3(\sigma, P, g) = 0,\end{aligned}\tag{33}$$

where $\boldsymbol{S}_3(-\sigma, P, g) - \boldsymbol{S}_3(\sigma, P, g) = \delta\boldsymbol{S}_3$ is (14). In the perturbation of the first and the third terms, one must ignore the terms at orders α'^4 and higher. The coefficients of the variations $\delta\sigma^{(1)}$, $\delta P^{(1)}$, and $\delta g^{(1)}_{ab}$ are given in (29), and those of the variations $\delta\sigma^{(2)}$, $\delta P^{(2)}$, and $\delta g^{(2)}_{ab}$ satisfy the constraint (31). After solving the constraint (31), one must replace the corresponding variations into the above constraint.

Adding all total-derivative terms to the constraint (33), and writing them in terms of independent couplings, one finds many algebraic equations involving the d-coefficients, the b-coefficients, the coefficients of the total-derivative terms, and the coefficients of the variations. Solving these equations, one finds 14 relations between the 37 unfixed coefficients of $\delta\sigma^{(2)}$, $\delta P^{(2)}$, and $\delta g^{(2)}_{ab}$. Moreover, one finds many coefficients in the variations $\delta\sigma^{(3)}$, $\delta P^{(3)}$, and $\delta g^{(3)}_{ab}$ are not fixed, and the remaining coefficients are fixed in terms of the unfixed coefficients, the d-coefficients and the b-coefficients. However, the equations fix the d-coefficients in the effective action (10) in terms of the b-coefficients, i.e.,

$$\begin{aligned}d_1 + 2d_4 &= -\frac{d_2}{4} + \frac{1}{256}(8b_3 - b_5 - 2b_6)\\ &\quad\times(16b_1 + 8b_3 - b_5 - 2b_6)\\ &\quad\times(8b_1 + 28b_2 + 108b_3 - 18b_6 + 3b_7)\\ d_5 &= -\frac{b_1}{4}(2b_1 + b_2)(16b_1 + 8b_3 - b_5 - 2b_6)\\ d_7 &= -\frac{b_1}{64}(16b_1 + 8b_3 - b_5 - 2b_6)\\ &\quad\times(8b_2 + 36b_3 - 6b_6 + b_7); \quad d_3 = d_6 = 0.\end{aligned}\tag{34}$$

The only unknown d-coefficient at order α'^3 is d_2.

As we have anticipated before, the coefficients d_5 and d_7 which are changed under field redefinitions in the heterotic theory depend on the form of effective action at order α'. However, the Riemann curvature couplings in (10) with coefficients d_1, d_4, and d_2 are not changed under field redefinitions; hence, we do not expect these coefficients to depend on the effective action (9). Therefore, we expect the coefficients $b_2, b_3, \cdots$, and b_8 in (9) not to be totally arbitrary. The invariance of the curvature terms at orders α'^2 and α'^3 under T-duality does not constrain these coefficients. However, the heterotic theory has other gravity couplings which result from the Green-Schwarz mechanism [25]. These couplings may constrain the parameters b_2, $b_3, \cdots$, and b_8.

Extension of the effective action at the leading order of α', i.e., Eq. (8), in the presence of the B-field is

$$S_0 = -\frac{2}{\kappa^2}\int d^{d+1}x e^{-2\Phi}\sqrt{-G}\left(R + 4\nabla_\alpha\Phi\nabla^\alpha\Phi - \frac{1}{12}H^2\right), \tag{35}$$

where $H = dB$. This action has been written in DFT formalism in Ref. [11]. In the heterotic theory, the Green-Schwarz mechanism [25] dictates that the B-field strength $H(B)$ must be replaced by the improved field strength $\hat{H}(B,\Gamma)$ that includes the Chern-Simons term built from the Christoffel connection,

$$\hat{H}_{\mu\nu\rho}(B,\Gamma) = 3(\partial_{[\mu}B_{\nu\rho]} + \alpha'\Omega(\Gamma)_{\mu\nu\rho}), \tag{36}$$

with the Chern-Simons three-form

$$\Omega(\Gamma)_{\mu\nu\rho} = \Gamma^\alpha_{[\mu|\beta|}\partial_\nu\Gamma^\beta_{\rho]\alpha} + \frac{2}{3}\Gamma^\alpha_{[\mu|\beta|}\Gamma^\beta_{\nu|\gamma|}\Gamma^\gamma_{\rho]\alpha}. \tag{37}$$

The replacement $H \to \hat{H}$ in S_0 produces the gravity coupling $\alpha'^2\Omega^2$ which should be invariant under T-duality.

The effective action at order α', i.e., Eq. (9), in the presence of a B-field is [9,26]

$$S_1 = \frac{-2b_1}{\kappa^2}\alpha'\int d^{d+1}x e^{-2\Phi}\sqrt{-G}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{2}R_{\alpha\beta\gamma\delta}H^{\alpha\beta\lambda}H^{\gamma\delta}{}_\lambda + \frac{1}{24}H_{\mu\nu\rho}H^\mu{}_\alpha{}^\beta H^\nu{}_\beta{}^\gamma H^\rho{}_\gamma{}^\alpha - \frac{1}{8}H_\mu{}^{\alpha\beta}H_{\nu\alpha\beta}H^{\mu\gamma\rho}H^\nu{}_{\gamma\rho} + \cdots\right), \tag{38}$$

where dots represent the terms which can be removed by appropriate field redefinitions. The DFT formulation of this action has been found in Refs. [42,43]. The H in the heterotic theory must be replaced by $\hat{H}$. This replacement produces the gravity couplings $\alpha'^3R_{\alpha\beta\gamma\delta}\Omega^{\alpha\beta\lambda}\Omega^{\gamma\delta}{}_\lambda$ and also some Ω^4 terms in which we are not interested in this paper because they are at order α'^5. The consistency of our calculations requires these gravity couplings to be invariant under the T-duality transformations, too.

Reduction of Ω^2 from ten-dimensional to nine-dimensional spacetime is

$$\begin{aligned}\Omega^2 = &-\frac{8}{9}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^j{}_c{}^k\Gamma_{ki}{}^l\Gamma_{lej} + \frac{8}{9}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^j{}_c{}^k\Gamma_{ke}{}^l\Gamma_{lij} - \frac{1}{6}\Gamma^{abc}\Gamma^{dei}\tilde{\nabla}_c\Gamma_{idj}\tilde{\nabla}_e\Gamma_{ba}{}^j\\ &-\frac{4}{3}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^j{}_c{}^k\tilde{\nabla}_e\Gamma_{kij} + \frac{4}{3}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^j{}_c{}^k\tilde{\nabla}_i\Gamma_{kej} + \frac{1}{3}\Gamma^{abc}\Gamma^{dei}\tilde{\nabla}_c\Gamma_{idj}\tilde{\nabla}^j\Gamma_{bae}\\ &-\frac{1}{6}\Gamma^{abc}\Gamma^{dei}\tilde{\nabla}_j\Gamma_{icd}\tilde{\nabla}^j\Gamma_{bae} - \frac{1}{6}\Gamma^{abc}\Gamma^d{}_b{}^e\tilde{\nabla}_i\Gamma_{edj}\tilde{\nabla}^j\Gamma_{ca}{}^i + \frac{1}{6}\Gamma^{abc}\Gamma^d{}_b{}^e\tilde{\nabla}_j\Gamma_{edi}\tilde{\nabla}^j\Gamma_{ca}{}^i.\end{aligned} \tag{39}$$

As can be seen, it contains no term which has σ. So, it is invariant under the Buscher rules (3). It must be also invariant under the T-duality transformations at order α', i.e.,

$$e^{-2P}\sqrt{-g}\Omega^2(P + \alpha'\delta P^{(1)}, g + \alpha'\delta g^{(1)}) - e^{-2P}\sqrt{-g}\Omega^2(P, g) = 0. \tag{40}$$

This constraint fixes the residual parameters in (29) to be zero, i.e., $A_6 = A_9 = 0$, and also fixes the coefficients b_4, b_5, and b_8 in terms of b_2, b_3, b_6, and b_7, i.e.,

$$b_4 = 4b_2; \qquad b_5 = 8b_3 - 2b_6; \qquad b_8 = -4(b_2 - b_6 + b_7). \tag{41}$$

Hence, the corrections to the Buscher rules at order α', i.e., Eq. (29), are fixed to be

$$\begin{aligned}\delta\sigma^{(1)} &= \left(b_2 + 3b_3 - \frac{1}{2}b_6\right)R + (2b_6 - b_7)\tilde{\nabla}_a\tilde{\nabla}^aP + \left(4b_1 - b_2 - 4b_3 + \frac{1}{2}b_6\right)\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma\\ \delta P^{(1)} &= 0\\ \delta g^{(1)}_{ab} &= 0.\end{aligned} \tag{42}$$

Note that the d-dimensional couplings in Ω^2 cannot be written in terms of Riemann curvatures; hence, the constraint (40) is independent of the constraint (33).

Let us compare the above transformation with the standard T-duality transformation at order α' [21] when the effective action has no Ricci nor scalar curvature. The constraint that the effective action at order α' in the heterotic theory must be invariant under the T-duality has been used in Ref. [21] to find the extension of the Buscher rules at order α' in the presence of a B-field and gauge field. In the absence of these fields, and for a diagonal metric, they are [21]

$$\begin{aligned}\tilde{g}_{ab} &= g_{ab} + \frac{g_{yy}\hat{\mathcal{G}}_{ya}\hat{\mathcal{G}}_{yb} - 2\hat{\mathcal{G}}_{yy}\hat{\mathcal{G}}_{y(a}g_{b)y}}{\hat{\mathcal{G}}_{yy}^2}\\ \tilde{\Phi} &= \Phi - \frac{1}{2}\log|\hat{\mathcal{G}}_{yy}|\\ \tilde{g}_{yy} &= e^{2\tilde{\sigma}} = \frac{e^{2\sigma}}{\hat{\mathcal{G}}_{yy}^2},\end{aligned} \tag{43}$$

where the α' correction appears in $\hat{\mathcal{G}}_{\mu\nu}$, i.e.,

$$\hat{\mathcal{G}}_{\mu\nu} = G_{\mu\nu} + \frac{1}{4}\alpha'\Omega_\mu{}^{\bar{a}\bar{b}}\Omega_\nu{}^{\bar{a}\bar{b}}. \tag{44}$$

The metric $G_{\mu\nu}$ is the ten-dimensional metric, and $\omega_\mu{}^{\bar{a}\bar{b}}$ is a torsionless spin connection, i.e.,

$$\Omega_\mu{}^{\bar{a}\bar{b}} = \omega_\mu{}^{\bar{a}\bar{b}} = e_\alpha{}^{\bar{a}}e^{\lambda\bar{b}}\Gamma^\alpha_{\mu\lambda} - e^{\lambda\bar{b}}\partial_\mu e_\lambda{}^{\bar{a}}.$$

Using the fact that fields are independent of the y-direction, one finds that $\Omega_a{}^{\bar{a}\bar{b}}\Omega_y{}^{\bar{a}\bar{b}} = 0 = \hat{\mathcal{G}}_{ay}$. One also finds $\hat{\mathcal{G}}_{yy} = e^{2\sigma}(1 - \frac{1}{2}\alpha'\tilde{\nabla}_a\sigma\tilde{\nabla}^a\sigma)$. Hence, the nine-dimensional metric and dilaton become invariant, and the transformation of σ becomes the same as the transformation (42) in which $b_2 = b_3 = b_6 = b_7 = 0$.

The $R\Omega^2$ couplings are at order α'^3, so to the order that we consider in this paper, the consistency requires it to be invariant under the Buscher rules. The reduction of this term to the d-dimensional spacetime is

$$\begin{aligned}R_{\alpha\beta\gamma\delta}\Omega^{\alpha\beta\lambda}\Omega^{\gamma\delta}{}_\lambda = {}& \frac{16}{9}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^j{}_c{}^k\Gamma_k{}^{lm}\Gamma_{lj}{}^n\tilde{R}_{eimn} + \frac{8}{9}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^{jkl}\tilde{R}_{eilm}\tilde{\nabla}_c\Gamma_{kj}{}^m\\ &+ \frac{1}{9}\Gamma^{abc}\Gamma^{dei}\tilde{R}_{cjil}\tilde{\nabla}_k\Gamma_{ed}{}^l\tilde{\nabla}^k\Gamma_{ba}{}^j + \frac{1}{9}\Gamma^{abc}\Gamma^{dei}\tilde{R}_{ckil}\tilde{\nabla}^k\Gamma_{ba}{}^j\tilde{\nabla}^l\Gamma_{edj}\\ &- \frac{2}{9}\Gamma^{abc}\Gamma^{dei}\tilde{R}_{cjil}\tilde{\nabla}^k\Gamma_{ba}{}^j\tilde{\nabla}^l\Gamma_{edk} + \frac{1}{9}\Gamma^{abc}\Gamma^d{}_b{}^e\tilde{R}_{ijkl}\tilde{\nabla}^j\Gamma_{ca}{}^i\tilde{\nabla}^l\Gamma_{ed}{}^k\\ &+ \frac{2}{9}\Gamma^{abc}\Gamma^{dei}\tilde{R}_{cjkl}\tilde{\nabla}_e\Gamma_{ba}{}^j\tilde{\nabla}^l\Gamma_{id}{}^k - \frac{2}{9}\Gamma^{abc}\Gamma^{dei}\tilde{R}_{cjkl}\tilde{\nabla}^j\Gamma_{bae}\tilde{\nabla}^l\Gamma_{id}{}^k\\ &- \frac{8}{9}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^{jkl}\tilde{R}_{eilm}\tilde{\nabla}^m\Gamma_{kcj} + \frac{8}{9}\Gamma^{abc}\Gamma_b{}^{de}\Gamma_{da}{}^i\Gamma^j{}_c{}^k\tilde{R}_{eilm}\tilde{\nabla}^m\Gamma_{kj}{}^l.\end{aligned} \tag{45}$$

Since σ does not appear in it, it is obviously invariant under the Buscher rules (3).

The constraints (41) simplify the equations in (34) as

$$\begin{aligned}d_1 + 2d_4 &= -\frac{d_2}{4}\\ d_5 &= -4b_1^2(2b_1 + b_2)\\ d_7 &= -\frac{b_1^2}{4}(8b_2 + 36b_3 - 6b_6 + b_7); \quad d_3 = d_6 = 0.\end{aligned} \tag{46}$$

As expected, the b-coefficients do not appear in the first equation. Moreover, for the specific field variables at order α', i.e., $b_2 = -2b_1$, $36b_3 - 6b_6 + b_7 = 16b_1$, the Riemann curvature couplings with coefficients d_5 and d_7 should be removed by the field redefinitions.

Since our calculations in the absence of a B-field cannot fix the coefficient d_4, we have to fix it by hand. In the superstring theory, we showed that $d_4 = 0$ precisely reproduces the known R^4 corrections to the type II supergravity. The difference between the superstring and the heterotic calculations is the presence of effective action at order α'. The presence of this action may cause the coefficient d_4 not to be zero in the heterotic theory. If we choose it to be $d_4 = -2b_1^3$, then the equations (46) produce the couplings (20) as well as the following couplings:

$$\begin{aligned}S_3^H = -\frac{2b_1^2}{\kappa^2}\int d^{d+1}xe^{-2\Phi}\sqrt{-G}\Big[&4b_1R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\zeta}{}^{\theta\iota}R_{\delta\eta\theta\iota} - 2b_1R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\theta\iota}R_{\zeta\eta\theta\iota}\\ &- 4(2b_1 + b_2)R_{\alpha\beta\gamma}{}^\zeta R^{\alpha\beta\gamma\delta}R_\delta{}^{\eta\theta\iota}R_{\zeta\eta\theta\iota} - \frac{1}{4}(8b_2 + 36b_3 - 6b_6 + b_7)(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})^2\Big].\end{aligned} \tag{47}$$

Since the last two terms above are changed under field redefinitions, we have to choose a specific field variable to compare them with the couplings in the literature. To compare the couplings with the couplings that have been found by the S-matrix method, one has to choose the effective action at order α' in specific field variables that do not change the graviton and dilaton propagators. That is, we have to choose the Gauss-Bonnet combinations for the curvature couplings at order α', i.e., $b_2 = -4b_1$, $b_3 = b_1$, to have standard graviton propagator, and also we have to choose $b_6 = b_7 = 0$ to have standard dilaton propagator. For these parameters, the above couplings become

$$S_3^H = -\frac{2b_1^3}{\kappa^2}\int d^{d+1}xe^{-2\Phi}\sqrt{-G}[4R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\zeta}{}^{\theta\iota}R_{\delta\eta\theta\iota} - 2R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\theta\iota}R_{\zeta\eta\theta\iota} + 8R_{\alpha\beta\gamma}{}^{\zeta}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\eta\theta\iota}R_{\zeta\eta\theta\iota} - (R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})^2]. \quad (48)$$

Using the tensor (21), one can write

$$t^{\mu_1\cdots\mu_8}\mathrm{Tr}(R_{\mu_1\mu_2}R_{\mu_3\mu_4})\mathrm{Tr}(R_{\mu_5\mu_6}R_{\mu_7\mu_8}) = 8R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\zeta}{}^{\theta\iota}R_{\delta\eta\theta\iota} - 4R_{\alpha\beta}{}^{\zeta\eta}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\theta\iota}R_{\zeta\eta\theta\iota} + 16R_{\alpha\beta\gamma}{}^{\zeta}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\eta\theta\iota}R_{\zeta\eta\theta\iota} - 2(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})^2.$$

Therefore, the effective actions that the T-duality constraint produces in the heterotic theory for the specific parameters $b_2 = -4b_1$, $b_3 = b_1$, and $b_6 = b_7 = 0$ are (22) and

$$S_3^H = -\frac{b_1^3}{\kappa^2}\int d^{d+1}xe^{-2\Phi}\sqrt{-G}[t^{\mu_1\cdots\mu_8}\mathrm{Tr}(R_{\mu_1\mu_2}R_{\mu_3\mu_4}) \times \mathrm{Tr}(R_{\mu_5\mu_6}R_{\mu_7\mu_8})], \quad (49)$$

which are exactly the couplings that have been found in Ref. [44].

We have seen that the gravity couplings resulting from the Green-Schwarz mechanism fix the residual T-duality parameters at order α' and also fix the parameters b_4, b_5, and b_8. There are also 23 T-duality parameters at order α'^2 that are not fixed by the constraint (33). These parameters may also be fixed by the gravity couplings resulting from the Green-Schwarz mechanism. Since this mechanism does not produce gravity couplings at order α'^4, one expects the T-duality transformation of $\alpha'^2\Omega^2$ at order α'^2 cancels the T-duality transformation of $\alpha'^3R\Omega^2$ at order α'. On the other hand, there is no σ in the reduction of $\alpha'^3R\Omega^2$, i.e., Eq. (45), and the T-duality transformation (42) at order α' does not change P and g_{ab}, so $\alpha'^3R\Omega^2$ is invariant under the T-duality transformation at order α'. Therefore, the T-duality transformation of $\alpha'^2\Omega^2$ at order α'^2 must be zero, i.e.,

$$e^{-2P}\sqrt{-g}\Omega^2(P+\alpha'\delta P^{(1)}+\alpha'^2\delta P^{(2)}, g+\alpha'\delta g^{(1)}+\alpha'^2\delta g^{(2)}) - e^{-2P}\sqrt{-g}\Omega^2(P,g) = 0. \quad (50)$$

This may further fix the parameters b_2, b_3, b_6, and b_7 in the effective action at order α' and the residual T-duality parameters at order α'^2. It would be interesting to perform these calculations in detail. It would be also interesting to extend the calculations in this paper which have no B-field to the case that the B-field is nonzero. That calculation would produce the B-field couplings at order α'^3, i.e., the extension of (38) to order α'^3, which is not known in the literature. The T-duality transformations at order α' in the presence of a B-field have been found in Refs. [21,22].

ACKNOWLEDGMENTS

This work is supported by Ferdowsi University of Mashhad under Grant No. 2/45608(1396/10/05).

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