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On the C.W.T on homogeneous spaces associated to Quasi invariant measure

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Abstract
Let G be a locally compact group and H be a closed subgroup of G. Consider the homogeneous space G/H equipped with a strongly quasi invariant measure. We introduce a square integrable representation of G/H and indicate the resolution of identity and among other things, we find the reproducing kernel Hilbert space.

Keywords Homogeneous space · Strongly quasi invariant measure · Square integrable representation · Admissible wavelet · Continuous wavelet transform · Reproducing kernel Hilbert space

Mathematics Subject Classification Primary 43A15; Secondary 43A85 · 65T60

1 Introduction and preliminaries

The study of wavelet transform has been an active area of harmonic analysis on the one hand, as well as physics, engineering, computer science, signal and image processing, on the other hand. Due to the wide variety of applications, there has been a great influx of researchers into the subject in different approaches [1,4,6]. In [2] the authors have studied continuous wavelet transform on homogeneous spaces possess a relatively invariant measure. Note that although generally it does not the case that any homogeneous space G/H possesses a relatively or G-invariant measure, but any such spaces has a strongly quasi invariant measure arises from a rho function and that all such measures are constructed in this manner. In this manuscript we study the con-

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tinuous wavelet transform on homogeneous space G/H considering with a strongly quasi invariant measure.

To be more precise, let G be a locally compact group, H be a closed subgroup of G. Consider G/H as a homogeneous space on which G acts from the left on it and $q : G \to G/H$ is the canonical quotient map. There is a useful linear map from $C_c(G)$ to $C_c(G/H)$. For $f \in C_c(G)$, there is a unique function $f^\sharp$ in $C_c(G/H)$ such that

$$f^\sharp(xH) = \int_H f(xh)dh,$$

for all $x \in G$. A rho-function for (G, H) is a non-negative locally integrable function ρ on G which satisfies

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g),$$

for $g \in G$, $h \in H$, where Δ_G and Δ_H are modular functions on G and H, respectively. A Radon measure ν on G/H is called strongly quasi- invariant measure if $\nu \neq 0$ and $\nu_g \sim \nu$, for every $g \in G$, where $\nu_g(E) = \nu(gE)$, for Borel subset E of G/H and $\sim$ denotes mutual absolute continuity of measures. For homogeneous space G/H, there exists a strongly quasi-invariant regular Borel measure ν such that the Radon–Nikodym derivative

$$\lambda(g, \omega) = \frac{d\nu_g}{d\nu}(\omega), \tag{1.1}$$

for $g \in G, \omega \in G/H$, is a continuous function on $G \times G/H$. The function $\lambda(g, \omega)$ is also satisfied

$$\lambda(g_1, q(g_2)) = \frac{\rho(g_1 g_2)}{\rho(g_2)} \tag{1.2}$$

in which ρ is a positive continuous rho function for the pair (G, H). It is also satisfies the identity

$$\lambda(g_1 g_2, \omega) = \lambda(g_1, g_2\omega)\lambda(g_2, \omega), \tag{1.3}$$

for $\omega \in G/H$, $g_1, g_2 \in G$ [3,5].

In this paper we mainly consider the continuous wavelet transform on a homogeneous space G/H with a strongly quasi invariant measure ν. Also in this case a reconstruction formula is established and as a result it is shown that the set of admissible vectors is path connected. Finally, we show that the range of this continuous wavelet transform in the setting is a reproducing kernel Hilbert space.

2 Main result

Throughout this section let G be a locally compact group and μ be the left Haar measure on it, H be a closed subgroup of G such that $\Delta_G|_H = \Delta_H$ and G/H be the corresponding homogeneous space with a strongly quasi invariant measure ν. Let ϖ be a unitary irreducible representation of G/H on a Hilbert space $\mathfrak{H}$ (see [2]). For any two vectors ϕ, ψ in $\mathfrak{H}$ their scalar product $< \phi|\psi >$ will be taken to be antilinear in the first variable ϕ and linear in the second variable ψ.

The following definition is a possible extension of an admissible wavelet for homogeneous space G/H equip with a strongly quasi invariant measure.

Definition 2.1 Let ϖ be a unitary irreducible representation of G/H on a Hilbert space $\mathfrak{H}$. A vector $\zeta \in \mathfrak{H}$ is said to be admissible if

$$I(\zeta) = \int_{G/H} \lambda(g^{-1}, gH)| < \varpi(gH)\zeta|\zeta > |^2 d\nu(gH) < \infty, \tag{2.1}$$

in which $\lambda(., gH)$ defined in (1.1) and ν is a strongly quasi invariant measure on G/H. Also, the irreducible representation ϖ is called square integrable. For admissible nonzero vector ζ in $\mathfrak{H}$, we shall write $c(\zeta) = \frac{I(\zeta)}{\|\zeta\|^2}$.

Note that, Definition 2.1 is well-defined. Indeed, by (1.3) we have

$$\lambda((gh)^{-1}, ghH) = \lambda(h^{-1}g^{-1}, ghH) = \lambda(g^{-1}, gH)\lambda(h^{-1}, hH),$$

on the other hand, using (1.2) and $\Delta_G|_H = \Delta_H$, we get $\lambda(h^{-1}, hH) = 1$. So, $\lambda((gh)^{-1}, ghH) = \lambda(g^{-1}, gH)$.

We intend to show that the reconstruction formula holds in the setting. At the first we prove the Lemma 2.2 which is needed in the Theorem 2.3.

Lemma 2.2 *Let ζ be an admissible vector and ϖ be a square integrable representation of G/H on Hilbert $\mathfrak{H}$. Denote,*

$$\mathfrak{A} = \{\phi \in \mathfrak{H};\ \int_{G/H} \lambda(g^{-1}, gH)| < \varpi(gH)\zeta|\phi > |^2 d\nu(gH) < \infty\}.$$

Then $\mathfrak{A}$ is invariant under ϖ.

Proof for $\phi \in \mathfrak{H}$ and $k \in G$ we have,

$$\begin{aligned}
&\int_{G/H} \lambda(g^{-1}, gH)| < \varpi(gH)\zeta|\varpi(kH)\phi > |^2 d\nu(gH) \\
&= \int_{G/H} \lambda(g^{-1}, gH)| < \varpi(k^{-1}gH)\zeta|\phi > |^2 d\nu(gH) \\
&= \int_{G/H} \lambda(g^{-1}k^{-1}, kgH)| < \varpi(gH)\zeta|\phi > |^2 d\nu_k(gH)
\end{aligned}$$

$$= \int_{G/H} \frac{\lambda(g^{-1}, gH)}{\lambda(k, gH)} | < \varpi(gH)\zeta|\phi > |^2 \lambda(k, gH) d\nu(gH)$$
$$= \int_{G/H} \lambda(g^{-1}, gH)| < \varpi(gH)\zeta|\phi > |^2 d\nu(gH).$$

Thus $\mathfrak{A}$ is invariant under ϖ. □

Theorem 2.3 *Let ϖ be a square integrable representation of G/H on $\mathfrak{H}$. For any admissible vector $\zeta \in \mathfrak{H}$, the mapping $w_\zeta : \mathfrak{H} \to L^2(G/H, \nu)$ given by $w_\zeta(\phi)(gH) = (\frac{\lambda(g^{-1},gH)}{c(\zeta)})^{1/2} < \zeta_{gH}|\phi >$, for $\phi \in \mathfrak{H}$, $g \in G$, in which $\zeta_{gH} = \varpi(gH)\zeta$, is a linear isometry onto a closed subspace $\mathfrak{H}_\zeta$ of $L^2(G/H, \nu)$ on $\mathfrak{H}$. Also the resolution of identity*

$$< \phi|\psi >= \frac{\lambda(g^{-1}, gH)}{c(\zeta)} \int_{G/H} < \zeta_{gH}|\phi > \overline{< \zeta_{gH}|\psi >} d\nu(gH), \qquad (2.2)$$

holds.

Proof The domain $D(w_\zeta)$ of w_ζ is the set of all vectors $\phi \in \mathfrak{H}$ such that $\frac{1}{c(\zeta)} \int_{G/H} \lambda(g^{-1}, gH)| < \zeta_{gH}|\phi > |^2 d\nu(gH) < \infty$. In Lemma 2.2 has been shown $D(w_\zeta)$ is invariant under ϖ and hence dense in $\mathfrak{H}$, since ϖ is irreducible. Moreover, w_ζ intertwines ϖ and the left type regular representation $\varrho_g : G \to U(L^2(G/H, \upsilon))$ defined by $\varrho_g \Phi(kH) = \lambda(g^{-1}, kH)^{1/2} \Phi(g^{-1}kH)$. Indeed,

$$\begin{aligned}
&\varrho_g(w_\zeta\phi)(kH) = \lambda(g^{-1}, kH)^{1/2} w_\zeta\phi(g^{-1}kH) \\
&= \frac{1}{c(\zeta)^{1/2}} \lambda(g^{-1}, kH)^{1/2} \lambda(g^{-1}k)^{-1}, g^{-1}kH)^{1/2} < \zeta_{g^{-1}kH}|\phi > \\
&= \frac{1}{c(\zeta)^{1/2}} \lambda(g^{-1}, kH)^{1/2} \lambda(k^{-1}g, g^{-1}kH)^{1/2} < \zeta_{g^{-1}kH}|\phi > \\
&= \frac{1}{c(\zeta)^{1/2}} \lambda(g^{-1}, kH)^{1/2} \left(\frac{\lambda(k^{-1}, kH)}{\lambda(g^{-1}, kH)}\right)^{1/2} < \zeta_{g^{-1}kH}|\phi > \\
&= \frac{1}{c(\zeta)^{1/2}} \lambda(k^{-1}, kH)^{1/2} < \zeta_{g^{-1}kH}|\phi > \\
&= \frac{1}{c(\zeta)^{1/2}} \lambda(k^{-1}, kH)^{1/2} < \zeta_{kH}|\varpi(gH)\phi > \\
&= w_\zeta(\varpi(gH)\phi)(kH).
\end{aligned}$$

We prove next that, as a linear map w_ζ is closed. Let $\{\phi_n\}_{n=1}^{n=\infty} \subseteq D(w_\zeta)$ be a sequence converging to $\phi \in \mathfrak{H}$ and let the corresponding sequence $\{w_\zeta\phi_n\}_{n=1}^{n=\infty} \subseteq L^2(G/H, \upsilon)$ converging to $\Phi \in L^2(G/H, \upsilon)$. Then by the continuity of the scalar product in $\mathfrak{H}$,

$$lim_{n\to\infty} w_\zeta\phi_n(gH) = lim_{n\to\infty} \left(\frac{\lambda(g^{-1}, gH)}{c(\zeta)}\right)^{1/2} < \zeta_{gH}|\phi_n >$$

$$= \left(\frac{\lambda(g^{-1}, gH)}{c(\zeta)} \right)^{1/2} < \zeta_{gH} | \phi > .$$

Thus, since $w_\zeta \phi_n \to \Phi$ in $L^2(G/H, \upsilon)$ and $w_\zeta \phi_n(gH) \to (\frac{\lambda(g^{-1},gH)}{c(\zeta)})^{1/2} < \zeta_{gH}|\phi >$, pointwise. So, $(\frac{\lambda(g^{-1},gH)}{c(\zeta)})^{1/2} < \zeta_{gH}|\phi >= \Phi$, almost every where. Whence,

$$\frac{1}{c(\zeta)} \int_{G/H} \lambda(g^{-1}, gH)| < \zeta_{gH}|\phi > |^2 d\nu(gH) < \infty,$$

implying that $\phi \in D(w_\zeta)$ and $w_\zeta \phi = \Phi$, i.e w_ζ is closed. Using the extended Schur's lemma on homogeneous spaces [2], we establish the boundedness of $w_\zeta : D(w_\zeta) \to L^2(G/H, \upsilon)$. Hence $D(w_\zeta) = \mathfrak{H}$ and furthermore w_ζ is a multiple of an isometry,

$$\|w_\zeta \phi\|^2 = \alpha \|\phi\|^2,$$

where $\alpha \in \mathbb{R}^+$. To fix α, take $\phi = \zeta$. Then

$$\alpha = \frac{\|w_\zeta \phi\|^2}{\|\phi\|^2} = \frac{I(\zeta)}{c(\zeta)\|\phi\|^2} = 1.$$

Thus, w_ζ is an isometry, i.e. $w_\zeta^* w_\zeta = I$, which implies that (2.2) holds. □

As an interesting consequence of Theorem 2.3, we conclude that:

Corollary 2.4 *Let ζ be a non-zero vector in $\mathfrak{H}$. The vector ζ in $\mathfrak{H}$ is admissible if and only if*

$$\int_{G/H} \lambda(g^{-1}, gH)| < \zeta_{gH}|\phi > |^2 d\nu(gH) < \infty,$$

for every ϕ in $\mathfrak{H}$

Corollary 2.5 *The set of admissible vectors is vector subspace of $\mathfrak{H}$.*

Now, we define the continuous wavelet transform on homogeneous space G/H with respect to strongly quasi invariant measure ν.

Definition 2.6 Let ϖ be a representation of G/H on Hilbert space $\mathfrak{H}$ and ζ be an admissible wavelet for ϖ. We define the continuous wavelet transform associated to the admissible wavelet ζ as the linear operator $w_\zeta : \mathfrak{H} \to \mathfrak{C}(G/H)$ defined by

$$w_\zeta \phi(gH) = \left(\frac{\lambda(g^{-1}, gH)}{c(\zeta)} \right)^{1/2} < \zeta_{gH}|\phi >, \tag{2.3}$$

for all $\phi \in \mathfrak{H}$, $g \in G$, where $c(\zeta)$ is the wavelet constant associated to ζ as in (2.1). Note that if ϖ is a square integrable representation of G/H on $\mathfrak{H}$ and ζ is an admissible wavelet for ϖ, then w_ζ is a bounded linear operator from $\mathfrak{H}$ into $L^2(G/H, \nu)$. Note that ν is a strongly quasi invariant measure on G/H.

As an application of Theorem 2.3 we can easily conclude that the continuous wavelet transform is an isometry. Furthermore, it turn out that the range of the continuous wavelet transform is a Hilbert space.

Corollary 2.7 *Let ϖ be a square integrable representation of G/H on $\mathfrak{H}$. Then the range of continuous wavelet transform is a Hilbert space.*

At this point we recall that *a reproducing kernel* on a measure space (B, μ) is a function k from $B \times B$ to $\mathbb{C}$ such that

$$(i)\ \ k(a,a) > 0 \tag{2.4}$$
$$(ii)\ \ k(a,b) = \overline{k(b,a)} \tag{2.5}$$
$$(iii)\ \ k(a,c) = \int_B k(a,\tau)k(\tau,c)d\mu(\tau) \tag{2.6}$$

for all $a, b, c \in B$. A Hilbert space $\mathcal{H}$ of functions on a measure space B is called a *reproducing kernel Hilbert space* if there exists a reproducing kernel k such that

$$x(a) = \int_B x(b)k(a,b)d\mu(b), \tag{2.7}$$

for all $x \in \mathcal{H}, a \in B$.

We are going to show that the range of the continuous wavelet transform defined in (2.3) is a reproducing kernel Hilbert space. In fact, we show first in the following lemma that the function defined by

$$\tilde{\mathcal{K}}_\zeta(gH, kH) = \sqrt{\lambda(g^{-1}, kH)\lambda(g^{-1}, gH)}\mathcal{K}_\zeta(g^{-1}kH), \tag{2.8}$$

in which, $\mathcal{K}_\zeta(gH) = \frac{(\lambda(g^{-1}, gH))^{1/2}}{c(\zeta)} < \zeta_{gH}|\zeta >$ is a reproducing kernel.

Proposition 2.8 *Let ϖ be a square integrable representation of G/H on $\mathfrak{H}$ and ζ be an admissible wavelet for ϖ. Then $\tilde{\mathcal{K}}_\zeta$ defined as (2.8) is a reproducing kernel.*

Proof It is clear that $\tilde{\mathcal{K}}_\zeta(gH, gH) > 0$. Now we show that $\tilde{\mathcal{K}}_\zeta(gH, kH) = \overline{\tilde{\mathcal{K}}_\zeta(kH, gH)}$.

$$\begin{aligned}
&\tilde{\mathcal{K}}_\zeta(gH, kH)\\
&= \frac{1}{c(\zeta)}\lambda(g^{-1}, kH)^{1/2}\lambda(g^{-1}, gH)^{1/2}\lambda((g^{-1}k)^{-1}, g^{-1}kH)^{1/2} < \zeta_{g^{-1}kH}|\zeta >\\
&= \frac{1}{c(\zeta)}\lambda(g^{-1}, kH)^{1/2}\lambda(g^{-1}, kk^{-1}gH)^{1/2}\frac{\lambda(k^{-1}, kH)^{1/2}}{\lambda(g^{-1}, kH)^{1/2}} < \zeta_{kH}|\zeta_{gH} >\\
&= \frac{1}{c(\zeta)}\frac{\lambda(g^{-1}k, k^{-1}gH)^{1/2}}{\lambda(k, k^{-1}gH)^{1/2}}\lambda(k^{-1}, kH)^{1/2} < \zeta|\zeta_{k^{-1}gH} >
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{c(\zeta)} \frac{\lambda(g^{-1}k, k^{-1}gH)^{1/2}}{\frac{\lambda(e,gH)^{1/2}}{\lambda(k^{-1},gH)^{1/2}}} \lambda(k^{-1}, kH)^{1/2} \overline{< \zeta_{k^{-1}gH}|\zeta >} \\
&= \frac{1}{c(\zeta)} \lambda(k^{-1}, gH)^{1/2} \lambda(k^{-1}, kH)^{1/2} \lambda(g^{-1}k, k^{-1}gH)^{1/2} \overline{< \zeta_{k^{-1}gH}|\zeta >} \\
&= \lambda(k^{-1}, gH)^{1/2} \lambda(k^{-1}, kH)^{1/2} \overline{\mathcal{K}_\zeta}(k^{-1}gH) \\
&= \overline{\tilde{\mathcal{K}}_\zeta(kH, gH)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\int_{G/H} \tilde{\mathcal{K}}_\zeta(gH, tH) \tilde{\mathcal{K}}_\zeta(tH, kH) d\nu(tH) \\
&= \frac{1}{c(\zeta)^2} \int_{G/H} \lambda(g^{-1}, tH)^{1/2} \lambda(g^{-1}, gH)^{1/2} \lambda(t^{-1}g, g^{-1}tH)^{1/2} < \zeta_{g^{-1}tH}|\zeta > \\
&\lambda(t^{-1}, kH))^{1/2} \lambda(t^{-1}, tH)^{1/2} \lambda(k^{-1}t, t^{-1}kH)^{1/2} < \zeta_{t^{-1}kH}|\zeta > d\nu(tH) \\
&= \frac{1}{c(\zeta)^2} \int_{G/H} \lambda(g^{-1}, tH)^{1/2} \lambda(g^{-1}, gH)^{1/2} \frac{\lambda(t^{-1}, tH)^{1/2}}{\lambda(g^{-1}, tH))^{1/2}} < \zeta_{g^{-1}tH}|\zeta > \\
&\lambda(t^{-1}, kH)^{1/2} \lambda(t^{-1}, tH)^{1/2} \frac{\lambda(k^{-1}, kH)^{1/2}}{\lambda(t^{-1}, kH)^{1/2}} < \zeta_{t^{-1}kH}|\zeta > d\nu(tH) \\
&= \frac{1}{c(\zeta)^2} \int_{G/H} \lambda(g^{-1}, gH)^{1/2} \lambda(k^{-1}, kH)^{1/2} \lambda(t^{-1}, tH) \\
&< \zeta_{tH}|\zeta_{gH} >< \zeta_{kH}|\zeta_{tH} > d\nu(tH) \\
&= \frac{1}{c(\zeta)} \lambda(g^{-1}, gH)^{1/2} \lambda(k^{-1}, kH)^{1/2} \frac{\lambda(g^{-1}, kH)^{1/2}}{\lambda(g^{-1}, kH)^{1/2}} < \zeta_{kH}|\zeta_{gH} > \\
&= \frac{1}{c(\zeta)} \lambda(g^{-1}, gH)^{1/2} \lambda(g^{-1}, kH)^{1/2} \lambda(k^{-1}g, g^{-1}kH)^{1/2} < \zeta_{g^{-1}kH}|\zeta > \\
&= \lambda(g^{-1}, gH)^{1/2} \lambda(g^{-1}, kH)^{1/2} \mathcal{K}_\zeta g^{-1}kH)) \\
&= \tilde{\mathcal{K}}_\zeta(gH, kH).
\end{aligned}
$$

$\square$

Theorem 2.9 *Let ϖ be a square integrable representation of G/H on $\mathfrak{H}$ and ζ be an admissible wavelet for ϖ. Then range of the continuous wavelet transform is a reproducing kernel Hilbert space with reproducing kernel $\tilde{\mathcal{K}}$ as defined in (2.8) such that*

$$
\Psi(gH) = \int_{G/H} \Psi(kH) \tilde{\mathcal{K}}_\zeta(gH, kH) d\nu(kH), \tag{2.9}
$$

for all Ψ in the range of the continuous wavelet transform and $g \in G$.

Proof By Proposition 2.8, it is enough to show that (2.9). Let $\Psi \in Range(w_\zeta)$. Then there exists an element $\phi \in \mathfrak{H}$ such that $\Psi = w_\zeta \phi$. Using reconstruction formula we get

$$
\begin{aligned}
&\int_{G/H} \Psi(kH)\tilde{\mathcal{K}}_\zeta(gH, kH)d\nu(kH) \\
&= \int_{G/H} \Psi(kH)\lambda(g^{-1}, kH)^{1/2}\lambda(g^{-1}, gH)^{1/2}\mathcal{K}_\zeta(g^{-1}kH)d\nu(kH) \\
&= \int_{G/H} \Psi(gkH)\lambda(g^{-1}, gkH)^{1/2}\lambda(g^{-1}, gH)^{1/2}\mathcal{K}_\zeta(kH)\lambda(g, kH)d\nu(kH) \\
&= \frac{1}{c(\zeta)}\int_{G/H} \Psi(gkH)\frac{\lambda(e, kH)^{1/2}}{\lambda(g, kH)^{1/2}}\lambda(g^{-1}, gH)^{1/2}\lambda(k^{-1}, kH)^{1/2} \\
&< \zeta_{kH}|\zeta > \lambda(g, kH)d\nu(kH) \\
&= \frac{1}{c(\zeta)}\int_{G/H} \Psi(gkH)\lambda(g^{-1}, gH)^{1/2}\lambda(k^{-1}, kH)^{1/2}\lambda(g, kH)^{1/2} \\
&< \zeta_{kH}|\zeta > d\nu(kH) \\
&= \frac{1}{c(\zeta)}\int_{G/H} w_\zeta\phi(gkH)\lambda(g^{-1}, gH)^{1/2}\lambda(k^{-1}, kH)^{1/2}\lambda(g, kH)^{1/2} \\
&< \zeta_{kH}|\zeta > d\nu(kH) \\
&= \frac{1}{c(\zeta)^{3/2}}\int_{G/H} \lambda(k^{-1}g^{-1}, gkH)^{1/2} < \zeta_{gkH}|\phi > . \\
&\lambda(g^{-1}, gH)^{1/2}\lambda(k^{-1}, kH)^{1/2}\lambda(g, kH)^{1/2} < \zeta_{kH}|\zeta > d\nu(kH) \\
&= \frac{1}{c(\zeta)^{3/2}}\int_{G/H} \frac{\lambda(k^{-1}, kH)^{1/2}}{\lambda(g, kH)^{1/2}} < \zeta_{kH}|\zeta_{g^{-1}H}\phi > . \\
&\lambda(g^{-1}, gH)^{1/2}\lambda(k^{-1}, kH)^{1/2}\lambda(g, kH)^{1/2} < \zeta_{kH}|\zeta > d\nu(kH) \\
&= \frac{1}{c(\zeta)^{1/2}}\lambda(g^{-1}, gH)^{1/2} < \varpi(g^{-1}H)\phi|\zeta > \\
&= w_\zeta\phi(gH) \\
&= \Psi(gH).
\end{aligned}
$$

□

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