Quantum Fisher information for unitary processes

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Quantum Fisher information for unitary processes

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Quantum Fisher information plays a central role in the field of quantum metrology. In this paper we study the problem of quantum Fisher information of unitary processes. Associated to each parameter θ_i of unitary process $U(\boldsymbol{\theta})$, there exists a unique Hermitian matrix $M_{\theta_i} = i(U^\dagger \partial_{\theta_i} U)$. Except for some simple cases, such as when the parameter under estimation is an overall multiplicative factor in the Hamiltonian, calculation of these matrices is not an easy task to treat even for estimating a single parameter of qubit systems. Using the Bloch vector $\boldsymbol{m}_{\theta_i}$, corresponding to each matrix M_{θ_i}, we find a closed relation for the quantum Fisher information matrix of the qubit systems for an arbitrary number of estimation parameters. We extend our results and present an explicit relation for each vector $\boldsymbol{m}_{\theta_i}$ for a general Hamiltonian with arbitrary parametrization. We illustrate our results by obtaining the quantum Fisher matrix of the so-called angle-axis parameters of a general $SU(2)$ Hamiltonian. Using a linear transformation between two different parameter spaces of a unitary process, we provide a way to move from quantum Fisher information of a unitary process in a given parametrization to the one of the other parametrization. Knowing this linear transformation enables one to calculate the quantum Fisher information of a composite unitary process, i.e. a unitary process resulted from successive action of some simple unitary processes. We apply this method for a spin-half system and obtain the quantum Fisher matrix of the coset parameters in terms of the one of the angle-axis parameters.

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I. INTRODUCTION

Estimation theory is an important topic in different areas of physics. Quantum metrology trying to improve estimation precision by using quantum strategy such as entanglement [1–3] and discord [4–6]. Many applications of quantum metrology have been found, such as gravitational radiation [7–9], quantum frequency standards [10–12], quantum imaging [13–15], atomic clocks [16–20]. Estimation precision in quantum metrology is described by the Cramer-Rao inequality [21–26]

$$\delta\theta = \sqrt{\mathrm{Var}(\hat{\theta})} \geq \frac{1}{\sqrt{N F_\theta}}, \tag{1}$$

where lower bound is related to the inverse of the quantum Fisher information (QFI). The estimation precision for separable states is bounded by the standard quantum limit $\Delta\theta \sim \frac{1}{\sqrt{N}}$, however for the maximally entangled states, GHZ and NOON states, it is bounded by the Heisenberg limit $\Delta\theta \sim \frac{1}{N}$ [27–29]. In general, there are three stages in quantum metrology: the first is the preparation of the input state, the so-called prob state. In the second stage the input state is encoded with an unknown parameter θ. Encoding can occur in a noisy [30–42] or noiseless scenario. Finally, the third stage is information extraction, carried out by measuring on the output states. Fisher information is at the heart of metrology and gives us knowledge about the unknown parameters from the probability distribution. It can be obtained directly from its definition $F_\theta = \sum_x P_\theta(x)[\partial_\theta \ln P_\theta(x)]^2$, for discrete outcomes x [43], where $P_\theta(x)$ is probability distribution obtained by measuring the encoded prob states. The maximum of F_θ over all possible measurements is the so-called quantum Fisher information. A further optimization expression of the QFI for parameter θ is given as [44–46]

$$F_\theta = \sum_{i=0}^{s-1} \frac{(\partial_\theta p_i)^2}{p_i} + \sum_{i=1}^{S} p_i F_{\theta,i} - \sum_{i\neq j} \frac{8 p_i p_j}{p_i + p_j} |\langle \varphi_i | \partial_\theta \varphi_j \rangle|^2,$$

where $F_{\theta,i} = 4\left(\langle \partial_\theta \varphi_i | \partial_\theta \varphi_i \rangle - |\langle \varphi_i | \partial_\theta \varphi_i \rangle|^2\right)$. Quantum Fisher information is related to the Bures [23, 47–49] and Hellinger [50] distances which are referred to as two different extensions from classical Fisher information. For a general unitary transformation $\rho_\theta = U(\theta)\rho_0 U^\dagger(\theta)$, the QFI can be expressed by [46, 51]

$$F_\theta = \sum_{i=0}^{s-1} 4 p_i \langle \Delta^2 \mathcal{H} \rangle_i - \sum_{i\neq j} \frac{8 p_i p_j}{p_i + p_j} |\langle \varphi_i | \mathcal{H} | \varphi_j \rangle|^2, \tag{2}$$

where $\mathcal{H} := i(\partial_\theta U^\dagger)U$ is a Hermitian operator [52, 53], can be expressed by [54]

$$\mathcal{H} = i \sum_{n=0}^{\infty} \mathcal{F}_n H_\theta^{\times n} (\partial_\theta H_\theta), \tag{3}$$

and the coefficients $\mathcal{F}_n$ are defined by

$$\mathcal{F}_n = \frac{(it)^{n+1}}{(n+1)!}.$$

Equation (2) provides a representation for the QFI of the parameter θ of a unitary process, however, its calculation requires calculation of, in general, the infinite series of Eq. (3) which is not an easy task to treat even for the simplest case of qubit systems. In this paper we consider the QFI of a unitary process, and provide a new representation for QFI of a general $SU(2)$ process. In this representation we associate to each parameter θ_i a

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real vectors $\boldsymbol{m}_{\theta_i} \in \mathbb{R}^3$. The formulation is independent of the parametrization of the process in a sense that it takes a covariant form for any parametrization of the process. We then provide an explicit relation for the vectors $\boldsymbol{m}_{\theta_i}$ for a general Hamiltonian with arbitrary parametrization. Furthermore, we present a linear transformation between two different parameter spaces of a unitary process, enables us to interplay between their corresponding QFI matrices. Using this linear transformation, one can go from either parametrization to another one, in particular, one can obtain the QFI of the coset parameters in terms of the one of the angle-axis parameters.

This paper is organized as follows. In section II, we have a brief review on the QFI and provide a representation for the QFI of a general unitary process in terms of the matrices $\boldsymbol{M}_{\theta_i}$. Section III is devoted to the QFI of unitary process of qubit systems. In this section we introduce vectors $\boldsymbol{m}_{\theta_i}$, associated with matrices $\boldsymbol{M}_{\theta_i}$, and present a closed relation for QFI matrix in terms of these vectors. An analytical closed relation to evaluate these vectors for general Hamiltonian and arbitrary estimation parameters is also provided in this section. In section IV, we present a linear transformation between two different parameter spaces of a unitary process and provide a way to move from QFI of a unitary process in a given parametrization to the one of the other parametrization. The utility of this transformation is examined by providing an example in qubit systems. The paper is concluded in section V with a brief discussion.

II. QUANTUM FISHER INFORMATION

From various different versions of QFI, the so-called symmetric logarithmic derivative (SLD) Fisher information is the one which has attracted much attention. For a single parameter θ, the SLD Fisher information is defined by [23–26]

$$F_\theta = \mathrm{Tr}\left(\rho_\theta L_\theta^2\right), \tag{4}$$

where ρ_θ is the density matrix depending on θ, and L_θ is the SLD operator determined by the equation

$$\partial_\theta \rho_\theta = \frac{1}{2}\left\{L_\theta, \rho_\theta\right\}, \tag{5}$$

where $\{,\}$ denotes anticommutator. For a multiparameter scenario $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_n\}$, the quantum Fisher information matrix is defined by

$$F_{\theta_i\theta_j} = \frac{1}{2}\mathrm{Tr}\left(\rho\left\{L_{\theta_i}, L_{\theta_j}\right\}\right), \tag{6}$$

where L_{θ_i} is the SLD operator for the parameter θ_i, given by

$$\partial_{\theta_i}\rho_{\boldsymbol{\theta}} = \frac{1}{2}\left\{L_{\theta_i}, \rho_{\boldsymbol{\theta}}\right\}, \tag{7}$$

and L_{θ_j} is defined similarly.

Suppose the set of parameters $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_n\}$ is encoded in a quantum state $\rho_{\boldsymbol{\theta}}$ by the action of a unitary operator $U(\boldsymbol{\theta})$ on the initially $\boldsymbol{\theta}$-independent state ρ_0, i.e.

$$\rho_{\boldsymbol{\theta}} = U\left(\boldsymbol{\theta}\right)\rho_0 U^\dagger\left(\boldsymbol{\theta}\right). \tag{8}$$

By taking derivative of the above equation with respect to θ_i we get $\partial_{\theta_i}\rho_{\boldsymbol{\theta}} = \frac{\partial U(\boldsymbol{\theta})}{\partial\theta_i}\rho_0 U^\dagger(\boldsymbol{\theta}) + U(\boldsymbol{\theta})\rho_0\frac{\partial U^\dagger(\boldsymbol{\theta})}{\partial\theta_i}$ which can be written as

$$\partial_{\theta_i}\rho_{\boldsymbol{\theta}} = -i[K_{\theta_i}, \rho_{\boldsymbol{\theta}}], \tag{9}$$

where the Hermitian matrices K_{θ_i}, $i = 1, \cdots, n$, are defined by [52]

$$K_{\theta_i} = i\frac{\partial U(\boldsymbol{\theta})}{\partial\theta_i}U^\dagger(\boldsymbol{\theta}). \tag{10}$$

Equation (9), similar to the time-evolution Schrodinger equation, describes the θ_i-evolution of $\rho_{\boldsymbol{\theta}}$ generated by the Hermitian operator K_{θ_i} for $i = 1, \cdots, n$. Moreover, corresponding to each K_{θ_i} we can define

$$M_{\theta_i} = iU^\dagger(\boldsymbol{\theta})\frac{\partial U(\boldsymbol{\theta})}{\partial\theta_i}, \tag{11}$$

where are related to each other by unitary operator $U(\boldsymbol{\theta})$ in the sense of $M_{\theta_i} = U^\dagger(\boldsymbol{\theta})K_{\theta_i}U(\boldsymbol{\theta})$. Note that using $\frac{\partial(U^\dagger(\boldsymbol{\theta})U(\boldsymbol{\theta}))}{\partial\theta_i} = 0$, one can see that $M_\theta = -\mathcal{H}$, where $\mathcal{H}$ is given in Eq. (2).

Using the eigenspectral of ρ_0 as

$$\rho_0 = \sum_{a=0}^{s-1} p_a|\varphi_a\rangle\langle\varphi_a|, \tag{12}$$

where $\{p_a\}_{a=0}^{s-1}$ and $\{|\varphi_a\rangle\}_{a=0}^{s-1}$ are the set of eigenvalues and eigenvectors of ρ_0, respectively, and s is the dimension of the support of ρ_0, one can write the eigenspectral of $\rho_{\boldsymbol{\theta}}$ as

$$\rho_{\boldsymbol{\theta}} = \sum_{a=0}^{s-1} p_a|\tilde{\varphi}_a\rangle\langle\tilde{\varphi}_a|, \tag{13}$$

where $|\tilde{\varphi}_a\rangle = U\left(\boldsymbol{\theta}\right)|\varphi_a\rangle$ denotes eigenvectors of $\rho_{\boldsymbol{\theta}}$. In this basis Eqs. (7) and (9) read, respectively, as (for $i = 1, \cdots, n$)

$$\begin{aligned}(\partial_{\theta_i}\rho_{\boldsymbol{\theta}})_{\tilde{a}\tilde{b}} &= \frac{1}{2}(p_a + p_b)\left(L_{\theta_i}\right)_{\tilde{a}\tilde{b}} \\ &= i(p_a - p_b)\left(K_{\theta_i}\right)_{\tilde{a}\tilde{b}},\end{aligned}$$

where we have defined $(L_{\theta_i})_{\tilde{a}\tilde{b}} = \langle\tilde{\varphi}_a|L_{\theta_i}|\tilde{\varphi}_b\rangle$ and $(K_{\theta_i})_{\tilde{a}\tilde{b}} = \langle\tilde{\varphi}_a|K_{\theta_i}|\tilde{\varphi}_b\rangle$. Using this, one can find the matrix elements of the SLD operators in the $\boldsymbol{\theta}$-parametrization in terms of the matrix elements of the corresponding matrices K_{θ_i} as

$$(L_{\theta_i})_{\tilde{a}\tilde{b}} = 2i\frac{(p_a - p_b)}{p_a + p_b}(K_{\theta_i})_{\tilde{a}\tilde{b}}. \tag{14}$$

This can be used in Eq. (6) to find matrix elements of the QFI matrix in the $\boldsymbol{\theta}$-representation as

$$\begin{aligned} F_{\theta_i\theta_j} &= \sum_a \sum_b 2\frac{(p_a-p_b)^2}{p_a+p_b}(K_{\theta_i})_{\tilde{a}\tilde{b}}(K_{\theta_j})_{\tilde{b}\tilde{a}} \\ &= \sum_a \sum_b 2\frac{(p_a-p_b)^2}{p_a+p_b}(M_{\theta_i})_{ab}(M_{\theta_j})_{ba}, \end{aligned} \quad (15)$$

where $(M_{\theta_i})_{ab} = \langle\varphi_a|M_{\theta_i}|\varphi_b\rangle$. Equation (15) gives a closed relation for the QFI matrix of an arbitrary unitary process $U(\boldsymbol{\theta})$, provided that we could calculate the infinitesimal generators K_{θ_i} (or M_{θ_i}) associated to each parameter θ_i ($i = 1, \cdots, n$). This, however, is not an easy task to handle in general except for some special cases such as when the estimation parameters are $K_{\theta_i} = \theta_i K_i$. In the next section we concern our attention to qubit systems and provide a closed relation for the QFI of arbitrary parameters of a general Hamiltonian.

III. QFI OF SU(2) PROCESSES

For the simplest case of qubit systems we will provide a closed relation for Eq. (15) in terms of the Bloch vector representation of the M-matrices. To do so, first suppose that the initial state ρ_0 is diagonal in the computational basis $\{|0\rangle, |1\rangle\}$. In this case Eq. (15) reduces to

$$\begin{aligned} F_{\theta_i\theta_j} &= 2(p_0-p_1)^2\left((M_{\theta_i})_{01}(M_{\theta_j})_{10} + (M_{\theta_i})_{10}(M_{\theta_j})_{01}\right) \\ &= 2(p_0-p_1)^2\left(\mathrm{Tr}[M_{\theta_i}M_{\theta_j}] - 2(M_{\theta_i})_{00}(M_{\theta_j})_{00}\right) \end{aligned} \quad (16)$$

where in the last line we have used the fact that M-matrices are traceless. Now, to each Hermitian traceless 2×2 matrix M_{θ_i}, one can associate a real vector $\boldsymbol{m}_{\theta_i} \in \mathbb{R}^3$ by $M_{\theta_i} = \boldsymbol{\sigma}\cdot\boldsymbol{m}_{\theta_i}$. In this representation we have $(M_{\theta_i})_{00} = (\boldsymbol{\sigma}\cdot\boldsymbol{m}_{\theta_i})_{00} = \hat{\boldsymbol{z}}\cdot\boldsymbol{m}_{\theta_i}$ and $M_{\theta_i}M_{\theta_j} = (\boldsymbol{\sigma}\cdot\boldsymbol{m}_{\theta_i})(\boldsymbol{\sigma}\cdot\boldsymbol{m}_{\theta_j}) = (\boldsymbol{m}_{\theta_i}\cdot\boldsymbol{m}_{\theta_j})\mathbb{1}_2 + i\boldsymbol{\sigma}\cdot(\boldsymbol{m}_{\theta_i}\times\boldsymbol{m}_{\theta_j})$. We find

$$F_{\theta_i\theta_j} = 4(p_0-p_1)^2\left[\boldsymbol{m}_{\theta_i}\cdot\boldsymbol{m}_{\theta_j} - (\hat{\boldsymbol{z}}\cdot\boldsymbol{m}_{\theta_i})(\hat{\boldsymbol{z}}\cdot\boldsymbol{m}_{\theta_j})\right] \quad (17)$$

In general, however, we are interested in the QFI of the unitary process $U(\boldsymbol{\theta})$ starting from an arbitrary initial state ρ_0 with associated orthonormal eigenbasis $\{|\varphi_0\rangle, |\varphi_1\rangle\}$. To do this we define $|\varphi_a\rangle = \Omega(\theta,\phi)|a\rangle$, for $a = 0, 1$, with

$$\Omega(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & -\mathrm{e}^{-i\phi}\sin\frac{\theta}{2} \\ \mathrm{e}^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \quad (18)$$

One can easily show that $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are eigenvectors of $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ corresponding to the eigenvalues $+1$ and -1, respectively, where $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{t}}$. Starting from this initial state transforms the M-matrices as $M_{\theta_i} \to M_{\theta_i} = \Omega^\dagger(\theta,\phi)M_{\theta_i}\Omega(\theta,\phi)$. Associated to this unitary transformation the $\boldsymbol{m}$-vectors rotate as $\boldsymbol{m}_{\theta_i} \to O^{\mathrm{t}}\boldsymbol{m}_{\theta_i}$, where the orthogonal matrix O is defined by $O_{ij} = \frac{1}{2}\mathrm{Tr}[\sigma_i\Omega\sigma_j\Omega^\dagger]$. Obviously, $\boldsymbol{m}_{\theta_i}\cdot\boldsymbol{m}_{\theta_j}$ remains invariant under such transformation and that $\hat{\boldsymbol{z}}\cdot O^{\mathrm{t}}\boldsymbol{m}_{\theta_i} = O\hat{\boldsymbol{z}}\cdot\boldsymbol{m}_{\theta_i}$. Moreover, simple calculation shows that for the unitary transformation (18), $O\hat{\boldsymbol{z}}$ is nothing but the unit vector $\hat{\boldsymbol{n}}$ defined above. We therefore arrive at the following proposition for the QFI matrix of the unitary process $U(\boldsymbol{\theta}) \in SU(2)$.

Proposition 1 *To each parameter θ_i of the unitary process $U(\boldsymbol{\theta})$ one can associate a unique vector $\boldsymbol{m}_{\theta_i}$ defined by $[\boldsymbol{m}_{\theta_i}]_k = \frac{1}{2}\mathrm{Tr}\{\sigma_k M_{\theta_i}\}$, where M_{θ_i} is given by Eq. (11). Using this, the QFI matrix takes the following form*

$$\begin{aligned} F_{\theta_i\theta_j} &= 4(p_0-p_1)^2\left[\boldsymbol{m}_{\theta_i}^{\mathrm{t}}\Lambda_{\hat{\boldsymbol{n}}}\boldsymbol{m}_{\theta_j}\right] \\ &= 4(p_0-p_1)^2\left[\boldsymbol{m}_{\theta_i}\cdot\boldsymbol{m}_{\theta_j} - (\hat{\boldsymbol{n}}\cdot\boldsymbol{m}_{\theta_i})(\hat{\boldsymbol{n}}\cdot\boldsymbol{m}_{\theta_j})\right] \end{aligned} \quad (19)$$

where $\Lambda_{\hat{\boldsymbol{n}}} = \mathbb{1}_3 - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^{\mathrm{t}}$ is a two-dimensional projection operator orthogonal to $\hat{\boldsymbol{n}}$.

This simple form shows that the QFI of a unitary process is composed of two independent contributions; First, each parameter θ_i of the unitary process $U(\boldsymbol{\theta})$ is contributed in the Fisher information via the vector $\boldsymbol{m}_{\theta_i} \in \mathbb{R}^3$, and second, the role of the initial state ρ_0 is played by the Bloch vector $\hat{\boldsymbol{n}}$ and the eigenvalues p_0, p_1. However, looking at Eq. (19) shows that although $\boldsymbol{m}_{\theta_i}$ are vectors in $\mathbb{R}^3$, their role in the QFI matrix is played effectively in a two dimensional subspace perpendicular to $\hat{\boldsymbol{n}}$. To see this note that $\boldsymbol{m}_{\theta_i}^{\mathrm{t}}\Lambda_{\hat{\boldsymbol{n}}}\boldsymbol{m}_{\theta_j} = (\Lambda_{\hat{\boldsymbol{n}}}\boldsymbol{m}_{\theta_i})^{\mathrm{t}}(\Lambda_{\hat{\boldsymbol{n}}}\boldsymbol{m}_{\theta_j})$ and $\Lambda_{\hat{\boldsymbol{n}}}\boldsymbol{m}_{\theta_i} \in \mathrm{Range}\{\Lambda_{\hat{\boldsymbol{n}}}\}$. Accordingly, initial states with different Bloch vectors result in different subspaces, hence different QFI matrix, in general. It turns out that the QFI matrix is invariant under orthogonal transformation on the vectors $\boldsymbol{m}_{\theta_i}$, i.e. $\boldsymbol{m}_{\theta_i} \longrightarrow R\boldsymbol{m}_{\theta_i}$, provided that the Bloch vector of the initial state is changed as $\hat{\boldsymbol{n}} \longrightarrow R^{\mathrm{t}}\hat{\boldsymbol{n}}$. In view of this, the maximum of QFI matrix over all initial states, if exist, is invariant under orthogonal transformation performed on $\boldsymbol{m}_{\theta_i}$. For instance, for a single parameter θ, Eq. (19) reduces to $F_\theta = 4(p_0-p_1)^2\left[|\boldsymbol{m}_\theta|^2 - (\hat{\boldsymbol{n}}\cdot\boldsymbol{m}_\theta)^2\right]$, implies that the QFI attains its maximum value $F_\theta^{\max} = 4|\boldsymbol{m}_\theta|^2$, gained by any initial pure state $|\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\rangle$ with $\hat{\boldsymbol{n}}$ laying in the plane perpendicular to $\boldsymbol{m}_\theta$.

Another consequence of Eq. (19) is that the lack of independency of the vectors $\boldsymbol{m}_{\theta_i}$ leads to vanishing determinant of the QFI matrix, meaning that the variances of the set of parameters cannot be estimated simultaneously through the Cramer-Rao bound. Moreover, for the eigdecomposition of $\rho_{\boldsymbol{\theta}}$, given by Eq. (13), one can see that $F(\rho_{\boldsymbol{\theta}}) = (p_0-p_1)^2 F(\tilde{\varphi}_a)$ for $a = 0, 1$, as such $F(\rho_{\boldsymbol{\theta}})/\sum_{a=0}^{1} p_a F(\tilde{\varphi}_a) = (p_0-p_1)^2 \le 1$, implies convexity of the QFI in this case.

Having Eq. (19) as a relation for QFI matrix in terms of the vectors $\boldsymbol{m}_{\theta_i}$, it is now the time to present a relation to calculate the required vectors $\boldsymbol{m}_{\theta_i}$ for a general Hamiltonian. The following proposition provides an explicit representation for vectors $\boldsymbol{m}_{\theta_i}$ of a general qubit Hamiltonian.

Proposition 2 *For a unitary process generated by the Hamiltonian [55]*

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}, \quad (20)$$

the associated $\boldsymbol{m}$-vectors are given by the following relation

$$\boldsymbol{m}_{\theta_i} = \frac{\partial|\boldsymbol{\alpha}|}{\partial\theta_i} t\, \hat{\boldsymbol{\alpha}} + \sin\left(|\boldsymbol{\alpha}|t\right)\cos\left(|\boldsymbol{\alpha}|t\right)\frac{\partial\hat{\boldsymbol{\alpha}}}{\partial\theta_i} - \sin^2\left(|\boldsymbol{\alpha}|t\right)\left(\hat{\boldsymbol{\alpha}} \times \frac{\partial\hat{\boldsymbol{\alpha}}}{\partial\theta_i}\right), \tag{21}$$

where we have defined the unit vector $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/|\boldsymbol{\alpha}|$.

Before we proceed further to provide a proof for the above equation, we have to stress here that a similar relation, however with different derivation, is provided in Ref. [55].

Proof Using the equation [56]

$$\frac{\partial}{\partial\theta_i}\mathrm{e}^{-iHt} = \int_0^1 \mathrm{e}^{-isHt}\left(-it\frac{\partial}{\partial\theta_i}H\right)\mathrm{e}^{-i(1-s)Ht}\mathrm{d}s, \tag{22}$$

in the definition of $[\boldsymbol{m}_{\theta_i}]_k$, we get

$$\begin{aligned}[\boldsymbol{m}_{\theta_i}]_k &= \frac{i}{2}\mathrm{Tr}\left\{\sigma_k U^\dagger(\boldsymbol{\theta})\frac{\partial U(\boldsymbol{\theta})}{\partial\theta_i}\right\} \\ &= \frac{t}{2}\int_0^1 \mathrm{d}s\ \mathrm{Tr}\left\{V(s)\sigma_k V^\dagger(s)\frac{\partial H}{\partial\theta_i}\right\} \\ &= t\sum_{l=1}^3 \frac{\partial\alpha_l}{\partial\theta_i}\int_0^1 \mathrm{d}s\ O_{lk}(s), \end{aligned}\tag{23}$$

where we have defined $V(s) = \mathrm{e}^{-i(1-s)Ht}$, and $O_{lk}(s) = \frac{1}{2}\mathrm{Tr}\left\{V(s)\sigma_k V^\dagger(s)\sigma_l\right\}$ is the orthogonal matrix corresponding to the unitary matrix $V(s)$. Now, for $H = \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}$, one can use $V(s) = \mathbb{1}_2\cos[\tau] - i\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sin[\tau]$ with $\tau = (1-s)|\boldsymbol{\alpha}|t$, so that

$$O_{lk}(s) = \cos[2\tau]\delta_{kl} - \sin[2\tau]\varepsilon_{ktl}\hat{\alpha}_t + 2\sin^2[\tau]\hat{\alpha}_k\hat{\alpha}_l, \tag{24}$$

where ε_{ktl} is the so-called Levi-Civita symbol and summation over repeated indices is understood. Using this in Eq. (23) and after calculating the integrals, we get

$$\boldsymbol{m}_{\theta_i} = \frac{1}{2|\boldsymbol{\alpha}|}\left\{\sin(2|\boldsymbol{\alpha}|t)\frac{\partial\boldsymbol{\alpha}}{\partial\theta_i} - (1-\cos(2|\boldsymbol{\alpha}|t))\left(\hat{\boldsymbol{\alpha}}\times\frac{\partial\boldsymbol{\alpha}}{\partial\theta_i}\right) + \left[(2|\boldsymbol{\alpha}|t - \sin(2|\boldsymbol{\alpha}|t))\left(\hat{\boldsymbol{\alpha}}\cdot\frac{\partial\boldsymbol{\alpha}}{\partial\theta_i}\right)\right]\hat{\boldsymbol{\alpha}}\right\}. \tag{25}$$

Finally, using $\boldsymbol{\alpha} = |\boldsymbol{\alpha}|\hat{\boldsymbol{\alpha}}$ and noting that $\hat{\boldsymbol{\alpha}}\cdot\frac{\partial\hat{\boldsymbol{\alpha}}}{\partial\theta_i} = 0$, we find Eq. (21).

Note that in Eq. (21) we have not fixed the parameters under estimation, in a sense that both amplitude and direction of the vector $\boldsymbol{\alpha}$ can be depend on each of the parameters θ_i. Moreover, the $\boldsymbol{m}$-vectors are generally not orthogonal nor normalized. In the following we will consider the so-called angle-axis parametrization of the $SU(2)$ group and show that for such a set of parameters the associated $\boldsymbol{m}$-vectors are orthogonal.

*E*xample.—Consider a system described by the Hamiltonian (20), with $\boldsymbol{\alpha}$ described by the following relation [55]

$$\boldsymbol{\alpha} = r\hat{\boldsymbol{\alpha}}, \quad \hat{\boldsymbol{\alpha}} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)^{\mathrm{t}}. \tag{26}$$

The unitary evolution generated by this Hamiltonian is given by $U(r,\vartheta,\varphi) = \mathrm{e}^{-iHt}$. Taking $\theta_1 = r$, $\theta_2 = \vartheta$, and $\theta_3 = \varphi$ as the parameters under estimation, one can easily find from Eq. (21)

$$\boldsymbol{m}_r = t\hat{\boldsymbol{\alpha}}_0, \tag{27}$$

$$\boldsymbol{m}_\vartheta = \sin rt\left[\cos rt\ \hat{\boldsymbol{\alpha}}_1 - \sin rt\ \hat{\boldsymbol{\alpha}}_2\right], \tag{28}$$

$$\boldsymbol{m}_\varphi = \sin\vartheta\sin rt\left[\sin rt\ \hat{\boldsymbol{\alpha}}_1 + \cos rt\ \hat{\boldsymbol{\alpha}}_2\right], \tag{29}$$

where

$$\hat{\boldsymbol{\alpha}}_0 = \hat{\boldsymbol{\alpha}}, \qquad \hat{\boldsymbol{\alpha}}_1 = \frac{\partial\hat{\boldsymbol{\alpha}}}{\partial\vartheta}, \qquad \hat{\boldsymbol{\alpha}}_2 = \frac{1}{\sin\vartheta}\frac{\partial\hat{\boldsymbol{\alpha}}}{\partial\varphi}, \tag{30}$$

form an orthonormal basis. Clearly, such a set of $\boldsymbol{m}$-vectors is orthogonal and can be written as

$$\boldsymbol{m}_r = \mathcal{R}\mathfrak{m}_r, \quad \boldsymbol{m}_\vartheta = \mathcal{R}\mathfrak{m}_\vartheta, \quad \boldsymbol{m}_\varphi = \mathcal{R}\mathfrak{m}_\varphi, \tag{31}$$

with

$$\mathfrak{m}_r = t\,(0\ ,\ 0\ ,\ 1)^{\mathrm{t}}, \tag{32}$$

$$\mathfrak{m}_\vartheta = \sin rt\,(\cos rt\ ,\ -\sin rt\ ,\ 0\ )^{\mathrm{t}}, \tag{33}$$

$$\mathfrak{m}_\varphi = \sin\vartheta\sin rt\,(\sin rt\ ,\ \cos rt\ ,\ 0)^{\mathrm{t}}, \tag{34}$$

Above, $\mathcal{R} = \mathcal{R}_z(\varphi)\mathcal{R}_y(\vartheta)$ where $\mathcal{R}_z(\varphi)$ and $\mathcal{R}_y(\vartheta)$ denote rotations about z and y axes, respectively

$$\mathcal{R}_z(\varphi) = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{35}$$

$$\mathcal{R}_y(\vartheta) = \begin{pmatrix}\cos\vartheta & 0 & \sin\vartheta\\ 0 & 1 & 0\\ -\sin\vartheta & 0 & \cos\vartheta\end{pmatrix}. \tag{36}$$

Note that vectors $\mathfrak{m}_r$, $\mathfrak{m}_\vartheta$, and $\mathfrak{m}_\varphi$ are independent of the azimuthal angle φ. With these $\boldsymbol{m}$-vectors in hand, one can easily use Eq. (19) to calculate QFI of the parameters r, ϑ, and φ for an arbitrary initial state. We find

$$F_{\theta_i\theta_j} = 4(p_0-p_1)^2\left[\mathfrak{m}_{\theta_i}^{\mathrm{t}}\Lambda_{\hat{\boldsymbol{n}}'}\mathfrak{m}_{\theta_j}\right],$$

where $\Lambda_{\hat{\boldsymbol{n}}'} = \mathbb{1}_3 - \hat{\boldsymbol{n}}'\hat{\boldsymbol{n}}'^{\mathrm{t}}$ with $\hat{\boldsymbol{n}}' = \mathcal{R}^{\mathrm{t}}\hat{\boldsymbol{n}}$.

As a particular case consider a spin-half system in a magnetic field B described by the Hamiltonian

$$H_\vartheta = B\left(\sin\vartheta\sigma_1 + \cos\vartheta\sigma_3\right). \tag{37}$$

This Hamiltonian can be obtained from Eqs. (20) and (26) by setting $r\to B$, $\vartheta\to\vartheta$ and $\varphi\to 0$. Suppose that the magnetic field B is known and ϑ is the parameter under estimation. In this case we find $\mathfrak{m}_\vartheta = \sin Bt\,(\cos Bt\ ,\ -\sin Bt\ ,\ 0\ )^{\mathrm{t}}$, so that

$$\boldsymbol{m}_\vartheta = \sin Bt\,(\cos\vartheta\cos Bt\ ,\ -\sin Bt\ ,\ -\sin\vartheta\cos Bt)^{\mathrm{t}} \tag{38}$$

Using this in Eq. (19) one can easily find the QFI. In this case the maximum Fisher information leads [57]

$$F_\vartheta^{\max} = 4|\boldsymbol{m}_\vartheta|^2 = 4\sin^2(Bt), \tag{39}$$

where happens for any initial pure state with Bloch vector $\hat{\boldsymbol{n}}$ perpendicular to $\boldsymbol{m}_\vartheta$. If both B and ϑ are parameters under estimations, the $\boldsymbol{m}_\vartheta$ is given by the same Eq. (38), and $\boldsymbol{m}_B$ is defined by

$$\boldsymbol{m}_B = t\left(\sin\vartheta\ ,\ 0\ ,\ \cos\vartheta\right)^{\mathrm{t}}. \tag{40}$$

For instance, if the initial state is taken in the spin $\hat{\boldsymbol{z}}$-direction, one can find the QFI matrix as

$$F = 4(p_0 - p_1)^2 \tag{41}$$
$$\times \begin{pmatrix} \sin^2(Bt)\left(1-\sin^2\vartheta\cos^2(Bt)\right) & \frac{t}{4}\sin(2\vartheta)\sin(2Bt) \\ \frac{t}{4}\sin(2\vartheta)\sin(2Bt) & t^2\sin^2\vartheta \end{pmatrix}.$$

IV. QFI OF A UNITARY PROCESS WITH TWO DIFFERENT PARAMETRIZATIONS

Let us consider a unitary transformation parameterized in terms of two different classes of parameters $\boldsymbol{\alpha} = \{\alpha_k\}$ and $\boldsymbol{\beta} = \{\beta_l\}$, i.e. $U(\boldsymbol{\alpha}) = U(\boldsymbol{\beta})$. Now the question is that if we start with the same initial state ρ_0 and encode these parameters on the state as

$$\begin{aligned}\rho_{\boldsymbol{\alpha}} &= U(\boldsymbol{\alpha})\rho_0 U^\dagger(\boldsymbol{\alpha}) \\ &= U(\boldsymbol{\beta})\rho_0 U^\dagger(\boldsymbol{\beta}) = \rho_{\boldsymbol{\beta}},\end{aligned} \tag{42}$$

what is the relation between the QFI matrices of these two parametrizations?

To address this question we should first find a relation between M-matrices with respect to these classes of parameters. To do so, we write Eq. (10) for α_k and β_l as $M_{\alpha_k} = iU^\dagger\frac{\partial U}{\partial\alpha_k}$ and $M_{\beta_l} = iU^\dagger\frac{\partial U}{\partial\beta_l}$, respectively. By using $\frac{\partial U}{\partial\beta_l} = \sum_k \frac{\partial U}{\partial\alpha_k}\frac{\partial\alpha_k}{\partial\beta_l}$, we get

$$M_{\beta_l} = i\sum_k U^\dagger\frac{\partial U}{\partial\alpha_k}\frac{\partial\alpha_k}{\partial\beta_l} = \sum_k M_{\alpha_k}S_{\alpha_k,\beta_l}, \tag{43}$$

where we have defined the transfer matrix S with matrix elements $S_{\alpha_k,\beta_l} = \frac{\partial\alpha_k}{\partial\beta_l}$. Similarly, if we write Eq. (14) for α_k and β_l and by using Eq. (43) we find a relation between SLD matrices with respect to these classes of parameters

$$(L_{\beta_l})_{\tilde{a}\tilde{b}} = \sum_k (L_{\alpha_k})_{\tilde{a}\tilde{b}}\,S_{\alpha_k,\beta_l}. \tag{44}$$

Finally the relation between various parametrizations of the QFI can be expressed by

$$F_{\beta_l\beta_{l'}} = \sum_k\sum_{k'} S_{\alpha_k,\beta_l}F_{\alpha_k\alpha_{k'}}S_{\alpha_{k'},\beta_{l'}}, \tag{45}$$

where can be written in a more compact form as

$$F^{\{\beta_l\}} = S^{\mathrm{t}}F^{\{\alpha_k\}}S. \tag{46}$$

For the simplest case of qubit systems, the relation between QFI of different parametrizations can be expressed in terms of a relation between $\boldsymbol{m}$-vectors of the corresponding parameters. Actually, if $\boldsymbol{m}_{\alpha_k}$ and $\boldsymbol{m}_{\beta_l}$ denote the $\boldsymbol{m}$-vectors of a unitary process in the $\{\alpha_k\}$ and $\{\beta_l\}$ parametrizations, respectively, we find from Eq. (43)

$$\boldsymbol{m}_{\beta_l} = \sum_k \boldsymbol{m}_{\alpha_k}S_{\alpha_k,\beta_l}. \tag{47}$$

In view of this, both $F_{\alpha_k\alpha_{k'}}$ and $F_{\beta_l\beta_{l'}}$ are given by the same relation (19) with their own $\boldsymbol{m}$-vectors replaced by $\boldsymbol{m}_{\theta_i}$. In order to show how the above algorithm works, in the example below we will obtain the QFI of a unitary process in the coset representation from the one in the canonical representation.

*E*xample.—Consider again a system described by the Hamiltonian (20) and parametrization (26). The unitary evolution generated by this Hamiltonian provides the canonical mapping of the algebra onto the group [58]. On the other hand, an arbitrary unitary matrix $U \in SU(2)$ can be written in a unique way as a product of two group elements [58]

$$U(\eta,\gamma,\xi) = \Omega^{(2)}(\gamma,\xi)\Omega^{(1)}(\eta), \tag{48}$$

where $\Omega^{(1)}(\eta) = \exp\{-i\eta\sigma_z/2\}$ is diagonal (in the computational basis $\{|0\rangle,|1\rangle\}$), corresponding to the one-dimensional Cartan subalgebra of $su(2)$, and $\Omega^{(2)}(\gamma,\xi) = \exp\{-i\gamma(\sigma_x\sin\xi + \sigma_y\cos\xi)/2\}$ is an arbitrary element of the two-dimensional quotient space $S^2 = SU(2)/U(1)$. The relation between the canonical parameters $\{r,\vartheta,\varphi\}$ and the aim parameters $\{\eta,\gamma,\xi\}$ is

$$r = \frac{1}{t}\cos^{-1}\left(\cos\frac{\gamma}{2}\cos\frac{\eta}{2}\right), \tag{49}$$

$$\vartheta = \cos^{-1}\left(\frac{\cos\frac{\gamma}{2}\sin\frac{\eta}{2}}{\sqrt{1-\cos^2\frac{\gamma}{2}\cos^2\frac{\eta}{2}}}\right), \tag{50}$$

$$\varphi = \tan^{-1}\left(\cot\left(\xi + \frac{\eta}{2}\right)\right), \tag{51}$$

where can be used to calculate the transfer matrix S. After calculating S, and regarding that we have an explicit expression for $\boldsymbol{m}$-vectors in the parameters $\{r,\vartheta,\varphi\}$, Eq (31), one can invoke Eq. (47) and get

$$\boldsymbol{m}_\eta = \frac{1}{2}\left(0,0,1\right)^{\mathrm{t}}, \tag{52}$$

$$\boldsymbol{m}_\gamma = \frac{1}{2}\left(\sin\Gamma,\cos\Gamma,0\right)^{\mathrm{t}}, \tag{53}$$

$$\boldsymbol{m}_\xi = \sin\frac{\gamma}{2}\left(\cos\frac{\gamma}{2}\cos\Gamma, -\cos\frac{\gamma}{2}\sin\Gamma, -\sin\frac{\gamma}{2}\right)^{\mathrm{t}}, \tag{54}$$

where $\Gamma = \xi + \eta$. Having these $\boldsymbol{m}$-vectors in hand, one can easily use Eq. (19) to calculate QFI of the coset parameters for an arbitrary initial state. For instance, when the initial state is diagonal in the computational basis $\{|0\rangle,|1\rangle\}$, we get

$$F(\eta,\gamma,\xi) = (p_0-p_1)^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\gamma \end{pmatrix}. \tag{55}$$

This simple form for $F(\eta,\gamma,\xi)$, in particular vanishing $F_{\eta\eta}(\eta,\gamma,\xi)$, is not surprising as we have assumed that ρ_0 is diagonal in the Cartan basis of the algebra, so that ρ cannot encode any parameters of the Cartan subalgebra.

V. CONCLUSION

In this paper, we have considered the quantum Fisher information for unitary process with special attention to $SU(2)$ process. In particular, we have presented a new formulation to calculate QFI matrix in terms of vectors $\boldsymbol{m}_{\theta_i} \in \mathbb{R}^3$, associated to each estimation parameter θ_i. Our method gives a closed relation for the QFI matrix and reveals, simply, its features. Furthermore, for a general Hamiltonian with arbitrary parametrization, we have provided a closed relation to calculate vectors $\boldsymbol{m}_{\theta_i}$. The relation is expressed in terms of derivatives of the Hamiltonian parameters with respect to the parameters under estimation. As an application we choose angle-axis parameters, both as Hamiltonian parametrization and estimation parameters, and calculate QFI. The generalization of the method to dimensions higher than two is not straightforward and is under further consideration.

Finally, using a linear transformation between two different parameter spaces of a unitary process, we find a relation between QFI matrices of two different classes of estimation parameters. This can be used, in particular, to calculate the QFI of a unitary process in terms of the one of the same process but with different parametrization, provided that the linear transformation between two parameter spaces is known. For illustration, we have applied this method for a spin-half system and obtained the QFI matrix of the coset parameters in terms of the one of the angle-axis parameters

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[1] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[2] V. Giovannetti, S. Lloyd, and L. Maccone, Nature photonics **5**, 222 (2011).
[3] J. Huang, S. Wu, H. Zhong, and C. Lee, Annual Review of Cold Atoms and Molecules **2**, 365 (2014).
[4] I. Georgescu, Nature Physics **10**, 474 (2014).
[5] P. Giorda and M. G. Paris, Phys. Rev. Lett. **105**, 020503 (2010).
[6] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[7] V. Braginsky and S. Vyatchanin, Physics Letters A **324**, 345 (2004).
[8] R. X. Adhikari, Rev. Mod. Phys. **86**, 121 (2014).
[9] J. M. McGuirk, G. T. Foster, J. B. Fixler, M. J. Snadden, and M. A. Kasevich, Phys. Rev. A **65**, 033608 (2002).
[10] G. Santarelli, P. Laurent, P. Lemonde, A. Clairon, A. G. Mann, S. Chang, A. N. Luiten, and C. Salomon, Phys. Rev. Lett. **82**, 4619 (1999).
[11] J. J. . Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).
[12] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. **79**, 3865 (1997).
[13] M. Tsang, Phys. Rev. Lett. **102**, 253601 (2009).
[14] V. Giovannetti, S. Lloyd, L. Maccone, and J. H. Shapiro, Phys. Rev. A **79**, 013827 (2009).
[15] G. Brida, M. Genovese, and I. R. Berchera, Nature Photonics **4**, 227 (2010).
[16] V. Bužek, R. Derka, and S. Massar, Phys. Rev. Lett. **82**, 2207 (1999).
[17] A. André, A. S. Sørensen, and M. D. Lukin, Phys. Rev. Lett. **92**, 230801 (2004).
[18] A. Louchet-Chauvet, J. Appel, J. J. Renema, D. Oblak, N. Kjaergaard, and E. S. Polzik, New Journal of Physics **12**, 065032 (2010).
[19] J. Borregaard and A. S. Sørensen, Phys. Rev. Lett. **111**, 090801 (2013).
[20] E. M. Kessler, P. Kómár, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, Phys. Rev. Lett. **112**, 190403 (2014).
[21] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory* (North Holland & Amsterdam, 1982).
[22] C. W. Helstrom, *Quantum detection and estimation theory* (Academic press, 1976).
[23] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[24] S. L. Braunstein, C. M. Caves, and G. J. Milburn, annals of physics **247**, 135 (1996).
[25] D. Petz, Journal of Physics A: Mathematical and General **35**, 929 (2
[26] D. Petz, Quantum Information Theory and Quantum Statistics , 143 (2008).
[27] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[28] C. Lee, Phys. Rev. Lett. **97**, 150402 (2006).
[29] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009).
[30] M. Sarovar and G. Milburn, Journal of Physics A: Mathematical and General **39**, 8487 (2006).
[31] A. Monras and M. G. A. Paris, Phys. Rev. Lett. **98**, 160401 (2007).
[32] Y. Watanabe, T. Sagawa, and M. Ueda, Phys. Rev. Lett. **104**, 020401 (2010).
[33] X.-M. Lu, X. Wang, and C. P. Sun, Phys. Rev. A **82**, 042103 (2010).
[34] B. Escher, R. de Matos Filho, and L. Davidovich, Nature Physics **7**, 406 (2011).
[35] J. Ma, Y.-x. Huang, X. Wang, and C. P. Sun, Phys. Rev. A **84**, 022302 (2011).
[36] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **109**, 233601 (2012).
[37] K. Berrada, S. Abdel-Khalek, and A.-S. Obada, Physics Letters A **376**, 1412 (2012).
[38] W. Zhong, Z. Sun, J. Ma, X. Wang, and F. Nori, Phys. Rev. A **87**, 022337 (2013).
[39] K. Berrada, Phys. Rev. A **88**, 035806 (2013).
[40] F. Ozaydin, Physics Letters A **378**, 3161 (2014).

[41] S. Alipour, M. Mehboudi, and A. T. Rezakhani, Phys. Rev. Lett. **112**, 120405 (2014).
[42] M. Ban, Quantum Information Processing **14**, 4163 (2015).
[43] R. A. Fisher, Proceedings of the Cambridge Philosophical Society **22**, 700 (1925).
[44] J. Xiao-Xing, L. Jing, Z. Wei, and W. Xiao-Guang, Communications in Theoretical Physics **61**, 115 (2014).
[45] S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, Phys. Rev. A **83**, 021804 (2011).
[46] L. Jing, J. Xiao-Xing, Z. Wei, and W. Xiao-Guang, Communications in Theoretical Physics **61**, 45 (2014).
[47] D. Bures, Transactions of the American Mathematical Society **135**, 199 (1969).
[48] A. Uhlmann, Reports on Mathematical Physics **24**, 229 (1986).
[49] M. Hübner, Physics Letters A **163**, 239 (1992).
[50] S. Luo and Q. Zhang, Phys. Rev. A **69**, 032106 (2004).
[51] L. Jing, J. Xiao-Xing, Z. Wei, and W. Xiao-Guang, Communications in Theoretical Physics **61**, 45 (2014).
[52] S. Boixo, S. T. Flammia, C. M. Caves, and J. Geremia, Phys. Rev. Lett. **98**, 090401 (2007).
[53] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Phys. Rev. Lett. **110**, 050402 (2013).
[54] J. Liu, X.-X. Jing, and X. Wang, Scientific reports **5** (2015).
[55] X.-X. Jing, J. Liu, H.-N. Xiong, and X. Wang, Phys. Rev. A **92**, 012312 (2015).
[56] R. M. Wilcox, Journal of Mathematical Physics **8** (1967), 10.1063/1.1
[57] S. Pang and T. A. Brun, Phys. Rev. A **93**, 059901 (2016).
[58] R. Gilmore, *Lie groups, Lie algebras, and some of their applications* (Courier Corporation, 2012).